Exam MFE/Exam 3F Review

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1 Put-Call Parity

1.1 Review

1.1.1 Forwards

Definition. *A forward is an agreement to buy something at a future date for a certain price.*

We will use the notation $F_{t,T}$ to indicate the price to be paid at time T in a forward agreement made at time t to buy an item at time T. Notice that no payment is made at time t; the only payment made is at time T. If you purchase a forward on a stock at time t, you will pay $F_{t,T}$ at time T and receive the stock.

Denote the price of the stock at time T as S_T.

Denote the continuously compounded risk-free interest rate as r.

Definition. *An arbitrage is a set of transactions which when combined have no cost, no possibility of loss and at least some possibility of profit.*

Here are assumptions over the entire review notes :

1. It is impossible to borrow or lend any amount of money at the risk-free rate
2. There are no transaction charges or taxes
3. Arbitrage is impossible

Forwards on stock

We will consider 3 possibilities for the stock :

1. The stock pays no dividends
2. The stock pays discrete dividends
3. The stock pays continuous dividends

Forwards on non-dividend paying stock $a)$ Method #1 : Buy stock at time t and hold it to time T $b)$ Method #2 : Buy forward on stock at time t and hold it to time T

$$F_{t,T} = S_t e^{r(T-t)}$$

Forwards on a stock with discrete dividends $a)$ Method #1 : Buy stock index at time t and hold it to time T $b)$ Method #2 : Buy $e^{\delta T}$ forwards on stock index at time t and hold it to time T

$$F_{t,T} = S_t e^{r(T-t)} - CumValue(Div)$$

, where $CumValue(Div)$ is the accumulated value at time T of dividends from time t to time T.

Forwards on a stock index with continuous dividends $a)$ Method #1 : Buy stock index at time t and hold it to time T $b)$ Method #2 : Buy $e^{\delta T}$ forwards on stock index at time t and hold it to time T

$$F_{t,T} = S_t e^{(r-\delta)(T-t)}$$

.

Forwards on currency For forwards on currency, it is assumed that each currency has its own risk-free interest rate. The risk-free interest rate for the foreign currency plays the role of a continuously compounded dividend on a stock.

$$F_{t,T} = x_t e^{(r_d - r_f)(T-t)}$$

1.1.2 Call and put options

Call option
1. premium : $C(K,T)$ where K is the strike price at expiration T.
2. the purchaser's payoff : $max(0, S_T - K)$

Put option
1. premium : $P(K,T)$
2. the purchaser's payoff : $max(0, K - S_T)$

European options only exercise at expiration

American options exercise at any time up to expiration

1.1.3 *Combinations of options

When we buy X, we are said to be long X, and when we sell X, we are said to be short X.

Spreads : buying an option and selling another option of the same kind

Bull spreads A bull spread pays off if the stock moves up in price, but subject to a limit. To create a bull spread with calls, buy a K_1-strike call and sell a K_2-strike call, $K_2 > K_1$. To create a bull spread with puts, buy a K_1-strike put and sell a $K_2 - strike$ put, $K_2 > K_1$.

Bear spreads A bear spread pays off if the stock moves down in price, but subject to a limit. To create a bear spread with calls, buy a K_1-strike call and sell a K_2-strike call, $K_2 < K_1$. To create a bull spread with puts, buy a K_1-strike put and sell a $K_2 - strike$ put, $K_2 < K_1$.

Ratio spreads A ratio spread involves buying n of one-option and selling m of another option of the same kind, where $m \neq n$. It is possible to make the net initial cost of this strategy zero.

Box spreads A box spread is a four option strategy consisting of buying a bull spread of calls with strikes K_2 and K_1 ($K_2 > K_1$) and buying a bear spread of puts with strikes K_2 and K_1. It has a definite profit $K_2 - K_1$.

Butterfly spreads A butterfly spread is a there-option strategy, all options of the same type, consisting of buying n bull spreads with strike prices K_1 and $K_2 > K_1$ and selling m bull spreads with strike prices K_2 and $K_3 > K_2$, with m and n selected so that if the stock price S_T at expiry is greater than K_1, the payoffs net to zero.

Calendar spreads Calendar spreads involve buying and selling options of the same kind with different expiry dates.

Collars : buying one option and selling an option of the other kind

In a collar you sell a call with strike K_2 and buy a put with strike $K_1 < K_2$.

Straddles : buying two options of different kinds

In a straddle, you buy a call and a put, both of them at-the-money ($K = S_0$). To lower the initial cost, you can buy a put with strike price K_1 and buy a call with strike price $K_2 > K_1$; then this strategy is called a strangle.

1.2 Put-call parity

assumption Buy a European call option and sell a European put option, both having the same underlying asset, the same strike, and the same time to expiry.

$$C(K,T) - P(K,T) = e^{-rT}(F_{0,T} - K)$$

1.2.1 Stock put-call parity

non-dividend paying stock

$$C(K,T) - P(K,T) = S_0 - Ke^{-rT}$$

prepaid forward

$$C(K,T) - P(K,T) = F^P_{0,T} - Ke^{-rT}$$

dividend paying stock a) discrete dividend

$$C(K,T) - P(K,T) = S_0 - PV_{0,T}(Divs) - Ke^{-rT}$$

b) continuous dividend

$$C(K,T) - P(K,T) = S_0 e^{-\delta T} - Ke^{-rT}$$

1.2.2 Synthetic stocks and Treasuries

Note : Creating a synthetic Treasury is called a conversion. Selling a synthetic Treasury by shorting the stock, buying a call and selling a put, is called a reverse conversion.

