PMATH 351: Real Analysis

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Contents

1 Axiom of Choice & Cardinality

1.1 Notation

$\mathbb{N}$ set of natural numbers, $\{1, 2, 3, \dots\}$

$\mathbb{Z}$ set of integers, $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

$\mathbb{Q}$ set of rationals, $\{\frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N}, gcd(a, b) = 1\}$

$\mathbb{R}$ set of reals

inclusion $A \subset$ or $A \subseteq B$

proper inclusion $A \subsetneq B$

Definition.

- Let X be a set $P(X) = \{A | A \subset X\}$ is the power set of X.
- A, B sets. The union of A and B is $A \cup B = \{x | x \in A \text{ or } x \in B\}$. If $I \neq \emptyset$, $\{A_\alpha\}_{\alpha \in I}$ are sets, $A_\alpha \subseteq X$,$\forall \alpha$,
$$\bigcup_{\alpha \in I} A_\alpha = \{x | x \in A_\alpha \text{ for some } \alpha \in I\}$$
- Similarly for intersections
- Let $A, B \in X$, $B \setminus A = \{b \in B | b \notin A\}$. If $B = X$, $X \setminus A = A^C$ is the complement of A (in X). Note: $(A^C)^C = A, A^C = B^C \iff A = B$

Theorem. De Morgan's Laws:

1. $(\bigcup_{\alpha \in I}) A_\alpha)^C = \bigcap_{\alpha \in I} A_\alpha^C$

   Proof. $x \in (\bigcup_{\alpha \in I}) A_\alpha)^C \iff x \notin \bigcup_{\alpha \in I}) A_\alpha \iff \forall \alpha \in I, x \notin A_\alpha \iff x \in \bigcap_{\alpha \in I} A_\alpha^C$ □

2. $(\bigcap_{\alpha \in I}) A_\alpha)^C = \bigcup_{\alpha \in I} A_\alpha^C$

1.2 Products & Axiom of Choice

Definition. Let X, Y be sets. The product of X and Y is $X \times Y = \{(x, y) | x \in X, y \in Y\}$. Let $X_1, X_2, \dots, X_n$ be sets. The product of $\{X_1, X_2, \dots, X_n\}$ is

$$X_1 \times X_2 \cdots \times X_n = \prod_{i=1}^{n} X_i = \{(x_1, x_2, \dots, x_n) | x_i \in X_i, \forall i = 1, 2, \dots\}$$

An element $(x_1, \dots, x_n)$ is called an n-tuple and x_i is called the ith coordinate.

Theorem. If $X_i = X, \forall i = 1, \cdots, n$, $\prod_{i=1}^n X_i = X^n$. If X is a set, $|X|$ is the number of elements of X. If $\{X_1, \cdots X_n\}$ is a finite collection of sets

$$\left| \prod_{i=1}^{n} X_i \right| = \prod_{i=1}^{n} |X_i|$$

If $X_i = X$, $\forall i, |X^n| = |X|^n$

How do we define the product of an arbitrary family of sets?

$(x_1, \dots, x_n) \in \prod_{i=1}^n X_i$, then $(x_1, x_2, \dots, x_n)$ determines a function

$$f_{(x_1,\dots,x_n)} : \{1, 2, \dots, n\} \to \bigcup_{i=1}^{n} X_i$$

i.e. $f_{(x_1,\dots,x_n)}(i) = X_i$

On the other hand, if we have a function

$$f : \{1, 2, 3, \dots, n\} \to \bigcup_{i=1}^{n} X_i$$

with $f(i) \in X_i$. We define $(x_1, \dots, x_n) \in \prod_{i=1}^n X_i$ by $x_i \in X_i = f(i), \forall i = 1, \dots, n$

$$\prod_{i=1}^{n} X_i = \left\{ f\{1, 2, \dots, n\} \to \bigcup_{i=1}^{n} X_i | f(i) \in X_i \right\}$$

Definition. Given a collection $\{X_\alpha\}_{\alpha \in I}$ of sets, we define

$$\prod_{\alpha \in I} X_\alpha := \{f : I \to U_{\alpha \in I} X_\alpha | f(\alpha) \in X_\alpha\}$$

Axiom. Zermlo's Axiom of Choice. Given a non-empty collection $\{X_\alpha\}_{\alpha \in I}$ if non-empty sets, $\prod_{\alpha \in I} X_\alpha = \emptyset$.

Axiom. Axiom of Choice: Given a non-empty set X, there exists a function $f : \mathcal{P}(x) \backslash \emptyset \to X$ for every $A \subseteq X, A \neq \emptyset, f(A) \in A$

1.3 Relations and Zorn's Lemma

Definition. X, Y are sets A relation is a subset of $X \times Y$. We write xRy if $(x, y) \in R$.

1. Reflexive if $xRx, \forall x \in X$
2. Symmetric if $xRy \implies yRx$
3. Anti-symmetric xRy and $yRx \implies x = y$
4. Transitive if xRy and $yRz \implies xRz$

Example:

1. $x = \mathbb{R}, xRy \iff x \subseteq y$. It is reflexive, antisymmetric, transitive.
2. X set. We define a relation on $\mathcal{P}(X)$. $ARB \iff A \subseteq B$
3. R^* relation on $\mathcal{P}(x)$. $ARB \iff A \supseteq B$

Definition. A relation R on a set X is a partial order if it is reflexive, anti-symmetric and transitive. (X, R) is a partially order set or poset.

A partial relation R on X is a total order if $\forall x, y \in X$, either xRy or yRx. (X, R) is a totally order set or a chain.

Definition. $(X, \leq)$ poset. Let $A \in X$. $x \in X$ is an upper bound for A if $a \leq x, \forall a \in A$. A is bounded above if it has an upper bound. $x \in X$ is the least upper bound (or supermum) for A if x is an upper bound and y is an upper bound, then $x \leq y$. $x = lub(A) = sup(A)$. If $x = lub(A)$ and $x \in A \implies x = max(A)$ is the maximum of A.

Axiom. Least Upper bound axiom for $\mathbb{R}$: Consider $\mathbb{R}$ with usual order $\leq$. $A \subseteq \mathbb{R}$, $A \neq \emptyset$. If A is bounded above, the A has a least upper bound.

Example

1. $(\mathcal{P}(X), \subseteq)$, $\{A_\alpha\}_{\alpha \in I}, A_\alpha \subseteq X, A_\alpha \neq \emptyset$. X is an upper bound for $\{A_\alpha\}_{\alpha \in I}$. $\emptyset$ is a lower bound, $lub(\{A_\alpha\}_{\alpha \in I}) = \bigcup_{\alpha \in I} A_\alpha$, and $glb(\{A_\alpha\}_{\alpha \in I}) = \bigcap_{\alpha \in I} A_\alpha$

2. $(\mathcal{P}(X), \supseteq)$

Definition. $(X, \leq)$ poset, $x \in X$ is maximal if $x \leq y$ implies $x = y$.

- $(\mathbb{R}, \leq)$ has no maximal element
- $(\mathcal{P}(X), \leq) \implies X$ is maximal.
- $(\mathcal{P}(X), \geq) \implies \emptyset$ is maximal.

Proposition. Every finite, non-empty poset has a maximal element but there are poset with no maximal element.

Lemma. Zorn's Lemma: $(X, \leq)$ non-empty poset. If every totally order subset $\mathcal{C}$ of X has an upper bound, then $(X, \leq)$ has a maximal element. Let $\mathcal{V}$ be a non-zero vector space. Let $\mathcal{L} = \{A \leq \mathcal{V} | A \text{ is linearly independent}\}$.

Note: A basis B for $\mathcal{P}$ is a maximal element on $(\mathcal{L}, \leq)$.

Theorem. Every non-zero vector space $\mathcal{V}$ has a basis.

Proof. Let $\mathcal{C} = \{A_\alpha | \alpha \in I\}$ be a chain in $\mathcal{L}$.

Let $A = \bigcup_{\alpha \in I} A_\alpha$. Claim: A is linearly independent. Let $\{x_1, x_2, \cdots, x_n\} \subseteq A$, $\{\beta_1, \beta_2, \cdots, \beta_n\} \subseteq \mathbb{R}$. Then $\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n = 0$.

For each $i = 1, 2, \cdots, n, \exists \alpha_i | x_i \in A_{\alpha_i}$.

Assume, $A_{\alpha_1} \subseteq A_{\alpha_2} \subseteq \cdots \subseteq A_{\alpha_n}$ ($\mathcal{L}$ is a chain, change name of index if needed). Therefore, $\{x_1, x_2, \cdots, x_n\} \subseteq A_{\alpha_n}$ and A_{α_n} is linearly independent. Hence, $\{x_1, x_2, \cdots, x_n\}$ is linearly independent. Lastly, $\beta_i = 0, \forall i$. Then A is linearly independent. A is an upper bound for $\mathcal{C}$ on $\mathcal{L}$. By Zorn's lemma, $\mathcal{L}$ has a maximal element. □

Definition. A poset $(X, \leq)$ is well-ordered, if every non-empty subset A has a least element.

Examples

- $(\mathbb{N}, \leq)$ is well-ordered.
- $\mathbb{Q} = \{\frac{n}{m} | n \in \mathbb{Z}, m \in \mathbb{N}, gcd(n, m) = 1\}$.

We can construct a well-order on $\mathbb{Q}$. $\phi : \mathbb{Q} \to \mathbb{N}$ by $\phi(\frac{n}{m}) = \begin{cases} 2^n 3^m & n > 0 \\ 1 & n = 0 \\ 5^{-n} 7^m & n < 0 \end{cases}$. ϕ is 1-to-1. $\frac{n}{m} \leq \frac{p}{q} \iff \phi(\frac{n}{m}) \leq \phi(\frac{p}{q})$

$(\mathbb{Q}, \leq)$ is well-ordered.

Axiom. Well-ordering principle: Given any set $X \neq \mathbb{Q}$, there exists a partial order $\leq$ such that $(X, \leq)$ is well-ordered.

Theorem. TFAE:

1. Axiom of Choice
2. Zorn's lemma
3. Well-ordering principle

1.4 Equivalence Relations & Cardinality

Definition. A relation $\sim$ on a set X is an equivalence relation if

1. Reflexive
2. Symmetric
3. Transitive

Given $x \in X$, let $[x] = \{y \in X | x \sim y\}$ be the equivalence class of x.

Proposition. Let $\sim$ be an equivalence relation on X

1. $[x] \neq \emptyset, \forall x \in X$
2. For each $x, y \in X$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.
3. $X = \bigcup_{x \in X} [x]$

Definition. If X is a set, a partition of X is a collection $\mathcal{P} = \{A_\alpha \subseteq X | \alpha \in I\}$.

1. $A_\alpha \neq \emptyset, \forall \alpha$.
2. If $\beta \neq \alpha \implies A_\alpha \cup A_\beta = \emptyset$
3. $X = \bigcup_{\alpha \in I} A_\alpha$.

Note:

Given $\sim$ on X $\implies$ $\sim$ induces a partition on X. Given a partition on X ($\mathcal{P} = \{A_\alpha | \alpha \in I\}$) we define an equivalence relation on X:

$$x \sim y \iff x, y \in A_\alpha, \text{for some } \alpha$$

Example: Define $\sim$ on $\mathcal{P}(X)$ by $A \sim B \iff \exists$ a 1-to-1 and onto function $f : A \to B$. $\sim$ is an equivalence relation.

Definition. Two sets X and Y are equivalent if there exists a 1-to-1 and onto function $f : X \to Y$. In this case, we write $X \sim Y$. We say that X and Y have the same cardinality, $|X| = |Y|$.

A set X is finite if $X =$ or $X \sim \{1, 2, \cdots, n\}$ for some $n \in \mathbb{N}$, $|X| = n$. Otherwise, X is infinite.

Can X be equivalent to both $\{1, 2, \cdots, n\}$ and $\{1, 2, \cdots, m\}$, with $n \neq m$? If $X \sim \{1, 2, \cdots, n\}$ and $X \sim \{1, 2, \cdots, m\} \implies \{1, 2, \cdots, n\} \sim \{1, 2, \cdots, m\}$.

Proposition. The set $\{1, 2, \cdots, m\}$ is not equivalent to any proper subset of itself.

Proof. Induction on m

$m = 1$: The only proper subset of $\{1\}$ is $\emptyset$. and $\{1\} \sim \emptyset$.

$m = k$ Statement holds for $\{1, 2, \cdots, k\}$. Assume $\exists S \subsetneq \{1, 2, \cdots, k, k+1\}$ and $f : \{1, 2, \cdots, k+1\} \to S$, 1-to-1 and onto.

Two cases:

1. If $k + 1 \in S \implies f_{\{1,2,\cdots,k\}} : \{1, 2, \cdots, k\} \to S \backslash \{f(k + 1)\} \subsetneq \{1, 2, \cdots, k\}$. This is impossible.
2. If $k + 1 \in S$, $f(k + 1) = k + 1$, then $f_{\{1,2,\cdots,k\}} \{1, 2, \cdots, k\} \to S \backslash \{k + 1\} \subsetneq \{1, 2, \cdots, k\}$. This is impossible.

   If $f(k + 1) = j$ and $f(i) = k + 1$. Define $f^* : \{1, 2, \cdots, k + 1\} \to S, f^*(l) = \begin{cases} k+1 & l = k+1 \\ j & l = i \\ f(l) & \text{otherwise} \end{cases}$. This is impossible

□

Corollary. If X is finite, then X is not equivalent to any proper subset of itself.

Example:
$f : \mathbb{N} \to \mathbb{N}\backslash\{1\} = n \to n + 1$ is 1-to-1 and onto. Hence $\mathbb{N} \sim \mathbb{N}\backslash\{1\}$.

Definition. A set X is countable if X is finite or $X \sim \mathbb{N}$. Otherwise, uncountable.X is countable infinite if $X \sim \mathbb{N}, |X| = |\mathbb{N}| = \aleph_0$

Proposition. Every infinite set contains a countable infinite subsets.

Proof. By Axiom of Choice, $\exists f : \mathcal{P}(X)\backslash\emptyset \to X, f(A) \in A$. $x_1 = f(X)$ and $x_2 = f(X\backslash\{x_1\}) \cdots x_{n+1} = f(X\backslash\{x_1, x_2, \cdots, x_n\})$
$A = \{x_1, x_2, \cdots, x_{n+1}, \cdots\}X$ is countable infinite. □

Corollary. A set X is infinite if and only if it is equivalent to a proper subset of itself.

Theorem. (Cantor-Schroeder-Berstein) (CSB) Assume that $A_2 \subseteq A_1 \subseteq A_0$. If $A_2 \sim A_0$, then $A_1 \sim A_0$.

Corollary. Assume $A_1 \subseteq A$ and $B_1 \subseteq B$. If $A \sim B_1$ and $B \sim A_1$, then $A \sim B$. $f : A \to B_1$ is 1-to-1 and onto and $g : B \to A_1$ is 1-to-1 and onto. $A_2 = g(f(A) = g(B) \subseteq A_1 \subseteq$ ad $g \circ f$ is 1-to-1 and onto on A_2. Hence $A_2 \sim A \to^{CSB} A_1 \sim A$ and $A_1 \sim B$. Hence $A \sim B$.

Corollary. An infinite set X is countable infinite if and only if there exists a 1-to-1 function $f : X \to \mathbb{N}$.

Proposition. Assume there exists $g : X \to Y$ onto. Then there exists a 1-to-1 function $f : Y \to X$.

Proof. By axiom of choice, $\exists h : \mathcal{P}(x)\backslash\emptyset \to X, h(A) \in A, A \neq \emptyset, A \subseteq X$. $\forall y \in Y$, define $f(y) = h(g^{-1}(\{y\})) \in X$. $f : Y \to X$. Check f is 1-to-1. □

Corollary. X, Y sets. TFAE

1. $\exists f : X \to Y$, 1-to-1
2. $\exists g : Y \to X$, is onto
3. $|Y| \succeq |X|$

Theorem. $[0, 1]$ is uncountable.

Proof. Assume $[0, 1]$ is countable

$$[0, 1] = \{a_1, a_2, \cdots, a_n, \cdots\}$$

each real number has a unique decimal expansion if we don't allow $.99\bar{9}$ (∞ times 9)

$$a_1 = 0.a_{11}a_{12}a_{13}\cdots$$

$$a_2 = 0.a_{21}a_{22}a_{23}\cdots$$

$$a_3 = 0.a_{31}a_{32}a_{33}\cdots$$

$$\vdots$$

Let $b \in [0, 1)$, $b = 0.b_1b_2\cdots$ where $b_n := \begin{cases} 1 & a_{n1} \neq 1 \\ 2 & a_{nn} = 1 \end{cases}$ Well, $b \neq a_n, \forall n$. It is impossible. Then $[0, 1]$ is uncountable.

□

Corollary. $\mathbb{R}$ is uncountable. $\mathbb{R} \sim (0, 1)$. Note $|\mathbb{R}| = c$.

Theorem. Comparability theorem for cardinals: Given X, Y sets, either $|X| \preceq |Y|$ or $|Y| \preceq |X|$.

1.5 Cardinal Arithmetic

1.5.1 Sums of Cardinals

Definition. Let X, Y be disjoint sets, then

$$|X| + |Y| = |X \cup Y|$$

Examples

1. $X = \{1, 3, 5, \cdots\}, Y = \{2, 4, 6, \cdots\}$. $|X| + |Y| = \aleph_0 + \aleph_0 = \aleph_0$.

Theorem. If X is infinite, then

$$|X| + |Y| = max\{|X|, |Y|\}$$

In particular,

$$|X| + |X| = |X|$$

$X_1, \cdots, X_n$ countable sets. Then $|\bigcup_{i=1}^{n} X_i| = \aleph_0$.

Theorem. $\{X_i\}_{i=1}^{\infty}$ countable collection of countable sets, then $X = \bigcup_{i=1}^{\infty} X_i$ is countable.

Note: we can assume $X_i \cap X_j = \emptyset$ if $i \neq j$. Otherwise, let $E_1 = X_1, E_2 = X_2 \backslash X_1, \cdots, E_n = X_n \backslash \cup_{i=1}^{n-1} X_i$. Assume $\{X_i\}_{i=1}^{\infty}$ is pairwise disjoint if $X_i \neq \emptyset$, let $X_i = \{x_{i1}, x_{i2}, \cdots\}$ countable. Let $f : X = \cup_{i=1}^{\infty} X_i \to \mathbb{N}$ 1-to-1 such that $f(x_{ij}) = 2^i 3^j$.

1.5.2 Product of cardinals

Let X, Y be two sets

$$|X| \cdot |Y| = |X \times Y|$$

Theorem. If X is infinite and $Y \neq \emptyset$, then

$$|X| \cdot |Y| = \max\{|X|, |Y|\}$$

In particular,

$$|X| \cdot |X| = |X|$$

1.5.3 Exponentiation of Cardinals

Recall: Given a collection $\{Y_x\}_{x \in X}$ of non-empty sets, we defined

$$\prod_{x \in X} Y_x = \{f : X \to \bigcup_{x \in X} Y_x | f(x) \in Y_x\}$$

If $\forall x \in X$, $Y_x = Y$ for some set Y, $Y^X = \prod_{x \in X} Y_x = \prod_{x \in X} Y = \{f : X \to Y\}$.

Definition. Let X, Y non empty sets, we define

$$|Y|^{|X|} = |Y^X|$$

Theorem. X, Y, Z non-empty sets.

1. $|Y|^{|X|} |Y|^{|Z|} = |Y|^{|X|+|Z|}$
2. $(|Y|^{|X|})^{|Z|} = |Y|^{|X|+|Z|}$

Example ($2^{\aleph_0} = c$) $2^{\aleph_0} = |\{0,1\}^{\mathbb{N}}| = |\{\{a_n\}_{n\in\mathbb{N}} | a_n = 0 \text{ or } a_n = 1\}$.
$2^{\aleph_0} \preceq c$: $f\{0,1\}^{\mathbb{N}} \to [0,1]$ is 1-to-1 such that $\{a_n\} \to \sum_{n=1}^{\infty} \frac{a_n}{3^n}$.
$2^{\aleph_0} \succeq c$: $g : [0,1] \to \{0,1\}^{\mathbb{N}}$ is 1-to-1. $\alpha = \sum_{n=1}^{\infty} \frac{a_n}{2^n} \to \{a_n\}$.
Hence done.
Given a set X, we want to find $|\mathcal{P}(X)| = 2^{|X|}$.
Let $A \subseteq X$, $\chi_A : X \to \{0,1\}$, such that $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$. This is called characteristics function of A. $X_A \in \{0,1\}^X$. If $f \in \{0,1\}^X, A = \{x \in X | f(x) = 1\}$. Hence $\chi_A = f$. Let $\Gamma : P(X) \to \{0,1\}^{\mathbb{N}}$. Hence Γ is a bijection. Therefore $|\mathcal{P}(X)| = 2^{|X|}$.

Theorem. $|\mathcal{P}(X)| \succ |X|$ for any $X \neq \emptyset$ (Russel's Paradox)

It is enough to show that there is no onto function $X \to \mathcal{P}(X)$. Assume to the contrary: there exists $f : X \to \mathcal{P}(X)$ onto.

$A = \{x \in X | x \notin f(X)\}.\exists x_0 \in X | f(x_0) = A$. If $X_0 \in A$: $\implies x_0 \notin f(x_0) = A$. Impossible. If $X \notin A :\implies x_0 \in f(x_0) = A$. OK

2 Metric spaces

Definition. Let $X \neq \emptyset$. A metric on X is a function $d : X \times X \to \mathbb{R}$.

1. $d(x,y) \geq 0, \forall x,y \in X$. $d(x,y) = 0 \implies x = y$.
2. $d(x,y) = d(y,x), \forall x,y \in X$.
3. $d(x,y) \leq d(x,z) + d(z,y), \forall x,y,z \in X$.

(X,d) is a metric space.

Examples

1. $X = \mathbb{R}$ $d(x,y) = |x-y|$ "usual metric on $\mathbb{R}$"
2. X any non-empty set $d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$ "discrete metric"
3. $X = \mathbb{R}^n$. $d_2((x_1,x_2,\cdots,x_n),(y_1,y_2,\cdots,y_n)) = \sqrt{\sum_{i=1}^n (x_i-y_i)^2}$. d_2 verifies $1), 2)$. This is called "Euclidean Metric".