Create an investment equivalent to a stock with continuous dividend rate of r Buy $e^{\delta T}$ call options and sell $e^{\delta T}$ put options, and buy a treasury for $Ke^{(\delta-r)T}$.

Create an investment equivalent to a stock with discrete dividend Buy a call, sell a put and lend $PV(dividends) + Ke^{-rT}$.

Create a synthetic Treasury in continuous dividend case Buy $e^{-\delta T}$ shares of the stock and buy a put and sell a call. The treasuries' maturity value is K.

Create a synthetic Treasury in discrete case Buy a stock and a put and sell a call. The treasuries' maturity value is $K + CumValue(dividends)$.

1.2.3 Synthetic Options

Situation There exists some misplace for options. Suppose the price of a European call based on put-call parity is C, but the price it is actually selling at is $C' < C$

Consequence Buy the underpriced call option and sell a synthesized call option

$$C(S,K,t) = Se^{-\delta t} - Ke^{-rt} + P(S,K,t)$$

. Well, you can pay $C(S,K,t)$ for the option and keep the rest.

1.2.4 Exchange options

Strike asset Q_t

Underlying asset S_t

$$C(S_t, Q_t, T-t) - P(S_t, Q_t, T-t) = F^P_{t,T}(S) - F^P_{t,T}(Q)$$

or

$$C(S_t, Q_t, T-t) - C(Q_t, S_t, T-t) = F^P_{t,T}(S) - F^P_{t,T}(Q)$$

1.2.5 Currency options

$$C(x_0, K, T) - P(x_0, K, T) = x_0 e^{-r_f T} - K e^{-r_d T}$$

How to change a call option to purchase pounds with dollars to a put to sell dollars for pounds?

Here is one for the general case :

$$K P_d(\frac{1}{x_0}, \frac{1}{K}, T) = C_d(x_0, K, T)$$

or

$$K x_0 P_f(\frac{1}{x_0}, \frac{1}{K}, T) = C_d(x_0, K, T)$$

Remark. *Doing exchange options or currency options, you have to be clear about what is strike asset and underlying asset. For instance, in dollar-denominated exchange, dollar is strike asset and the foreign currency is underlying asset; in buying stock A in exchange for stock B, Stock A is underlying asset and Stock B is strike asset.*

While calculating premium for options, you need to normalize it to one unit.

2 Comparing Options

2.1 Bounds for Option Prices

$$S \geq C_{Amer}(S, K, T) \geq C_{Eur}(S, K, T) \geq \max(0, F^P_{0,T}(S) - K e^{-rT})$$

$$K \geq P_{Amer}(S, K, T) \geq P_{Eur}(S, K, T) \geq \max(0, K e^{-rT} - F^P_{0,T}(S))$$

2.2 Early exercise of American options

implicit options Every call option has an implicit put option built in it and similarly every put option has an implicit call option built in it.

Call options on non dividend paying stocks Early exercising will have couple of disadvantage : you lose protection against the price of the stock going below the strike price; you must pay K earlier and lose interest on the strike price. Hence early excessing is not rational.

Consider the following :

$$C_{Eur}(S_t, K, T-t) = P_{Eur}(S_t, K, T-t) + (S_t - K) + K(1 - e^{-r(T-t)}) \geq S_t - K$$

Above means the value of the option is greater than the exercise value.

Call options on dividend paying stock

$$C_{Eur}(S_t, K, T-t) = P_{Eur}(S_t, K, T-t) + (S_t - K) + K(1 - e^{-r(T-t)}) - PV_{t,T}(Div)$$

The put must be worth at least zero. Early exercise will not be rational if $PV_{t,T}(Div) < K(1 - e^{-r(T-t)})$.

We can have a similar decomposition for put option :

$$P_{Eur}(S_t, K, T-t) = C_{Eur}(S_t, K, T-t) + (S_t - K) - K(1 - e^{-r(T-t)}) + PV_{t,T}(Div)$$

The second term is exercise value. If the other terms add up to a negative, it may be rational to exercise early.

To summarize : For Stocks without dividends :

- An American call option is worth the same as a European call option
- An American put option may be worth more than a European option.

2.3 Time to expiry

For two European call options on a non-dividend-paying stock, the one with the longer time to expiry must be worth at least as much as the other one. Well, we can make a similar statement for American call options : a longer-lived American call option with a strike price increasing at the risk-free rate must be worth at least as much as a shorted lived option. Similar for European put options with non-dividend-paying stock.

Here is a summary of relationships of option prices and time to expiry

1. An American option with expiry T and strike price K must cost at least as much as one with expiry t and strike price K.
2. A European call option on a non-dividend paying stock with expiry T and strike price K must cost at least as much as one with expiry t and strike price K.
3. A European option on a non-dividend paying stock with expiry T and strike price $Ke^{-r(T-t)}$ must cost at least as much as one with expiry t and strike price K.