Definition. Let V be a vector space. A norm on V is a function $\|\cdot\| : V \to \mathbb{R}$ such that

1. $\|x\| \geq 0, \forall x \in V$. $\|x\| = 0 \iff x = 0$
2. $\|\alpha x\| = |\alpha|\|x\|, \forall \alpha \in \mathbb{R}, \forall x \in V$.
3. $\|x+y\| \leq \|x\| + \|y\|, \forall x,y \in V$

$(V, \|\cdot\|)$ is normed vector space.

Remark: $(V, \|\cdot\|)$ normed vector space. $\|\cdot\|$ induces a metric on V. $d_{\|\cdot\|}(x,y) = \|x-y\|$

1. $d_{\|\cdot\|}(x,y) = \|x-y\| \geq 0, \forall x,y \in V$. $\|x-y\| = 0 \implies x = y$.
2. $d_{\|\cdot\|}(x,y) = \|x-y\| = |-1|\|y-x\| = d_{\|\cdot\|}(y,x)$
3. $d_{\|\cdot\|}(x,y) = \|x-y\| \leq \|x-z\| + \|z-y\|$

Examples

1. $X = \mathbb{R}^n$, $\|(x_1, \cdots, x_n)\|_2 = (\sum_{i=1} |x_i|^2)^{1/2}$. $d_{\|\cdot\|_2} = d_2$. This is a 2-norm or Euclidean norm.
2. $X = \mathbb{R}^n$, $1 < p < \infty$. $\|(x_1, x_2, \cdots, x_n)\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ This is called p-norm.
3. $X = \mathbb{R}^n$, $\|(x_1, \cdots, x_n)\|_\infty = \max\{|x_i|\}$. This is called $\infty-$norm.
4. $\|(x_1, \cdots, x_n)\|_1 = \sum_{i=1}^n |x_i|$. This is called 1-norm.

Remark: Let p, $1 < p < \infty$, and $q, \frac{1}{p} + \frac{1}{q} = 1$. Then $1 + \frac{p}{q} = p \implies \frac{p}{q} = p - 1 \implies \frac{p}{p-1} = q \implies \frac{q}{p} = q - 1 \implies \frac{1}{p-1} = \frac{q}{p} = q - 1$.

Lemma. Let $\alpha, \beta > 0$, $1 < p < \infty$. If $\frac{1}{p} + \frac{1}{q} = 1$, then $\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$ (Young's inequality)

$u = t^{p-1} \implies t = u^{\frac{1}{p-1}} = u^{q-1}$. $\alpha\beta \leq \int_0^\alpha t^{p-1} dt + \int_0^\beta u^{q-1} du = \frac{\alpha^p}{p} + \frac{\beta^q}{q}$.

Theorem. Hdder's Inequality: Let $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n) \in \mathbb{R}^n$. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{i=1}^n |a_i b_i| \leq (\sum_{i=1}^n |a_i|^p)^{1/p} (\sum_{i=1}^n |b_i|^q)^{1/q}$$

Proof. Assume $a \neq 0 \neq b$.

Note: $\alpha, \beta > 0$,

$$\sum_{i=1}^n |(\alpha a_i)(\beta b_i)| = \alpha\beta \sum_{i=1}^n |a_i b_i|$$
$$(\sum_{i=1}^n |\alpha a_i|^p)^{1/p} = \alpha(\sum_{i=1}^n |a_i|^p)^{1/p}$$
$$(\sum_{i=1}^n |\beta b_i|^q)^{1/q} = \beta(\sum_{i=1}^n |b_i|^q)^{1/q}$$

Then the inequality holds for $a, b \in \mathbb{R}^n \iff$ it holds for $\alpha a, \beta b \in \mathbb{R}^n$ for some $\alpha\beta > 0$. By scaling if needed, we can assume

$$(\sum_{i=1}^n |a_i|^p)^{1/p} = 1, (\sum_{i=1}^n |b_i|^q)^{1/q} = 1$$

Lemma.

$$|a_i b_i| \leq \frac{|a_i|^p}{p} + \frac{|b_i|^q}{q}, \forall i = 1, \cdots, n$$

Hence $\sum_{i=1}^n |a_i b_i| \leq \frac{\sum_{i=1}^n |a_i|^p}{p} + \frac{\sum_{i=1}^n |b_i|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1$ ◻

Theorem. Minkowski's Inequality: Let $a = (a_1, a_2, \cdots, a_n), b = (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n$. Let $1 < p < \infty$, then

$$(\sum_{i=1}^n |a_i + b_i|^p)^{1/p} \leq (\sum_{i=1}^n |a_i|^p)^{1/p} + (\sum_{i=1}^n |b_i|^p)^{1/p}$$

Proof. Assume $a \neq 0 \neq b$. Let $q/\frac{1}{p}+\frac{1}{q}=1$.

$$\sum_{i=1}^{n}|a_i+b_i|^p = \sum_{i=1}^{n}|a_i+b_i||a_i+b_i|^{p-1}$$
$$\leq \sum_{i=1}^{n}|a_i||a_i+b_i|^{p-1}+\sum_{i=1}^{n}|b_i||a_i+b_i|^{p-1}$$
$$\sum_{i=1}^{n}|a_i||a_i+b_i|^{p-1} \leq (\sum_{i=1}^{n}|a_i|^p)^{1/p}(\sum_{i=1}^{n}(|a_i+b_i|^{p-1})^q)^{1/q} = (\sum_{i=1}^{n}|a_i|^p)^{1/p}(\sum_{i=1}^{n}|a_i+b_i|^p)^{1/q}$$
$$\text{Similarly, } \sum_{i=1}^{n}|b_i||a_i+b_i|^{p-1} \leq (\sum_{i=1}^{n}|b_i|^p)^{1/p}(\sum_{i=1}^{n}|a_i+b_i|^p)^{1/q}$$
$$\sum_{i=1}^{n}|a_i+b_i|^p \leq ((\sum_{i=1}^{n}|a_i|^p)^{1/p}+(\sum_{i=1}^{n}|b_i|^p)^{1/p})(\sum_{i=1}^{n}|a_i+b_i|^p)^{1/q}$$
$$(\sum_{i=1}^{n}|a_i+b_i|^p)^{1-1/p} \leq \|a\|_p+\|b\|_p$$

□

Examples: sequence space

1. Let $l_1 = \{\{x_n\}|\sum_{i=1}^{\infty}|x_n| < \infty\}$ Then $\|\{x_n\}\|_1 = \sum_{i=1}^{\infty}|x_n|$. Let $\{x_n\},\{y_n\} \in l_1$. Claim that $\{x_n+y_n\} \in l_1$. Let $k \in \mathbb{N}$

   $$\sum_{n=1}^{k}|x_n+y_n| \leq \sum_{n=1}^{k}|x_n|+\sum_{n=1}^{k}|y_n| \leq \sum_{n=1}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n| < \infty$$

   By MCT, $\{\sum_{i=1}^{k}|x_n+y_n|\}$ convergent then $\sum_{n=1}^{\infty}|x_n+y_n|$ convergent. Hence $\{x_n+y_n\} \in l_1$.

   Moreover,
   $$\|\{x_n+y_n\}\| \leq \|\{x_n\}\|_1+\|\{y_n\}\|_1$$

   This implies $\|\cdot\|_1$ is a norm.

2. Let $1 < p < \infty$,
   $$l_p = \{\{x_n\}|\sum_{i=1}^{\infty}|x_i|^p < \infty\}$$

   $\|\{x_n\}|_p = (\sum_{i=1}^{\infty}|x_i|^p)^{1/p}$ Prove that $\{x_n\},\{y_n\} \in l_p$ and then $\{x_n+y_n\} \in l_p$ and $\|\cdot\|_p$ is norm.

3. $l_\infty = \{\{x_n\}|sup\{|x_n|\} < \infty\}$. $\|\{x_n\}\|_\infty = sup\{|x_n|\}$. This is a norm.

Examples Continuous function space

1. $C([a,b]) = \{f : [a,b] \to \mathbb{R}|$ f is continuous$\}$. $\|f\|_\infty = \max\{|f(x)||x \in [a,b]\}$. Let $f,g \in C([a,b]), x \in [a,b]$.

   $$|(f+g)(x)| = |f(x)+g(x)| \leq |f(x)|+|g(x)| \leq sup_{x\in[a,b]}|f(x)|+\max_{x\in[a,b]}|g(x)| = \|f\|_\infty+\|g\|_\infty$$

   $$\|f+g\|_\infty = \max_{x\in[a,b]}|f(x)+g(x)| \leq \|f\|_\infty+\|g\|_\infty$$

2. $\mathcal{C}([a,b])$, $\|f\|_1 = \int_a^b |f(t)|dt$.

3. $\mathcal{C}([a,b])$, $\|f\|_p = (\int_a^b |f(t)|^p dt)^{1/p}$

Theorem. Holder's inequality II: Let $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1$. If $f, g \in \mathcal{C}[a,b]$.

$$\int_a^b |f(t)g(t)|dt \le (\int_a^b |f(t)|^p dt)^{1/p} (\int_a^b |g(t)|^q dt)^{1/q}$$

Theorem. Minkowski's Inequality II: If $f, g \in \mathcal{C}([a,b])$ and $1 < p < \infty$

$$(\int_a^b |(f+g)(t)|^p dt)^{1/p} \le (\int_a^b |f(t)|^p dt)^{1/p} + (\int_a^b |g(t)|^p dt)^{1/p}$$

Then $f \neq 0 \neq g$.

Proof.

$$\begin{aligned}
\int_a^b |f(t)+g(t)|^p dt &= \int_a^b |(f+g)(t)||(f+g)(t)|^{p-1} dt \\
&\le \int_a^b |f(t)||(f+g)(t)|^{p-1} dt + \int_a^b |g(t)||(f+g)(t)|^{p-1} dt \\
&\le (\int_a^b |f(t)|^p dt)^{1/p} (\int_a^b |f(t)+g(t)|^{(p-1)q} dt)^{1/q} \\
&+ (\int_a^b |g(t)|^p dt)^{1/p} (\int_a^b |f(t)+g(t)|^{(p-1)q} dt)^{1/q} \\
\int_a^b |f(t)+g(t)|^p dt &\le [(\int_a^b |f(t)|^p dt)^{1/p} + (\int_a^b |g(t)|^p dt)^{1/p}](\int_a^b |f(t)+g(t)|^p dt)^{1/q}
\end{aligned}$$

$$(\int_a^b |f(t)+g(t)|^p dt)^{1-1/q} \le \|f\|_p + \|g\|_p$$

□

Example: Bounded operators

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed linear spaces. Let $T : X \to Y$, linear. $\|T\| := \sup\{\|T(x)\|_Y | \|x\|_X \le 1, x \in X\}$. $B(X,Y) = \{T : X \to Y \text{ linear} | \|T\| < \infty\}$.

Claim: $B(X,Y)$ is a vector space and $\|\cdot\|$ is a norm.