2.4 Different strike prices

2.4.1 Three inequalities

Two ways to create an arbitrage :

1. Create a position which results in maximal immediate gain, and which cannot possibly lose as much as the initial gain in the future.

2. Create a position which results in minimal immediate gain, but which has the possibility of future gain.

Direction For a call option, the higher the strike price, the lower the premium. For a put option, the higher the strike price, the higher the premium. Algebraically, it is easy to see that $\frac{\partial C(S,K,T)}{\partial K} \leq 0$ and $\frac{\partial P(S,K,T)}{\partial K} \geq 0$.

To create the first arbitrage, sell one K_2 option and buy one K_1 option. To create the second one, sell one K_2-strike option and buy $C(S, K_2, T)/C(S, K_1, T)$ K_1-strike calls.

Slope The premium for a call option decreases more slowly than the strike price increases. The premium for a put option increases more slowly than the strike price increases. Algebraically, it means $\frac{\partial C(S,K,T)}{\partial K} \geq -1$ and $\frac{\partial P(S,K,T)}{\partial K} \leq 1$.

Convexity The rate of decrease in call premiums as a function of K decreases. The rate of increase in put premiums as a function of K increases. Algebraically, it means $\frac{\partial^2 C(S,K,T)}{\partial K^2} \geq 0$ and $\frac{\partial^2 P(S,K,T)}{\partial K^2} \geq 0$.

2.4.2 Options in the money

An option is in the money if it would have a positive payout if it could be exercised. A call option is in the money if the strike price is less than the underlying asset price. A put option is in the money if the strike price is more than the underlying asset price.

An option is out of money if the price of the underlying asset is different from the strike price in such a way that the option doesn't pay off.

An option is at the money if the strike price equals the underlying asset's price.

Given two American call options with the same expiry in the money, $C(S, K_2, T)$ and $C(S, K_1, T)$, with $K_2 > K_1$, if exercising $C(S, K_2, T)$ is optimal, then so is exercising $C(S, K_1, T)$, an option is even more in the money. The same holds true for puts.

3 Binomial Trees - Stock, One Period

$$\Delta = (\frac{C_u - C_d}{S(u-d)})e^{-\delta h}$$

$$B = e^{-rh}(\frac{uC_d - dC_u}{u-d})$$

The option premium is $\Delta S + B$.

Risk-neutral probabilities

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d}$$

Volatility

$$u = e^{(r-\delta)h+\sigma\sqrt{h}}, d = e^{(r-\delta)h-\sigma\sqrt{h}}$$

$$p^* = \frac{1}{1+e^{\sigma\sqrt{h}}}$$

4 Binomial Trees - General

Algorithm to get the option price for American or European call option.

1. Build the tree to the last period
2. Use the formula in previous chapter to calculate the value of the nodes in previous period. If it is American option, we use the max of the price and the maximum payoff; if it is European option, we use the max of the price and 0.
3. Eventually we will get the price for the option.

5 Risk-neutral Pricing and Utility

Here I will introduce couple of notations

- U_i denotes the current value of \$1 paid at the end of one year when the price of the stock is in state i, $i = H, L$. $U_H \leq U_L$ because of declining marginal utility. Since risk-neutral, $U_H = U_L = \frac{1}{1+r}$
- C_i denotes the cash flow of the stock at the end of one year in state i, $i = H, L$.
- Q_i denotes the current value of \$1 paid at the end of one year only if the price of the stock is C_i, $i = H, L, 0$ otherwise.

Let's denote p be the true probability of state H. $Q_H = pU_H, Q_L = (1p)U_L$.
$Q_H + Q_L = \frac{1}{1+r}$.
$C_0 = pU_HC_H + (1-p)U_LC_L = Q_HC_H + Q_LC_L = Q_HC_H + Q_LC_L$
Let's consider α is the rate of return. It is easy to get from the above formula.
$p^* = \frac{Q_H}{Q_H+Q_L} = Q_H(1+r) = pU_H(1+r)$

6 Lognormality and Alternative Tree

Alternative trees:

1. Cox-Ross-Rubinstein tree: $u = e^{\sigma\sqrt{h}}, d = e^{-\sigma\sqrt{h}}$

2. Lognormal tree: $u = e^{(r-\delta-0.5\sigma^2)h+\sigma\sqrt{h}}, d = e^{(r-\delta-0.5\sigma^2)h-\sigma\sqrt{h}}$ (Jarrow-Rudd)

3. Normal: $u = e^{(r-\delta)h+\sigma\sqrt{h}}, d = e^{(r-\delta)h-\sigma\sqrt{h}}$

All three tress have $\ln(u/d) = 2\sigma\sqrt{h}$

Estimating volatility

$$\hat{\sigma} = \sqrt{p}\sqrt{\frac{n}{n-1}(\frac{\sum_{i=1}^{n} x_i^2}{n} - \bar{x}^2)}$$

7 Modelling Stock Price with the Lognormal Distribution

For a stock whose price S_t follows a lognormal model:

- The expected value is $E[S_t|S_0] = S_0 e^{(\mu+0.5\sigma^2)t}$
- $\hat{d}_1$ and $\hat{d}_2$ are defined by $\hat{d}_1 = \frac{\ln(S_0/K)+(\alpha-\delta+0.5\sigma^2)t}{\sigma\sqrt{t}}$ $\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{t}$
- Probabilities of payoffs and partial expectation of stock prices are.

$$Pr(S_t < K) = N(-\hat{d}_2)$$

$$Pr(S_t > K) = N(\hat{d}_2)$$

$$PE[S_t|S_t < K] = S_0 e^{*(\alpha-\delta)t} N(-\hat{d}_1)$$

$$E[S_t|S_t < K] = \frac{S_0 e^{*(\alpha-\delta)t} N(-\hat{d}_1)}{N(-\hat{d}_2)}$$

$$PE[S_t|S_t > K] = S_0 e^{*(\alpha-\delta)t} N(\hat{d}_1)$$

$$E[S_t|S_t > K] = \frac{S_0 e^{*(\alpha-\delta)t} N(\hat{d}_1)}{N(\hat{d}_2)}$$

- Expected option payoffs are

  -call :

$$E[max(0, S_t - K)] = S_0 e^{*(\alpha-\delta)t} N(\hat{d}_1) - KN(\hat{d}_2)$$

  -put :

$$E[max(0, K - S_t)] = -S_0 e^{*(\alpha-\delta)t} N(-\hat{d}_1) + KN(-\hat{d}_2)$$

8 The Black-Scholes Formula

$$C(S, K, \sigma, r, T, \delta) = F^P(S)N(d_1) - F^P(K)N(d_2)$$

where

$$d_1 = \frac{\ln(F^P(S)/F^P(K)) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

This is couple of assumptions of the Black-Scholes formula

1. Continuously compounded returns on the stock are normally distributed and independent over time
2. Continuously compounded returns on the strike asset are known and constant
3. Volatility is known and constant.
4. Dividends are known and constant
5. There are no transaction costs or taxes
6. It is possible to short-sell any amount of stock and to borrow any amount of money at the risk-free rate.

9 Application of Black-Scholes Formula and Greeks

Delta $\Delta = \frac{\partial C}{\partial S}$ S shaped

Gamma $\Gamma = \frac{\partial^2 C}{\partial S^2}$ Symmetric hump, peak to left of strike price, further left with higher t.

Vega $0.01\frac{\partial C}{\partial \sigma}$ Asymmetric hump, peak similar to Γ

Theta $\theta = \frac{\partial C}{365\partial t}$ Upside-down hump for short lives, gradual decrease for long lives, unless δ is high. Almost always negative for calls, usually negative for puts unless far in-the-money.

Rho $\rho = 0.01\frac{\partial C}{\partial r}$ increasing curve

Psi $\psi = 0.01\frac{\partial C}{\partial \delta}$ decreasing curve

$$\Delta_{call} = e^{-\delta T}N(d_1)$$

$$\Delta_{put} = \Delta_{call} - e^{-\delta T} = -e^{-\delta T}N(-d_1)$$

9.1 Elasticity

$\Omega = \frac{S\Delta}{C}$

9.2 Sharpe ratio

$$\sigma_{option} = \sigma_{stock}|\Omega|$$
$$\frac{\gamma - r}{\sigma_{option}} = \frac{\alpha - r}{\sigma_{stock}}$$

9.3 Last Notes

Greek for portfolio = sum of the Greeks

Elasticity for portfolio = weighted average of the elasticities

Note: The volatility calculated by Black-Scholes formula is called implied volatility; the volatility calculated by history data is called historical volatility. We use the difference of the price of an option at date t_2 and the future value of an option of date t_1 at date t_2 to get the profit of an option overtime.

10 Delta Hedging

10.1 Overnight profit on a delta-hedge portfolio

Three components

1. The change in the value of the option
2. Δ times the change in the price of the stock
3. Interest on the borrowed money

1 and 2 is called mark-to-market.

$$\text{Profit} = C(S_0) - C(S_1) + \Delta(S_1 - S_0) - (e^{r/365} - 1)(\Delta S_0 - C(S_0))$$

10.2 The delta-gamma-theta approximation

$$C(S_1) = C(S_0) + \Delta\epsilon + \frac{1}{2}\Gamma\epsilon^2 + h\theta$$

$$\text{Market Maker Profit} = -(\frac{1}{2}\Gamma\epsilon^2 + \theta h + rh(\Delta S - C(S)) - \delta h \Delta S)$$

We'll plug

$$\epsilon = \pm\sigma S\sqrt{h}$$

and set the profit to be zero. Hence we will get the Black-Sholes equation.