- $T, S \in B(X,Y) \implies T + S \in B(X,Y)$, $x \in X, \|x\|_X \le$.

$$\begin{aligned}
\|(T+S)(x)\|_Y &= \|T(x) + S(x)\|_Y \\
&\le \|T(x)\|_Y + \|S(x)\|_Y \\
&\le \|T\| + \|S\|
\end{aligned}$$

$$\begin{aligned}
\|T+S\| = sup\|(T+S)(x)\| \le \|T\| + \|S\| < \infty, x \in X, \|x\|_X \le 1 \\
\implies T + S \in B(X,Y) \text{ and } \|T+S\| \le \|T\| + \|S\|
\end{aligned}$$

- $\alpha \in \mathbb{R}, T \in B(X,Y)$

$$\|\alpha T\| = \sup_{x \in X, \|x\|_X \le 1} \|\alpha T(x)\|_Y = |\alpha| \sup_{x \in X, \|x\|_X \le 1} \|T(x)\|_Y = |\alpha|\|T\| < \infty$$

$$\implies \alpha T \in B(X,Y) \text{ and } \|\alpha T\| = |\alpha|\|T\|$$

Note $B(X,Y) \le \mathcal{L}(X,Y)$, $0 \in B(X,Y) \implies B(X,Y)$ subspace of $\mathcal{L}(X,Y)$. $\|T\| \ge 0$ and $\|T\| = 0 \iff \|T(x)\|_Y = 0, \forall x \in X, \|x\|_X \le 1$.

2.1 Topology of Metric Spaces

Definition. Let (X, d) be a metric space. Let $x_0 \in X$ and $\epsilon > 0$. The open ball centered at x_0 with radius ϵ is

$$B(x_0, \epsilon) = \{x \in X | d(x, x_0) < \epsilon\}$$

The closed ball centered at x_0 with radius ϵ is

$$B[x_0, \epsilon] = \{x \in X | d(x, x_0) \leq \epsilon\}$$

A subset $U \subseteq X$ is open if $\forall x \in U, \exists \epsilon > 0 | B(x, \epsilon) \subseteq U$. A subset $F \subseteq X$ is closed if F^C is open.

Proposition. Let (X, d) be a metric space. Then

1. $X, \emptyset$ are open.
2. If $\{U_\alpha\}_{\alpha \in I}$ is a collection of open sets, then the union of all the sets in this collection is open =.
3. If $\{U_1, U_2, \cdots, U_n\}$ are open, then $\cap_{i=1}^n U_i$ is open.

Example

1. If $x \in X$, any $\epsilon > 0$, $B(x, \epsilon) \subseteq X \implies X$ is open. $\emptyset$ is "trivially" open.
2. If $x \in \cup_{\alpha \in I} U_\alpha$, then $\exists \alpha \in I$ such that $x \in U_{\alpha_0}$. Since U_α is an open set and $x \in U_{\alpha_0}, \exists \epsilon > 0$ such that $B(x, \epsilon) \subseteq U_\alpha \subseteq \cup_{\alpha \in I} U_\alpha \implies \cup_{\alpha \in I} U_\alpha$ is open.
3. If $x \in \cap_{i=1}^n U_i$, $\forall i \in \{1, \cdots, n\}, \exists \epsilon < 0$ such that $B(x, \epsilon) \subseteq U$, let $\epsilon = \min\{\epsilon | i = 1, \cdots, n\} > 0, B(x, \epsilon) \subseteq B(x, \epsilon_i), \forall i \implies B(x, \epsilon_i) \subseteq \cap_{i=1}^n B(x, \epsilon_i) \subseteq \cap_{i=1}^n U_i$.

Proposition. Let (X, d) be a metric space. Then

1. $X, \emptyset$ are closed
2. If $\{F_\alpha\}_{\alpha \in I}$ is addition of close sets, then $\cap_{\alpha \in I} F_\alpha$ is closed
3. If $F_1, \cdots, F_n$ are closed sets, then the union is also closed.

From this proposition, it flows that if (X, d) is a metric space. $\tau_j = \{U \subseteq X | U$ is open with respect to d$\}$. τ_j is a topology.

Proposition. Let (X, d) be a metric space, then

1. If $x_0 \in X, \epsilon > 0 \implies B(x_0, \epsilon)$ is open
2. $U \subseteq X$ is open $\iff$ U is the union of open balls
3. If $x_0 \in X, \epsilon > 0 \implies B[x_0, \epsilon]$ is closed
4. If $x \in X, \{x\}$ is closed. Every finite subset is closed.

Proof.

1. Let $x \in B(x_0, \epsilon)$, then $d(x, x_0) = \delta < \epsilon$ Let $\epsilon' = \epsilon - \delta$. Claim $B(x, \epsilon') \subseteq B(x, \epsilon)$. Let $x \in B(x, \epsilon')$ and $d(x_0, z) \leq d(x_0, x) + d(x, z) < \epsilon + \epsilon - \delta = \epsilon$ This proves that $B(x_0, \epsilon)$ is open.
2. $\implies$ follows (1). $\rightarrow$ If $x \in U, \exists \epsilon_x > 0$ such that $B(x, \epsilon_x) \subset U$, $\cup_{x \in U} B(x, \epsilon_x) = U$.
3. Let $x \in (B[x_0, \epsilon])^C$. $d(x, x_0) = \delta > \epsilon$. Let $\epsilon' = \delta \cdot \epsilon$. Claim $B(x, \epsilon') \subseteq (B[x_0, \epsilon])^C$. Let $z \in B(x, \epsilon')$ assume $z \in B[x_0, \epsilon], d(x, x_0 \leq d(x, z) + d(z, x_0) < \epsilon' + \epsilon = \delta - \epsilon + \epsilon = \delta$. This implies $z \in (B[x_0, \epsilon])^C$.

4. If $y \in \{x\}^C$, then $y \neq x$ and $d(y,x) > 0$ and $B(y, d(x,y)) \implies \{x\}^C$ is open.

$\square$

Open sets in $\mathbb{R}$.
Recall $I \subseteq \mathbb{R}$ is an interval if $x, y \in I$ and z such that $x < z < y \implies z \in I$.

- Open finite intervals (a, b)
- Closed finite intervals $[a, b]$.
- Half open finite set $(a, b]$.
- Open rays (a, ∞)
- Closed rays

Example: Cantor set

P_n is obtained from P_{n-1} by removing the open interval of length $1/3^n$ from the middle third of each of the 2^{n-1} subintervals of P_{n-1}. Each P_n is closed. It's the union of 2^n closed intervals of length $1/3^n$.

$$P = \bigcap_{n=1}^{\infty} P_n \text{Cantor (ternary) set)}$$

- P is closed
- P is uncountable ($x \in P \to x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with $a_n = 0, 2$.
- P contains no interval of positive length

Example: Discrete metric

X set, $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ $x \in X, B(x,2) = X, B(x,1) = \{x\}$ is an open set.
If $U = X, U = U_{x \in U}\{x\} = U_{x \in U} B(x,1)$ open. U is also closed.

2.2 Boundaries, interiors and closures

Definition. Let (X, d) metric space,

1. $A \subseteq \implies$ The closure of A is
$$A = \cap\{F \text{closed in } | A \subseteq F\}$$
It's the smallest closed set that contains A.
2. The interior of A is $int(A) = \cup\{U \text{ is open in } X | U \subseteq A\}$. It is the largest open set inside A.
3. Let $x \in X, N \subseteq X$, we say that N is a neighborhood of x ($N \subset \mathcal{N}_x$). If $x \in int(N)$.
4. Given $A \subseteq X, x \in X$ is a boundary point of A. If for every neighbor N of x, we have $N \cap A \neq \emptyset$ and $N \cap A^C \neq \emptyset$. Equivalently, x is a boundary point of A, if $\forall \epsilon > 0, B(x,\epsilon) \cap A \neq \emptyset$ and $B(x,\epsilon) \cap A^C \neq \emptyset$.
$$(\partial A) bdy(A) = \{x \in X | x \text{is a boundary point of A}\}$$

Proposition. (X, d) metric space, $A \subseteq X$

1. A is closed $\iff bdy(A) \subseteq A$

2. $\bar{A} = A \cup bdy(A)$.

Proof. 1. ($\implies$) A is close if and only if A^C is open. If $x \in A^C$, $\exists \epsilon > 0$ such that $B(x,\epsilon) \subseteq A^C$ and then $B(x,\epsilon) \cap A = \emptyset \implies x \notin bdy(A)$.

$\leftarrow$ Let $x \in A^C$, then $x \notin bdy(A)$. This implies $\exists \epsilon > 0$ such that $B(x,\epsilon) \cap A = \emptyset$. This implies $B(x,\epsilon) \subseteq A^C$. By definition, A^C is open.

2. Claim that $bdy(A) \subseteq \bar{A}$. Let $x \in (\bar{A})^C$. There exists $\exists \epsilon > 0$ such that $B(x,\epsilon) \cap \bar{A} = \emptyset$. This implies that $B(x,\epsilon) \cap A = \emptyset \implies x \notin bdy(A)$. This implies $F = bdy(A) \cup A \subseteq \bar{A}$. Claim that F is closed.

□

Definition. Let (X,d) metric space, $A \subseteq X$ and $x \in X$. We say that x is a limit point of A, if for all neighbor hood N of x, we have $N \cap (A \backslash \{x\}) \neq \emptyset$. Equivalently, $\forall \epsilon > 0, B(x,\epsilon) \cap (A \backslash \{x\}) \neq \emptyset$. The set of limit points of A is $Lim(A)$ cluster points.

Note: $A = [0,1] \subseteq \mathbb{R}, bdy(A) = \{0,1\}, Lim(A) = A$. For $B = \{x\} \subseteq \mathbb{R}, bdy(B) = B, Lim(B) = \emptyset$.

Proposition. Let (X,d) metric space, $A \subseteq X$

1. A is closed $\iff Lim(A) \subseteq A$
2. $\bar{A} = A \cup Lim(A)$.

Proposition. 1. $\bar{A} \subseteq \bar{B}$.

2. $int(A) \subseteq int(A)$.
3. $int(A) = A \backslash bdy(A)$.

Proposition. Let $A, B \subseteq (X,d)$ metric space.

1. $\overline{A \cup B} = \bar{A} \cup \bar{B}$
2. $int(A \cup B) = int(A) \cap int(B)$

Proof. 1. $A \cup B \subseteq \bar{A} \cup \bar{B}$. Hence, $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$

Conversely, $A \subseteq \overline{A \cup B} \implies \bar{A} \subseteq \overline{A \cup B}$. Similarly for B.

2. $int(A) \cap int(B) \subseteq A \cap B$. and $int(A) \cap int(B) \subseteq int(A \cap B)$.

Conversely, $int(A \cap B) \subseteq A \implies int(A \cap B) \subseteq int(A)$. Similar for B.

□

Definition. Let (X,d) metric space. $A \subseteq X$ is dense in X if $\bar{A} = X$. We say that (X,d) is separable if X has a countable subset A such that $\bar{A} = X$. Otherwise, X is non-separable.

Examples:

1. $\mathbb{R}$ is separable
2. $\mathbb{R}^n$ is separable.
3. l_1 is separable
4. l_∞ is non-separable.

Question:
Is $(C[a,b], \|\|_\infty)$ separable?