10.3 Greeks for binomial trees

$$\Delta(`S,0) = (\frac{C_u - C_d}{S(u-d)})e^{-\delta h}$$

$$\Gamma(S,0) \approx \Gamma(S,h) = \frac{\Delta(Su,h) - \Delta(Sd,h)}{Su - Sd}$$

$$C(Sud,2h) = C(S,0) + \Delta(S,0)\epsilon + 0.5\Gamma(S,0)\epsilon^2 + 2h\theta(S,0)$$

10.4 Rehedging

Boyle-Emanuel formula

$$R_{h,i} = \frac{1}{2}S^2\sigma^2(x_i^2 - 1)\Gamma h$$

$$Var(R_{h,i}) = \frac{1}{2}(S^2\sigma^2\Gamma h)^2$$

11 Other Options

11.1 Asian Options

Denote $A(S), G(S)$ as arithmetic mean of stock price overtime and geometric mean of stock price overtime.

1. Call with strike price $C = max(0, A(S) - K)$
2. Put with strike price $P = max(0, K - A(S))$
3. Strike call $C = max(0, S_t - A(S))$
4. Strike put $P = max(0, A(S) - S_t)$

$G(S)$ has similar option.

11.2 Barrier options

Knock-out : reach barrier, the option doesn't exercise. (down-and-out, up-and-out)
Knock-in : reach barrier, the option does exercise. (down-and-in, up-and-in)

Rebate option

Pay a little amount when hit the barrier.

$$\text{Knock-in option} + \text{Knock-out option} = \text{Ordinary option}$$

11.3 Maxima and Minima

$$max(S,K) = S + max(0, K - S) = K + max(S - K, 0)$$

$$max(cS, cK) = cmax(S,K), c > 0$$

$$max(-S, -K) = -min(S,K)$$

$$min(S,K) + max(S,K) = S + K$$

11.4 Compound options

$$CallOnCall(S,K,x,\sigma,r,t_1,T,\delta) - PutOnCall(S,K,x,\sigma,r,t_1,T,\delta) = C(S,K,\sigma,r,T,\delta) - xe^{-rt_1}$$

$$CallOnPut(S,K,x,\sigma,r,t_1,T,\delta) - PutOnPut(S,K,x,\sigma,r,t_1,T,\delta) = P(S,K,\sigma,r,T,\delta) - xe^{-rt_1}$$

11.5 American options on stocks with one discrete dividend

It is not rational to exercise early for American call options.

ex-dividend After the dividend is paid

cum-dividend Including the dividend; before the dividend is paid

Suppose we have one discrete dividend for the stock. Then at time t_1, the value of the option is

$$max(S_{t_1} + D - K, C(S_{t_1}, K, T - t_1))$$

Applying the maxima and minima, we will get

$$C^{American}(S,K,T) = S_0 - Ke^{-rt_1} + CallOnPut(S,K,D - (K(1 - e^{-r(T-t_1)}), t_1, T)$$

It is optimal to exercise if the value of the put is less than $D - K(1 - e^{-r(T-t_1)})$

11.6 Bermudan options

Exercise at specified date. Similar pricing strategy as American option with one discrete dividend payment.

11.7 All-or-nothing options

1. asset-or-nothing call option $S|S > K$

$$S_0 e^{-\delta T} N(d_1)$$

$$\Delta = e^{-\delta T} N(d_1) + e^{-\delta T} \frac{e^{-d_1^2/2}}{\sigma\sqrt{2\pi T}}$$

2. asset-or-nothing put option $S|S < K$

$$S_0 e^{-\delta T} N(-d_1)$$

$$\Delta = e^{-\delta T} N(-d_1) - e^{-\delta T} \frac{e^{-d_1^2/2}}{\sigma\sqrt{2\pi T}}$$

3. cash-or-nothing call option $c|S > K$

$$ce^{-rT} N(d_2)$$

$$\Delta = e^{-rT} \frac{e^{-d_2^2/2}}{S\sigma\sqrt{2\pi T}}$$

4. cash-or-nothing put option $c|S < K$

$$ce^{-rT} N(-d_2)$$

$$\Delta = -e^{-rT} \frac{e^{-d_2^2/2}}{S\sigma\sqrt{2\pi T}}$$

An ordinary option is the asset-or-nothing option and cash(-K)-or-nothing option.

11.8 Gap option

Put trigger to activate the exercise of the option.

A gap call option is $S|S > K_2 - K_1|S > K_2$ and a gap put option is $K_1|S < K_2 - S|S < K_2$

11.9 Exchange option

$$\sigma^2 = \sigma_S^2 + \sigma_Q^2 - 2\rho\sigma_S\sigma_Q$$

For the option price, we replace δ by δ_S and r by δ_Q.

11.10 Chooser options

A chooser options allow you to choose at time $t \leq T$ to take either a call or put option expiring at time T, both with the same strike price K.

At time t, the option worth $V = C(S, K, T) + P(S, K, T)$ if $t = T$.

If $t < T$, $V_t = max(C(S, K, T-t), P(S, K, T-t))$. By put-call parity and maxima and minima, we get

$$V = C(S, K, T) + e^{-\delta(T-t)} P(S, Ke^{-(r-\delta)(T-t)}, t)$$

11.11 Forward start options

This is basically a forward contract on an option.