2.3 Convergence of sequences and topology in a metric space

Definition. (X,d) metric space, $\{x_n\} \subseteq X$ sequence. We say that $\{x_n\}$ converges to a point $x_0 \in X$ if $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $d(x_n, x_0) < \epsilon$. Then x_0 is the limit of $\{x_n\}$, $\lim_n x_n = x_0, x_n \to x_0$. Equivalently, $\lim_n x_n = x_0 \iff \lim_n d(x_0, x) = 0$.

Proposition. (X,d) metric space, $\{x_n\} \subseteq X$. If $\lim x_n = x_0 = y_0$

Proposition.
1. $x_0 \in bdy(A) \iff \exists$ sequence $\{x_n\} \subseteq A, \{y_n\} \subseteq A^c$ such that $x_n \to x_0, y_n \to x_0$.
2. A is closed $\iff$ whenever $\{x_n\} \subseteq A$ with $x_n \to x_0 \implies x_0 \subseteq A$.

Proof.
1. $x_0 \in bdy(A), x_n \in B(x_0, \frac{1}{n}) \cap A$. $y_n \in B(x_0, \frac{1}{n}) \cap A^c$. Conversely, suppose $\{x_n\} \subseteq A, \{y_n\} \subseteq A^c, x_n \to x_0, y_n \to x_0$. Given $\varepsilon > 0, \exists N \in \mathbb{N}$, such that $x_n \in B(x_0, \varepsilon), \forall n \geq N \implies B(x_0, \epsilon) \cap A \neq \emptyset$. $\exists N' \in \mathbb{N}$, such that $x_n \in B(x_0, \varepsilon), \forall n \geq N' \implies B(x_0, \varepsilon) \cap A^c \neq \emptyset$. This implies $x_0 \in bdy(A)$.
2. A is closed, $\{x_n\} \subseteq A, x_n \to x_0$. Suppose $x_0 \in A^c \implies \exists \varepsilon > 0$, such that $B(x_0, \varepsilon) \cap A = \emptyset$ but since $x_n \to x_0, \exists N \in \mathbb{N}$, such that $d(x_0, x_n) < \varepsilon, \forall n \geq N$. Contradiction. Then $x_0 \in A$.

   Conversely, suppose A is not closed, Then $x_0 \in bdy(A) \backslash A$. By (1), $\exists \{x_n\} \subseteq A$ such that $x_n \to x_0 \implies x_0 \in A$. This is a contradiction. Then A is closed.

□

Proposition. Let (X,d) metric space, $\{x_n\} \subseteq X$. If $x_0 = \lim_{n\to\infty} x_n = y_0$, then $x_0 = y_0$.

Proof. Suppose $x_0 \neq y_0 \implies d(x_0, y_0) = \epsilon > 0$. $\frac{\epsilon}{2} > 0, \exists N \in \mathbb{N}$ such that $d(x_n, x_0) < \frac{\epsilon}{2}, \forall n \geq N$, $\exists N' \in \mathbb{N}$ such that $d(x_n, x_0) < \frac{\epsilon}{2}, \forall n \geq N'$, If $n = \max\{N, N'\}$, $\epsilon = d(x_0, y_0) \leq d(x_0, x_n) + d(x_n, y_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. □

Definition. We say that x_0 is a limit point of $\{x_n\}$ if $\exists$ a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0$. $lim^*(\{x_n\}) = \{x_0 \in X | x_0 \text{ is a limit point of } \{x_n\}\}$ $lim(\{x_n\}) \leftarrow \{x_n\}$ subset of X.

Example, $x_n = (-1)^n$.$lim^*(\{x_n\}) = \{-1, 1\}$. $lim(\{x_n\}) = \emptyset$.

Proposition. (X,d) metric space, $A \subseteq X$. $x_0 \in lim(A) \iff \exists \{x_n\} \subseteq A$, with $x_n \neq x_0$ and $x_n \to x_0$.

Proof. Let $x_0 \in lim(A), \forall n \in \mathbb{N}, \exists x_n \in N$ such that $\{x_n\} \cap B(x_0, \frac{1}{n})$ Hence $\{x_n\} \subseteq A, x_n \neq x_0, x_n \to x_0$.

Conversely. $\forall \epsilon > 0, A \backslash \{x_0\} \cap B(x_0, \epsilon) \neq \emptyset$. Since $\exists N \in \mathbb{N}$, such that $x_n \neq x_0 \in B(x_0, \epsilon), \forall n \geq N$. □

2.4 Induced metric and the relative topology

Definition. Let (X,d) metric space, $A \subseteq X$. Define $d_A : A \times A \to \mathbb{R}$ such that $d_A(x,y) = d(x,y), \forall x, y \in A$. d_A is a mtreic, and its called the induced metric. Let $\tau_A = \{W \subset A | W = U \cap A \text{ for some U open in X}\}$. τ_A is a topology in A called the relative topology in A inherited from τ_d on X.

Theorem. (X,d) metric space, $A \subseteq X$, Then $\tau_A = \tau_{d_A}$.

Proof. Let $W \subseteq A, W \in \tau_A$ and $x \in W$. $\exists U$ open in X such that $U \cap A = W$. $x \in U \implies \exists \epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$. $x \in B_{d_A}(x, \epsilon) \subseteq B_d(x, \epsilon) \subseteq U$. $x \in B_{d_A}(x, \epsilon) \subseteq U \cap A = W \in \tau_{d_A}$.

Let $W \subseteq A, W \in \tau_{d_A}, \forall x \in W, \exists \epsilon_x > 0$ such that $B_{d_A}(x, \epsilon_x) \in W$.

$$W = \bigcup_{x \in W} B_{d_A}(x, \epsilon_x)$$

$$X \supseteq U = \bigcup_{x \in W} B_d(x, \epsilon_x) \text{ open in X}$$

Now $W = A \bigcap U \implies W \in \tau_A$. □

2.5 Continuity

$(X, d_x), (Y, d_y)$ metric spaces, $f : X \to Y$ function $f(x)$ is continuous at $x_0 \in X$ if $\forall \epsilon > 0, \exists \delta > 0$ such that $x \in B(x_0, \delta)$ then $f(x) \in B(f(x_0), \epsilon)$. Otherwise, $f(x)$ is discontinuous at x_0 $f(x)$ is continuous if it is continuous at x_0, for all $x_0 \in X$.

Theorem. $(X, d_x), (Y, d_y)$ metric space, $f : X \to Y$ TFAE

1. $f(x)$ is continuous at $x_0 \in X$.
2. If W is a neighborhood of $g = f(x_0)$, then $v = f^{-1}(W)$ is a neighborhood of x_0.

Proof. From (1) to (2): $\exists \epsilon > 0$ such that $B(f(x_0), \epsilon) \subseteq W$. This implies $\exists \delta > 0$ such that $d(z, x_0) < \delta \implies d_X(f(z), f(x_0)) < \epsilon$. Therefore, $f(B(x_0, \delta)) \subseteq B(f(x_0), \epsilon) \subseteq W$. But $V = f^{-1}(W)$ Hence $x_0 \in B(x_0, \delta) \subseteq V \implies x_0 \in int(V)$.

From 2 to 1, let $\epsilon > 0$, Therefore, $B(f(x_0), \epsilon) = W$ neighborhood of $f(x_0)$. THen $f^{-1}(W)$ is a neighborhood of x_0, i.e. $x_0 \in int(f^{-1}(W))$ Therefore, $\exists \delta > 0$ such that $B(x_0, \delta) \subseteq f^{-1}(W)$. □

Theorem. Sequential Characterization of continuous $(X, d_x), (Y, d_y)$ metric space, $f : X \to Y$, TFAE

1. $f(x)$ is continuous at $x_0 \in X$.
2. If $\{x_n\} \subseteq X, x_n \to x_0 \implies f(x_n) \to f(x_0)$.

Proof. From 1 to 2, $f(x)$ is continuous at x_0, $\{x_n\} \subseteq X$, $x_n \to x_0$. Fix $\epsilon > 0$, then $\exists \delta > 0$ such that $d_x(x, x_0) < \delta \implies d_y(f(x), f(x_0)) < \epsilon$. Since $x_n \to x_0$. $\exists N \in \mathbb{N}$, such that if $n \geq N, d_x(x_n, x_0) < \delta \implies d_y(f(x_n), f(x_0)) < \epsilon$.

From 2 to 1, assume $f(x)$ is not continuous at x_0. $\exists \epsilon_0 > 0$, for every ball $B_x(x_0, \delta)$, $\exists x_\delta \in B_x(x_0, \delta)$ such that $d_Y(f(x_\delta), f(x_0) \geq \epsilon_0$. In particular, for each $n \in \mathbb{N}$, $x_n \in B_x(x_0, \frac{1}{n})$ Note: $x_n \to x_0$ but $d_Y(f(x_n), f(x_0)) \geq \epsilon_0$ i.e. $f(x_n)$ does not converge to $f(x_0)$. □

Theorem. $(X, d_x), (Y, d_y)$ metric space, $f : X \to Y$, TFAE

1. f(x) is continuous
2. If $W \subseteq Y$ is open, then $f^{-1}(W) = V \subseteq X$ is open
3. If $\{x_n\} \subseteq X, x_n \to x_0$ for some $x_0 \in X$, then $f(x_n) \to f(x_0) \in Y$.

Proof. 3 to 1 is done

1 to 2: Let $W \subseteq Y$ open and $V = f^{-1}(W)$. Let $x_0 \in V', f(x_0) \in W$ open. Therefore, W is a neighborhood of $f(x_0)$. By 1, $f^{-1}(W) = V$ is a neighborhood of x_0 i.e. $x_0 \in int(V)$ Then $V = int(V)$ is open.

2 to 3: let $\{x_n\} \subseteq X, x_n \to x_0$. Let $y_0 = f(x_0)$. Fix $\epsilon > 0$, if $W = B_y(y_0, \epsilon)$ open in Y. Then $f^{-1}(W) \subseteq X$ open. Note: $x_0 \in V \implies \exists \delta > 0$, such that $B_x(x_0, \delta) \subseteq V$. Since $x_n \to x_0, \exists N$ such that if $n \geq N$, then $d_x(x_n, x_0) < \delta$, i.e. $x_n \in V, \forall n \geq N$. Hence $f(x_n) \subseteq W, \forall n \geq N$. i.e. $d_y(f(x_n), f(x_0)) < \epsilon \iff f(x_n) \to f(x_0)$. □

Example: X a set, d discrete metric (Y, d_x) metric space, $f(X, d) \to (Y, d_Y)$ is continuous.

Definition. $f(X, d_X) \to (X, d_y)$: f is a homeomorphism if f is one-to-one and onto, and both f and f^{-1} are continuous. We say that (X, d_X) and (Y, d_Y) are homeomorphic.

Remark: $f : X \to Y$ is homeomorphic, $U \subseteq X$ is open $\iff f(U) \subseteq Y$ is open.

Two metric spaces (X, d_X) and (Y, d_Y) are equivalent if $\exists$ a one-to-one and onto map $f : X \to Y$ and two constants, $c_1, c_2 > 0$, such that $c_1 d_X(x_1, x - 2) \leq d_Y(f(x_1), f(x_2)) \leq c_2 d_X(x_1, x_2), \forall x_1, x - 2 \in X$.
Remark: If X and Y are equivalent, then they are homeomorphic.

2.6 Complete Metric Spaces: Cauchy sequences

Note: If $\{x_n\} \subset (X, d_X), x_n \to x_0 \in X$ then $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that if $n \geq N \implies d(x_0, x_n) < \epsilon/2$, If $n, m \geq N$, $d(x_n, x_m) \leq d(x_n, x_0) + d(x_0, x_m) < \epsilon/2 + \epsilon/2 < \epsilon$.

Definition. A sequence $\{x_n\} \subseteq (X, d_x)$ is cauchy in (X, d_x) if $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N$, $d(x_n, x_m) < \epsilon$.

Theorem. Let $\{x_n\} \subseteq (X, d_x)$ be a convergent sequence then $\{x_n\}$ is Cauchy.

Does every Cauchy sequence converge? $x_n = \frac{1}{n}, X = (0, 2)$ used metric $\{x_n\}$ is Cauchy but it does not converge.

Definition. A metric space (X, d_x) is complete if every Cauchy sequence converges. A set $A \subseteq X$ is bounded if $\exists M > 0$, and $x_0 \in X$ such that $A \subseteq B[x_0, M]$.

Proposition. Every Cauchy sequence is bounded $\{x_n\}$ is Cauchy. This implies $\exists N \in \mathbb{N}$ such that $\forall n, m \geq N, d(x_n, x_m) < 1$. In particular, $d(x_N, x_m) < 1, \forall m \geq N$. $M = \max\{d(x_1, x_N), \cdots, d(x_{N-1}, x_N), 1\}$. This implies $\{x_n\} \subseteq B[x_N, M]$.

Proposition. Assume $\{x_n\}$ is a Cauchy sequence with a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0$. Then $x_n \to x_0$. Then $x_n \to x_0$. Let $\epsilon > 0 \implies \exists N \in \mathbb{N}$ such that $n, m \geq N, d(x_n, x_m) < \epsilon/2$ since $x_{n_k} \to x_0, \exists k \in \mathbb{N}$ such that $\forall n_k \geq k, d(x_{n_k}, x_0) < \epsilon/2$. $M = \max\{N, k\}, \forall n \geq M, d(x_n, x_0) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x_0) < \epsilon/2 + \epsilon/2 < \epsilon$. Pick $n_k > M$.

2.7 Completeness of $\mathbb{R}, \mathbb{R}^n$ and l_p

Theorem. Bolzano-Weierstrass Theorem: every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Theorem. Completeness Theorem for $\mathbb{R}$. Every Cauchy sequence in $\mathbb{R}$ converges. $\{x_n\}$ is Cauchy $\implies$ $\{x_n\}$ is bounded $\implies$ $\{x_n\}$ has a convergent subsequence $\implies$ Then $\{x_n\}$ is convergent.

Theorem. Let $1 \leq p \leq \infty$, every Cauchy sequence in $(\mathbb{R}^n, \|\cdot\|_p)$ converges.

Lemma. Let $1 \leq p < \infty$, let $\{x_k\}$ be a Cauchy sequence in $(l_p, \|\cdot\|_p)$. Then for each $i \in \mathbb{N}$, the component sequence $\{x_{k,2}\}_k$ is Cauchy in $\mathbb{R}$.

Proof. Assume $\{x_k\}_{k\in\mathbb{N}} \subseteq (l_p, \|\cdot\|_p)$ is Cauchy. $x_k = \{x_{k,1}, \cdots, x_{k,n}\}$ Since each component sequence $\{x_{k,i}\}_k$ is Cauchy on $\mathbb{R}$. and $\mathbb{R}$ is complete. Let $x_{0,i} = l_m x_{k,i} \in \mathbb{R}$ Let $x_0 = \{x_{0,1}, \cdots, x_{0,i}, \cdots\}$.

Claim: $x_0 \in l_p$ and $x_k \to x_0$.

Let $\epsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that $k, m \geq N_0, \|x_m - x_k\|_p < \frac{\epsilon}{2}$.

Case 1 Let $p = \infty$, $k \geq N_0$, $|x_{m,i} - x_{k,i}| \leq \|x_m - x_k\|_\infty$, $\forall m \geq N_0, \forall i \in \mathbb{N}$. $k \geq N_0, |x_{0,i} - x_{k,i}| = \lim_{m\to\infty} |x_{m,i} - x_{k,i}| \leq \frac{\epsilon}{2} < \epsilon, \forall i \in \mathbb{N}$. This implies $\{x_{0,i} - x_{k,i}\}_i \in l_\infty$. Well $\{x_{k,i}\} \in l_\infty$. This implies $\{x_{0,i}\} \in l_\infty$. Therefore, $\|x_0 - x_k\|_\infty < \epsilon, \forall k \geq N_0$. This implies $x_k \to x_0$.

Case 2 Let $k \geq N_0$. For each $j \in \mathbb{N}$ such that $(\sum_{i=1}^j |x_{m,i} - x_{k,i}|^p)^{1/p} \leq \|x_m - x_k\|_[< \frac{\epsilon}{2}$. $(\sum_{i=1}^j |x_{0,i} - x_{k,i}|^p)^{1/p} = \lim_m (\sum_{i=1}^j |x_{m,i} - x_{k,i}|^p)^{1/p} \leq \frac{\epsilon}{2}$.

$$(\sum_{i=1}^{\infty} |x_{0,i} - x_{k,i}|^p)^{1/p} \leq \frac{\epsilon}{2} < \epsilon, \forall k \geq N_0$$

Then this implies $\{x_{0,i} - x_{k,i}\} \in l^p$ and $\{x_{k,i}\}_i \in l^p$. Then $\{x_{0,i}\} = x_0 \in l^p$. then $\|x_0 - x_k\|_p < \epsilon, \forall k \geq N_0$, then $x_k \to x_0$.

□

2.8 Completeness of $(\mathcal{C}_b(X), \|\cdot\|_\infty)$

Definition. $(X, d_x), (Y, d_y)$ metric space $\{f_n\}$ sequence of functions $f_n : X \to Y$. $\{f_n\}$ converges pointwise to $f_0 : X \to Y$ if $\lim_n f_n(x_0) = f_0(x_0), \forall x_0 \in X$. $\{f_n\}$ converges uniformly to $f_0 : X \to Y$ if $\forall \epsilon > 0, \exists N_0 \in \mathbb{N}$ such that $n \geq N_0, d_Y(f_n(x), f_0(x)) < \epsilon, \forall x \in X$.

Remark: $\{f_n\}$ such that $f_n \to^{uniform} f_0 \implies f_n \to^{pointwise} f_0(x), \forall x$. Let $f_n(x) = x^n$ on $[0, 1]$. $f_n(x) \to f_0(x), \forall x$, for $f_0(x) = 1, x = 1$ otherwise 0.

Theorem. $(X, d_x), (Y, d_y)$ metric space, $\{f_n\}$ such that $f_n : X \to Y$ and $f_n \to^{unit} f_0 : X \to Y$. If each f_n is continuous at x_0, so is f_0.

$f_n \to^{unit} f_0$. This implies $\exists N_0 \in \mathbb{N}$ such that $n \geq N_0, d_y(f_n(x), f_0(x)) < \frac{\epsilon}{3}, \forall x \in X$.

f_n continuous at $x_0, \forall n \implies$ in particular f_{N_0} is continuous at x_0. This means $\exists \delta > 0$ such that $x \in B(x_0, \delta) \implies d_y(f_{N_0}(x_0), f_{N_0}(x)) < \frac{\epsilon}{3}$.

Proof. If $x \in B(x_0, \delta)$,

$$d_Y(f_0(x_0), f_0(x)) \leq d_Y(f_0(x_0), f_{N_0}(x_0)) + d_Y(f_{N_0}(x_0), f_{N_0}(x)) + d_Y(f_{N_0}(x), f_0(x)) < \frac{\epsilon}{3} \times 3 = \epsilon$$

. □

Definition. (X, d_x) metric space, $\mathcal{C}_b(X) := \{f : X \to \mathbb{R} | f$ is continuous on X and f(x) is bounded$\}$.

$$\|f\|_\infty = sup\{|f(x)| x \in X\}$$

$(\mathcal{C}_b(X), \|\cdot\|_\infty)$ is a normed linear space.

Remark: let $\{f_n\} \subseteq \mathcal{C}_b(X), f_n(X, d_x) \to (\mathbb{R}, \text{usual metric})$. $f_n \to^{\|\|_\infty} f_0 \iff f_n \to^{uniform} f_0$.

Theorem. Completeness for $(\mathcal{C}_b(X), \|\cdot\|_\infty)$, $(\mathcal{C}_b(X), \|\cdot\|_\infty)$ is complete.

Let $\{f_n\}$ be a Cauchy sequence.For each $x_0 \in X$, $|f_n(x_0) - f_m(x_0)| \leq \|f_n - f_m\|_\infty$. It follows, that $\{f_n(x_0)\}$ is Cauchy in $\mathbb{R}$, $\forall x_0 \in X$. $f_0(x) = \lim_{n\to\infty} f_n(x), \forall x \in X$.

Claim: $f_n \to f_0$.

Let $\epsilon > 0$, choose N_0 such that $n, m \geq N_0 \implies \|f_n - f_m\|_\infty < \frac{\epsilon}{2}$. If $n \geq N_0$ and $x \in X$, then $|f_n(x) - f_0(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \frac{\epsilon}{2} < \epsilon$. Therefore, $f_n \to f_0 \implies f_0$ is continuous.

f_0 is bounded. $\{f_n\}$ is Cauchy, then $\{f_n\}$ is bounded. $\exists M > 0$ such that $\|f_n\|_\infty < M, \forall n \in \mathbb{N}$. $\exists n_0$ such that $|f_0(x) - f_{n_0}(x)| < 1, \forall x \in X$. Then $|f_0(x)| \leq f_0(x) - f_{n_0}(x)| + |f_{n_0}(x)| < 1 + M, \forall x \in X$. Hence $f_0 \in \mathcal{C}_b(X)$ and $f_n \to f_0$.

Remark: $\mathbb{N}$, discrete metric space. $(\mathcal{C}_b(\mathbb{N}), \|\cdot\|_\infty) = (l_\infty, \|\|_\infty)$ and $(\mathcal{C}_b(X), \|\cdot\|_\infty) \implies (l_\infty(X), \|\cdot\|_\infty)$

2.9 Characterizations of Complete Metric Spaces

Note: Theorem fails if we consider open intervals $\{(0, 1/n)\}$.

Note: Theorem fails if we consider unbounded intervals $\{[n, \infty)\}$.

Definition. Let $A \subseteq (X, d)$. $diam(A) := sup\{d(x, y) | x, y \in A\}$ is the diameter of A.

Proposition. Let $A \subseteq B \subseteq (X, d)$, Then:

1. $diam(A) \leq diam(B)$
2. $diam(A) = diam(\bar{A})$.

Proof. The second: $\leq$ from (1). If $diam(A) = \infty \implies diam(\bar{A}) = \infty$. Let $\epsilon > 0$, let $x, y \in \bar{A}$. this implies $\exists x_0, y_0 \in A$ such that $d(x, x_0) < \frac{\epsilon}{2}, d(y, y_0) < \frac{\epsilon}{2}$. $d(x, y) \leq d(x_1, x_0) + d(x_0, y_0) + d(y_0, y) \leq diamA + \epsilon$. Hence $diamA \leq diam\bar{A} \leq diamA + \epsilon, \forall \epsilon > 0$. □

Generalization of Nested Interval Theorem to (X, d) is complete.