If you can purchase a call option with strike price cS_t, at time t expiring at time T, then the value of the forward start option is

$$V = Se^{-\delta T} N(d_1) - cSe^{-r(T-t)-\delta t} N(d_2)$$

12 Monte Carlo Valuation

12.1 Generating Lognormal random numbers

Three steps:

1. Generate a standard normal random number z_j

   (a) Method 1: add up twelve u_i's and subtract 6. In other words,

   $$z_j = \sum_{i=1}^{12} u_i - 6$$

   (b) Method 2: the inversion method. In other words,

   $$z_j = N^{-1}(u_j)$$

2. Generate an $N(\mu, \sigma^2)$ random number n_j

   $$n_j = \mu + \sigma z_j$$

3. Generate the desired lognormal random number x_j

   $$x_j = e^{n_j}$$

- the true distribution of the growth in stock prices over one year is lognormal with annual parameters $\mu = \alpha - \mu - 0.5\sigma^2$
- the risk-neutral distribution of the growth in stock prices over one year is lognormal with annual parameters $\mu = r - \mu - 0.5\sigma^2$

12.2 Control Variate Method

$$X^* = \bar{X} + (E[Y] - \bar{Y})$$

$$Var(X^*) = Var(\bar{X}) + Var(\bar{Y}) - 2Cov(\bar{X}, \bar{Y})$$

For Boyle modification, we add β.

$$\beta = \frac{Cov(\bar{X}, \bar{Y})}{Var(\bar{Y})}$$

The minimum variance possible using a control variate is the variance of the naive estimate times the complement of the square of the correlation coefficient.

12.3 Other variance reduction techniques

Antithetic Variates The antithetic variate method uses, for every uniform number u_i, the uniform number $1 - u_i$ as well.

Stratified sampling Stratified sampling means breaking the sampling space into strata and then scaling the uniform numbers to be in these strata. Rather than picking uniform numbers randomly, you guarantee that each stratum has an appropriate amount of random numbers.

13 Brownian Motion

Properties

1. Memoryless : Given $X(t) = k$, the probability that $X(t+\mu) = 1$ given that $X(t) = k$ is the same as $Pr(X(\mu) = l - k)$

$$Pr(X(t + \mu) = l | X(t) = k) = Pr(X(\mu) = l - k)$$

2. $X(t)$ is random but the distance traversed is not random. $\sum \text{movement}^2 = t$.
3. $\frac{1}{2}(X(t) + t) \sim BIN(t, \frac{1}{2})$

Let $Z(t)$ be Brownian Motion

- $Z(0) = 0$
- $Z(t+s)|Z(t) \sim N(\mu = Z(t), \sigma^2 = S)$
- Increments are independent; $Z(t+s_1) - Z(t)$ is independent of $Z(t) - Z(t-s_2)$
- $Z(t)$ is continuous at t.

13.1 Arithmetic Brownian Motion

$$X(t) = X(0) + \alpha t + \sigma dZ(t)$$
$$X(t+s) - X(t) = st + \sigma(Z(t+s) - Z(t)) \sim N(s, \sigma^2 s)$$
$$X(t+s)|X(t) \sim N(X(t) + \alpha s, \sigma^2 s)$$

13.2 Geometric Brownian Motion

$$\ln(X(t)/X(0)) \sim N(\mu t, \sigma^2 t)$$

$$X(t)/X(0) \sim LOGNORM(e^{\mu t + 0.5\sigma^2 t}, e^{2\mu t + \sigma^2 t}(e^{\sigma^2 t} - 1))$$

From Geometric Brownian Motion to the associated arithmetic Brownian motion, subtract $0.5\sigma^2$

$$m = (\alpha - \delta - 0.5\sigma^2)t, v = \sigma\sqrt{t}$$

13.3 Covariance

Arithmetic
$$Cov(X(t), X(\mu)) = \sigma^2 min(t, \mu)$$

Geometric
$$Cov(X(t), X(\mu)) = X(0)^2 e^{(\alpha-\delta)(t+\mu)}(e^{t\sigma^2} - 1)$$

13.4 Additional Terms

Definition. *A diffusion process is a continuous process in which the absolute value of the random variable tends to get larger.*

Definition. *A martingale is a process $X(t)$ for which $E[X(t+s)|X(t)] = X(t)$. For Brownian motion, this follows from the second property above.*

14 Differentials

$$d(\ln X(t)) = (\xi - 0.5\sigma^2)dt + \sigma dZ(t)$$

15 It$\hat{o}$'s Lemma

$$dC = C_S dS + 0.5 C_{SS}(dS)^2 + C_t dt$$

15.1 Multiplication Rules

$$dt \times dt = dt \times dZ = 0$$

$$dZ \times dZ = dt$$

$$dZ \times dZ' = \rho dt$$

15.2 Ornstein-Uhlenbeck Process

$$dX(t) = \lambda(\alpha - X(t))dt + \sigma dZ(t)$$

$$X(t) = X(0)e^{-\lambda t} + \alpha(1 - e^{-\lambda t}) + \sigma \int_0^t e^{\lambda(s-t)} dZ(s)$$

16 The Black-Scholes Equation

$$0.5 S^2 \sigma^2 C_{SS} + C_S S(r - \delta) + C_t = rC$$

or

$$\Delta S(r - \delta) + 0.5 \Gamma S^2 \sigma^2 + \theta = rC$$

17 Sharpe Ratio

$$\phi(t, S(t)) = \frac{\alpha(t, S(t)) - r}{\sigma(t, S(t))}$$

The Sharpe Ratio may vary with time (t), with the risk-free rate $r(t)$ which itself may vary with time, or with the Brownian motion $Z(t)$ that is part of the $S(t)$. However at any time t, for two Itô processes depending on the same $Z(t)$ the Sharpe ratios are equal.