Theorem. Cantor's Intersection Theorem: Let (X, d) be a metric space TFAE

1. (X, d) is complete.
2. (X, d) satisfies the following proposition.
    - If $\{F_n\}$ is a sequence of non-empty closed sets. such that $F_{n+1} \subseteq F_n, \forall n$, and $\lim_n(diamF_n) = 0 \implies \bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Proof. 1 to 2: $\{F_n\}$ a sequence such that $F_n \neq \emptyset$, F_n is closed, $F_{n+1} \subseteq F_n$, $\lim(diamF_n) = 0$. For each n, choose $x_n \in F_n$. Let $\epsilon > 0$, $\exists N_0$ such that $diamF_{N_0} < \epsilon$. If $n, m \geq N_0, \implies x_n, x_m \in F_{N_0}$. $d(x_n, x_m) \leq diam(F_{N_0}) < \epsilon$. Hence $\{x_n\}$ is a Cauchy sequence and (X, d) is complete. Then $x_n \to_n x_0 \in X$.

For each n, $\{x_n, x_{n+1}, \cdots, x_{n+k}, \cdots\} \subseteq F_n$. Then $x_{n+k} \to_k x_0$ and F_n closed so $x_0 \in F_n, \forall n$. This implies $x_0 \in \bigcap_{n=1}^{\infty} F_n$.

2 to 1: let $\{x_n\} \subseteq X$. Cauchy. For each n, $A_n := \{x_n, x_{n+1}, \cdots\}$ Claim: $diam(A_n) \to_n 0$. Let $F_n = \bar{A}_n$, $A_{n+1} \subseteq A_n \implies F_{n+1} \subseteq F_n$. $diam(F_n) \to_n 0$.

This implies $\exists x_0 \in \bigcap_{n=1}^{\infty} F_n$, let $\epsilon > 0$, choose N_0 such that $diamF_{N_0} < \epsilon$. This implies $F_{N_0} \subseteq B(x_0, \epsilon)$. If $n \geq N_0$, $d(x_n, x_0) < \epsilon$. This implies $x_n \to_n x_0$. □

Definition. Define $(X, \|\cdot\|)$ normed space. $\{x_n\} \subseteq X$. A series with terms $\{x_n\}$ is a formal sum $\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \cdots$. For each $k \in \mathbb{N}$, define the kth-[artial sum of $\sum_{n=1}^{\infty} x_n$ by $s_k = \sum_{n=1}^{k} x_n \in X$. The series $\sum_{n=1}^{\infty} x_n$ converges if the sequence $\{s_k\}$ converges. Otherwise, diverge.

Definition. A normed linear space $(X, \|\cdot\|)$ which is complete under the metric induced is called a Banach space.

Theorem. Generalized Werestrass M-Test: Let $(X, \|\cdot\|)$ normed linear space TFAE

1. $(X, \|\cdot\|)$ is a Banach Space.
2. The space $(X, \|\cdot\|)$ satisfies the following property:
   Let $\{x_n\} \subseteq X$. If $\sum_{n=1}^{\infty} \|x_n\|$ converges in $\mathbb{R} \implies \sum_{n=1}^{\infty} x_n$ converges in $(X, \|\cdot\|)$.

Proof. 1 to 2: Let $T_k = \sum_{n=1}^{k} \|x_n\| \implies \{T_k\}$ is Cauchy. Given $\epsilon > 0, \exists N_0$ such that $k > m > N_0$

$$\sum_{n=m+1}^{k} \|x_n\| = |T_k - T_m| < \epsilon$$

Let $s_k = \sum_{n=1}^{k} x_n$, let $k > m > N_0$.

$$\|s_k - s_m\| = \|\sum_{n=m+1}^{k} x_n\| \leq \sum_{n=m+1}^{k} \|x_n\| < \epsilon$$

Therefore $\{s_k\}$ is Cauchy. This implies $\{s_k\}$ converges and then $\sum_{n=1}^{\infty} x_n$ converges.

2 to 1: Assume 2 holds and $\{x_n\}$ is Cauchy. Choose n_1 if $i, j > n_1 \implies \|x_1 - x_j\| < \frac{1}{2}$ and choose n_2, such that if $i, j > n_2 \implies \|x_i - x_j\| < \frac{1}{2^2}$.

If we have $n_k > n_{k-1} > \cdots > n_2 > n_1$ such that if $i, j > n_k \implies \|x_i - x_j\| < \frac{1}{2^k}$. Choose $n_{k+1} > n_k$ such that if $i, j > n_{k+1} \implies \|x_i - x_j\| < \frac{1}{2^{k+1}}$. By induction, $\{n_k\}_k$ is an increasing sequence of $\mathbb{N}$ such that $i, j > n_k \implies \|x_i - x_j\| < \frac{1}{2^k}$. In particular $\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k} \implies g_k = x_{n_k} - x_{n_{k+1}} \in X, \forall k$.

$$\sum_{k=1}^{\infty} \|g_k\| = \sum_{k=1}^{\infty} \|x_{n_{k+1}} - x_{n_k}\| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

Hence $\sum_{k=1}^{\infty} \|g_k\|$ converges. Hence $\sum_{k=1}^{\infty} g_k$ converges in $(X, \|\cdot\|) \iff \{s_k\}_k$ converges $s_k = \sum_{j=1}^{k} g_j$. $s_k = g_1 + g_2 + \cdots + g_k = x_{n_1} - x_{n_2} + x_{n_2} - x_{n_3} + \cdots + x_{n_k} - x_{n_{k+1}} = x_{n_1} - x_{n_{k+1}}$. $x_{n_{k+1}} \to x_{n_1} - \sum_{j=1}^{\infty} g_j$. Therefore $\{x_{n_k}\}$ converges and $\{x_n\}$ is Cauchy. Then $\{x_n\}$ converges. □

Example:
A continuous, nowhere differentiable function
Let $\phi(x) = \begin{cases} x & x \in [0,1] \\ 2 - x & x \in [1,2] \end{cases}$. Extend to $\mathbb{R}$ by $\phi(x) = \phi(x+2)$. Let $f(x) = \sum_{n=0}^{\infty} (\frac{3}{4})^n \phi(4^n x)$.

1. Claim 1: $f(x)$ is continuous on $\mathbb{R}$. $\sum_{n=1}^{\infty} (\frac{3}{4})^n \phi(4^n x) \leq \sum_{n=0}^{\infty} (\frac{3}{4})^n = L$. Then $f(x)$ is defined. $\sum_{n=1}^{k} (\frac{3}{4})^n \phi(4^n x) \leq \sum_{n=0}^{\infty} (\frac{3}{4})^n \to f(x)$.

2.10 Completion of Metric Space

Proposition. (X, d) complete metric space, let $A \subseteq X$, then (A, d_A) is complete $\iff$ A is closed in X.

Proof. Converse: assume $A \subseteq X$ is closed, $\{x_n\} \subseteq A$ Cauchy in (A, d_A).Then $\{x_n\}$ Cauchy in $(X, d) \implies \exists x_0$ such that $x_n \to x_0$ and A is closed so $x_0 \in A$.

$\implies$ Suppose A is not closed. This implies $\exists x_0 \in bdy(A) \setminus A$. This implies $\exists \{x_n\} \subseteq A$ such that $x_n \to_n x_0$. This means $\{x_n\}$ is Cauchy (A, d_A). This means A is not complete. Hence contradiction. □

Definition. $(X, d_x), (Y, d_y)$ metric spaces. A map $\phi : X \to Y$ is an isometry if $d_Y(\phi(x), \phi(y)) = d_X(x, y), \forall x, y \in X$. Note: If ϕ is an isometry, then ϕ is one-to-one. If ϕ is an isometry and ϕ is onto, we say that (X, d_X) and (Y, d_Y) are isometric. A completion of (X, d_X) is a pair $((Y, d_Y), \phi)$ such that (Y, d_Y) is a complete metric space, $\phi : X \to Y$ is an isometry and $\overline{\phi(X)} = Y$.

Theorem. (X, d) metric space. This implies $\exists$ an isometry such that

$$\phi : X \to (C_b(X), \|\cdot\|_\infty)$$

.

Proof. Fix $a \in X$, for $u \in X$, let $f_u : X \to \mathbb{R}$. Then $f_u(x) = d(u, x) - d(x, a)$. f_u is continuous such that f_u is bounded, $|f_u(x)| = |d(u, x) - d(x, a)| \leq d(u, a)$. This implies $f_u \in C_b(X)$. Let $\phi : X \to C_b(X)$ such that $u \to f_u$.

$$\begin{aligned} d(f_u, f_v) = \|f_u - f_v\|_\infty &= \sup_{x \in X} \{|f_u(x) - f_v(x)|\} \\ &= sup_{x \in X} \{|d(u, x) - d(x, a) - d(v, x) + d(x, a)|\} \leq d(u, v) \\ |f_u(v) - f_v(v)| = d(u, v) &\implies \|f_u - f_v\|_\infty = d(u, v) \end{aligned}$$

□

Corollary. Every metric space has a completion. Let $\phi : X \to (C_b(X), \|\cdot\|_\infty)$ and $Y = \phi(x)$. $((Y, d_Y), \phi)$ is complete.

2.11 Banach Contractive Mapping Theorem

Question: can we find $f \in C[0,1]$ such that $f(x) = e^x + \int_0^x \sin(t)/2f(t)dt$?

Strategy: define $\Gamma : C[0,1] \to C[0,1]$. $\Gamma(g)(x) = e^x + \int_0^x \sin(t)/2g(t)dt \in C([0,1])$. $\exists! f \in C[0,1]$ such that Γ fixes f, i.e., $\Gamma(f) = f$.

Definition. (X, d_X) metric space, let $\Gamma : X \to X$. We call $x_0 \in X$ a fixed point of Γ if $\Gamma(x_0) = x_0$. We say that Γ is Lipchitz if $\exists \alpha \geq 0$ such that $d(\Gamma(x), \Gamma(y)) \leq \alpha d(x,y), \forall x, y \in X$ and Γ is a contraction if $\exists k$ such that $0 \leq k < 1$ such that $d(\Gamma(x), \Gamma(y)) \leq kd(x,y), \forall x, y \in X$.

Theorem. Banach Contractive Mapping Theorem (or Banach fixed point Theorem). Let (X, d) be a complete metric space. This implies Γ has a unique fixed point $x_0 \in X$.