17.1 Risk-free Portfolio

If two assets with prices $X_1(t)$ and $X_2(t)$ follow

$$\frac{X_1(t)}{X_1(t)} = (\alpha_1 - \delta_1)dt + \sigma_1 dZ(t)$$

$$\frac{X_2(t)}{X_2(t)} = (\alpha_2 - \delta_2)dt + \sigma_2 dZ(t)$$

and the continuously compounded risk-free rate is r, a risk-free portfolio may be created by buying c_1 shares of X_1 and c_2 shares of X_2. (A negative sign on c_i implies shares are sold.) c_1 and c_2 may be determined in either of the following ways:

1. Set
$$c_1 X_1(0)\sigma_1 + c_2 X_2(0)\sigma_2 = 0$$

2. Set
$$\frac{c_1 X_1(0)\alpha_1 + c_2 X_2(0)\alpha_2}{c_1 X_1(0) + c_2 X_2(0)} = r$$

17.2 CAPM

The Capital Asset Pricing Model says that the return α_i on any asset is equal to the risk-free return plus β_i times the risk premium of the market.

$$\alpha_i = r + \beta_i(\alpha_M - r)$$

$$\beta_i = \frac{\rho_{i,M}\sigma_i}{\sigma_M}$$

$$\phi_i = \rho_{i,M}\phi_M$$

18 Risk-Neutral Pricing and Proportional Portfolios

We can translate a true Itô process into a risk neutral process. Simply change $dZ(t)$ to $d\tilde{Z}(t)$.

$$d\tilde{Z}(t) = dZ(t) + \eta dt$$

where η is Sharpe ratio.

η measures drift of the motion so if you want to get $Z(t)$, it is $-\eta \times t$

Notes: risk-neutral pricing is used for those that do not satisfy geometric Brownian motion.

18.1 Proportional portofolio

Let's analyze a portfolio consisting of a risky asset and a risk-free asset in which the risky asset is a constant proportion, φ, of the portfolio. This implies continuous rehanging. Let's call the process followed by the blended portfolio, $W(t)$

$$d\ln W(t) = (\varphi\alpha + (1-\varphi)r)dt + \sigma dZ(t)$$

Solving this we will get

$$W(t) = W(0)(\frac{S(t)}{S(0)})^{\varphi} e^{[(1-\varphi)(r+0.5\varphi\sigma^2]t}$$

if there are no dividends

$$W(t) = W(0)(\frac{S(t)}{S(0)})^{\varphi} e^{[\varphi\delta_S-\delta_w+(1-\varphi)(r+0.5\varphi\sigma^2]t}$$

if there are dividends

19 S^a

19.1 Valuing a forward on S^a

$$E[S(T)^a] = S(0)^a e^{[a(\alpha-\delta)+0.5a(a-1)\sigma^2]T}$$

Therefore

$$F_{0,T}(S^a) = S(0)^a e^{[a(r-\delta)+0.5a(a-1)\sigma^2]T}$$

$$F^P_{0,T}(S^a) = e^{-rT} S(0)^a e^{[a(r-\delta)+0.5a(a-1)\sigma^2]T}$$

19.2 The Itô's process for S^a

$$\frac{dC}{C} = (a(\alpha-\delta) + 0.5a(a-1)\sigma^2)dt + a\sigma dZ(t)$$

$$\gamma = a(\alpha - r) + r$$

20 Stochastic Integration

20.1 integration

$$\int_0^a dZ(t) = \lim_{n\to\infty}\sum_{i=1}^{n}(Z(\frac{ia}{n}) - Z(\frac{(i-1)a}{n}))$$

$$\int_a^b d(X(t)) = X(b) - X(a)$$

$$d(\int_0^t X(s)dZ(s)) = X(t)dZ(t)$$

(This is the an example, so when you see this similar expression, simply replace s by t)

Note: To solve Stochastic integral, we can get the integral by ordinary calculus and then apply Itô lemma.

20.2 Quadratic Variation

Quadratic variation

$$\lim_{n\to\infty}\sum_{i=1}^{n}(X(\frac{iT}{n}) - X(\frac{(i-1)T}{n}))^2$$

Total variation

$$\lim_{n\to\infty}\sum_{i=1}^{n}|X(\frac{iT}{n}) - X(\frac{(i-1)T}{n})|$$

Conclusion for Brownian motion: In any finite interval, the path crosses its starting point infinitely often with probability 1.

21 Binomial Tree Models for Interest Rates

$P_t(T, T+s)$ will be the notation for the price, to be paid at time T, for an agreement at time t to purchase a zero-coupon bond for 1 issued at time T maturing at time $T+s, t \leq T$. Let $F_{t,T}(P(T, T+s))$ be the forward price at time t for an agreement to buy a bond at time T maturing at time $T+s$. Above represent the same stuff.

$$F_{t,T}(P(T, T+s)) = \frac{P(t, T+s)}{P(t, T)}$$

21.1 The Black-Derman-Toy model

$$\sigma_h = \frac{\ln(r_{h1}/r_{h2})}{2\sqrt{h}}$$

For cap, the cap pays $Lmax(0, \frac{R_T - K_R}{1+R_T})$

22 The Black Formula for Bond Options

$$C(F, P(0,T), \sigma, T) = P(0,T)(FN(d_1) - KN(d_2))$$
$$P(F, P(0,T), \sigma, T) = P(0,T)(-FN(-d_1) + KN(-d_2))$$

where

$$d_1 = \frac{\ln(F/K) + 0.5\sigma^2 T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

and σ is the volatility of the T-year forward price oof the bond.

Black formula for caplets: each caplet is $1 + K_R$ puts with strike $\frac{1}{1+K_R}$

23 Equilibrium Interest Rate Models: Vasicek and Cox-Ingersoll-Ross

$$R(t,T) = \frac{\ln(1/P(t,T))}{T-t}$$

the continuously compounded interest rate for zero coupon bond. Let $r(t) = \lim_{T\to t} R(t,T)$ be the short rate. Hence r satisfies the Itô process

$$dr(t) = a(r)dt + \sigma(t)dZ(t)$$

.

23.1 Hedging formulas

Duration hedging bond 1 with bond 2:

$$N = -\frac{(T_1 - t)P(t,T_1)}{(T_2 - t)P(t,T_2)}$$

Delta hedging bond 1 with bond 2:

$$N = -\frac{P_r(r,t,T_1)}{P_r(r,t,T_2)} = -\frac{B(t,T_1)P(r,t,T_1)}{B(t,T_2)P(r,t,T_2)}$$

for Vasicek and Cox-Ingersoll-Ross

23.2 Differential equation for bond prices

If the interest rate process is

$$dr(t) = a(r)dt + \sigma(t)dZ(t)$$

, then

$$\frac{dP}{P} = \alpha(r,t,T)dt - q(r,t,T)dZ(t)$$

where

$$\alpha(r,t,T) = \frac{a(r)P_r + 0.5\sigma(r)^2 P_{rr} + P_t}{P}$$

$$q(r,t,T) = -\frac{P_r\sigma(r)}{P}$$

in general

$$\alpha(r,t,T) = -a(b-r)B(r,T) + 0.5\sigma(r)^2 B(t,T)^2 + \frac{P_t}{P}$$

$$q(r,t,T) = B(t,T)\sigma(r)$$

for Vasicek and Cox-Ingersoll-Ross

23.3 Black-Scholes equation analog for bond prices

$$0.5\sigma(r)^2 P_{rr} + (a(r) + \sigma(r)\phi(r,t))P_r + P_t - rP = 0$$

23.4 Sharpe Ratio

General

$$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

Vasicek

$$\phi(r,t) = \phi$$

Cox-Ingersoll-Ross

$$\phi(r,t) = \bar{\phi}\sqrt{r}/\bar{\sigma}$$

$\bar{\sigma} = \sigma(r)/\sqrt{r}$

23.5 Definition of interest rate models

General

$$dr(t) = a(r)dt + \sigma(t)dZ(t)$$

Rendleman-Bartter

$$dr(t) = ardt + \sigma tdZ(t)$$

Vasicek

$$dr(t) = a(b - r)dt + \sigma dZ(t)$$

Cox-Ingersoll-Ross

$$dr(t) = a(b - r)dt + \bar{\sigma}\sqrt{r}dZ(t)$$

23.6 Risk-neutral version of interest rate models

General

$$dr = (a(r) + \phi(r,t)\sigma(r))dt + \sigma(r)d\tilde{Z}(t)$$

Vasicek

$$dr = \tilde{a}(\tilde{b} - r)dt + \sigma d\tilde{Z}(t)$$

with

$$\tilde{a} = a, \tilde{b} = b + \sigma\phi/a$$

Cox-Ingersoll-Ross

$$dr = \tilde{a}(\tilde{b} - r)dt + \tilde{\sigma}\sqrt{r}d\tilde{Z}(t)$$

with

$$\tilde{a} = a - \phi, \tilde{b} = ba/(a - \bar{\phi})$$

23.7 Bond price in Vasicek and CIR models

$$P(r,t,T) = A(t,T)e^{-B(t,T)r(t)}$$

In Vasicek model, $B(t,T) = \frac{1-e^{-a(T-t)}}{a}$

23.8 Yield-to-maturity on infinitely-lived bond

Vasicek

$$\bar{r} = b + \sigma\phi/a - 0.5\sigma^2/a^2$$

Cox-Ingersoll-Ross

$$\bar{r} = 2ab/(a - \bar{\phi} + \gamma)$$

where $\gamma = \sqrt{(a - \bar{\phi})^2 + 2\bar{\sigma}^2}$

24 Additional Resources

Lognormal

$$E[X^k] = e^{km+0.5k^2v^2}$$

References

[Weishaus] Abraham Weishaus. Study Manual for Exam MFE/Exam 3F, Financial Economics, Ninth Edition