1. If such x_0 exists, it's unique: suppose $\Gamma(x_0) = x_0$ and $\Gamma(y_0) = y_0$, $\Gamma \neq 0$. This implies $d(x_0, y_0) = d(\Gamma(x_0), \Gamma(y_0)) \leq kd(x_0, y_0)$ This implies $d(x_0, y_0) = 0$.
2. Let $x_1 \in X$ and $x_2 = \Gamma(x_1), x_3 = \Gamma(x_2), \cdots, x_{n+1} = \Gamma(x_n)$.

$$d(x_2, x_3) = d(\Gamma(x_1), \Gamma(x_2)) \leq kd(x_1, x_2)$$

$$d(x_4, x_3) = d(\Gamma(x_3), \Gamma(x_2)) \leq kd(x_3, x_2) \leq k^2 d(x_1, x_2)$$

By induction, $d(x_{n+1}, x_n) \leq k^{n-1} d(x_1, x_2)$. If $m > n$, $d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n-2}, x_{n-1}) + d(x_{n-1}, x_n) \leq k^{m-2} d(x_2, x_1) + k^{m-3} d(x_2, x_1) + \cdots + k^n d(x_1, x_2) + k^{n-1} d(x_2, x_3) = \frac{k^{n-1}}{1-k} d(x_2, x_1)$.

Remark: If $d(\Gamma(x), \Gamma(y)) < d(x,y)$, theorem fails.

Example: Show that there exists a unique $f \in C[0,1]$ such that

$$f(x) = e^x + \int_0^x \frac{\sin(t)}{2} f(t)dt$$

Let $\Gamma(g)(x) = e^x + \int_0^x \frac{\sin(t)}{2} g(t)dt$. $(C[0,1], \|\cdot\|_\infty)$ is complete. Let $f(x), g(x) \in C[0,1]$ and $x \in [0,1]$.

$$\begin{aligned}
|\Gamma(g)(x) - \Gamma(f)(x)| &= |e^x + \int_0^x \frac{\sin(t)}{2} g(t)dt - e^x - \int_0^x \frac{\sin(t)}{2} f(t)dt| \\
&= |\int_0^x \frac{\sin(t)}{2}(g(t) - f(t))dt| \\
&\leq \int_0^x |\frac{\sin(t)}{2}||g(t) - f(t)|dt \leq \|g - f\|_\infty \int_0^1 \frac{1}{2}dt = \frac{1}{2}\|g - f\|_\infty \\
&\implies \|\Gamma(g) - \Gamma(f)\|_\infty \leq \frac{1}{2}\|g - f\|_\infty \implies \Gamma \text{is a contraction} \\
&\implies \exists | f(x) \in C[0,1]
\end{aligned}$$

Example: Show that there exists a unique $f_0(x) \in C[0,1]$ such that

$$f_0(x) = x + \int_0^x t^2 f_0(t)dt$$

Find a power series representation for $f_0(x)$. Let $\Gamma(g)(x) = x + \int_0^x t^2 g(t)dt$ Note $(C[0,1], \|\cdot\|_\infty)$ is complete. Let $f, g \in C[0,1], x \in [0,1]$.

$$
\begin{aligned}
|\Gamma(g)(x) - \Gamma(f)(x)| &= |\int_0^x t^2(g(t)-f(t))dt| \\
&\leq \int_0^1 t^2|g(t)-f(t)|dt \leq \|g-f\|_\infty \int_0^1 t^2 dt = \frac{1}{3}\|g-f\|_\infty, \forall x \in [0,1] \\
\|\Gamma(g) - \Gamma(f)\|_\infty &\leq \frac{1}{3}\|f-g\|_\infty, \forall f, g \in C[0,1]
\end{aligned}
$$

Therefore, Γ is a contraction. By BCM theorem, $\exists! f_0 \in C[0,1]$ such that $\Gamma(f_0) = f_0$.
Let $f_1 = 0, f_{n+1} = \Gamma(f_n)$. Therefore,

$$f_2(x) = x + \int_0^x t^2 \theta dt = x$$

$$f_3(x) = x + \int_0^x t^2 t dt = x + \frac{x^4}{4}$$

$$\cdots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{1, 47(3n+1)}$$

Theorem. Picard-Lindelof Theorem: Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and Lipchitz in y, i.e., $1 > \alpha \geq 0$, such that

$$|f(t,y) - f(t,z)| \leq \alpha|y-z|, \forall y, z \in \mathbb{R}$$

Let $y_0 \in \mathbb{R}$, $\implies !y(t) \in C[0,b]$ such that $y'(t) = f(t, y(t)) \forall t$ and $y(0) = y_0$.

2.12 Baire's Category Theorem

Example:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \backslash \mathbb{Q} \\ \frac{1}{n} & \text{if } x = \frac{m}{n}, m \in \mathbb{Z}, n \in \mathbb{N}, m \neq 0, gcd(m,n) = 1 \\ 1 & x = 0 \end{cases}$$

$f(x)$ is discontinuous at $x = r$, for all $r \in \mathbb{Q}$. $f(x)$ is continuous at $x = \alpha$, for all $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.

Definition. (X, d) metric space, $A \subseteq X$ is said to be on F_σ set if $A = \bigcup_{n=1}^\infty F_n$ where $\{F_n\}$ is a sequence of closed sets. This implies $A \subseteq X$ is said to be a G_δ set if $A = \bigcap_{n=1}^\infty U_n$ where $\{U_n\} \subseteq X$ is a sequence of open sets.

Remarks:

1. From DeMorgan's Law, A is $F_\sigma \iff A^c$ is G_δ.
2. $[0,1)$ is both F_σ and G_δ. $[0,1) = \bigcup_{n=1}^\infty [0, 1 - \frac{1}{n}]$ and $[0,1) = \bigcap_{r=1}^\infty (-\frac{1}{n}, 1)$.
3. $F \subseteq X$ closed. This implies F is G_δ. $U \subseteq X$ open. This implies U is F_σ.

Definition. $(X, d_X), (Y, d_Y)$ metric spaces and $f : X \to Y$. $D(f) = \{x \in X | f$ is not continuous $\}$.
$D_n(f) = \{x \in X | \forall \epsilon > 0, \exists y, z \in B(x, \delta)$ with $d_Y(f(y), f(z)) \geq \frac{1}{n}\}$.

Theorem. Let $f : (X, d_X) \to (Y, d_Y), \forall n \in \mathbb{N}, D_n(f)$ is closed in X. Moreover, $D(f) = \bigcup_{r=1}^\infty D_n(f)$. In particular, $D(f)$ is F_σ.

Proof. $(D_n(f))^c$ open and $x \in (D_n(f))^c \implies \exists\delta > 0, \forall y, z \in B(x,\delta), d_Y(f(y), f(z)) < \frac{1}{n}$. Let $v \in B(x,\delta), \eta = \delta \cdot d_X(x, v)$. Let $y, z \in B(v, \eta)$ If $y \in B(v,\eta) \implies d(y, x) \leq d(y, v) + d_X(v, x) < \delta - d_X(x, v) + d_X(v, x) < \delta$. This implies $y, z \in B(x, \delta) \implies d_Y(f(x), f(y)) < \frac{1}{n}$. Hence $B(x, \delta) \subseteq (D_n(f))^c \implies (D_n(f))^c$ is open. □

Definition. (X, d) metric space. A set $A \subseteq X$ is nowhere dense if $int(\bar{A}) = \emptyset$. A is of first category in X if $A = \bigcup_{n=1}^{\infty} A_n$ where each A_n is nowhere dense. Otherwise, A is of second category in X. A set C is residual in X if C^c is of first category in X.

Recall: A set $A \subseteq X$ is dense if $\bar{A} = X$. Equivalently, A is dense if $\forall W \subseteq X$ open, $W \cap A \neq \emptyset$. Suppose there exists $W \subseteq X$ open such that $W \cap A = \emptyset$. Let $x \in W \implies x \in X\backslash A$. But $\exists\delta$ such that $B(x, \delta) \subseteq W \implies x \notin \bar{A}$.

Let $x_0 \in X\backslash A$ (want $\exists\{x_n\} \subseteq A\backslash x_n \to x_0$) since $B(x, \frac{1}{n}) \cap A \neq \emptyset$. This implies $\exists x_n \in B(x, \frac{1}{n} \cap A \implies \{x_n\} \subseteq A, x_n \to x_0$.

Theorem. Baire Category Theorem 1, (X, d) complete metric space. Let $\{U_n\}$ be a sequence of open, dense sets. Then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.

Proof. Let $W \subseteq X$ be open and non-empty. Then $\exists x_1 \in X$ and $r_1 < 1, B(x_1, r_1) \subseteq B[x_1, r_1] \subseteq W \cap U$. And $\exists x_2 \in X, r_2 < \frac{1}{2}$ such that $B(x_2, r_2) \subseteq B[x_2, r_2] \subseteq B(x_1, r_1) \cap U_2$

Recursively, we find sequences $\{x_n\} \subseteq X$ and $\{r_n\} \subseteq \mathbb{R}$ such that $0 < r_n < \frac{1}{n}$ and $B(x_{n+1}, r_{n+1}) \subseteq B[x_{n+1}, r_{n+1}] \subseteq B(x_n, r_n) \cap U_{n+1}, \forall n \geq 1$ but $r_n \to 0$, $B[x_{n+1}, r_{n+1}] \subseteq B[x_n, r_n]$, X is complete. By Cantor intersection theorem, there exists $x_0 \in \bigcap_{n=1}^{\infty} B[x_n, r_n] \subseteq W$ and $B[x_n, r_n] \subseteq U_n, \forall n$. This means $x_0 \in W \cap (\bigcap_{n=1}^{\infty} U_n)$. This implies $\bigcap_{n=1}^{\infty} U_n$ is dense. □

Remarks:

1. The Cantor set is nowhere dense in $\mathbb{R}$, and has cardinality c.

2. A close set F is nowhere dense if and only if $U = F^c$ is dense.

Corollary. Baire Category Theorem II: every complete metric space (X, d) is of second category in itself. Assume X is of the first category, i.e. $\exists\{A_n\}$ sequence of nowhere dense sets such that $X = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \bar{A}_n$. Let $U_n = (\bar{A}_n)^c \implies U_n$ is open and dense.

But $\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} (\bar{A}_n)^c = (\bigcup_{n=1}^{\infty} \bar{A}_n)^c = X^c = \emptyset$. Hence contradiction.

Corollary. $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$. Suppose $\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$, where each U_n is open. Let $F_n = (U_n)^c, \forall n$. $\mathbb{Q} \subseteq U_n, \forall n$ and $\bar{Q} = \mathbb{R}$ then $\bar{U}_n = \mathbb{R}$. Therefore, F_n is nowhere dense, for all n. Consider $\mathbb{Q} = \{r_1, r_2, \cdots\}$ Let $S_n = F_n \cup \{r_n\}$ closed and nowhere dense. Then $\mathbb{R} = \bigcup_{n=1}^{\infty} S_n$.

Then $\mathbb{R} = \bigcup_{n=1}^{\infty} S_n$, if $x \in \mathbb{Q} \implies x = r_n$ for some n. This implies $x \in S_n$. If $x \in \mathbb{R}\backslash\mathbb{Q} \implies x \in \bigcup_{n=1}^{\infty} U_n^c$.Hence $x \in F_n$ for some n, $x \in S_n$.

Corollary. There is no function $f : \mathbb{R} \to \mathbb{R}$ for which $D(f) = \mathbb{R}\backslash\mathbb{Q}$.

Definition. $(X, d_x), (Y, d_y)$ metric space, $\{f_n : X \to Y\}$ sequence of function $f_n \to f_0$ pointwise on X. We say that f_n converges uniformly at $x_0 \in X$ if $\forall\epsilon > 0$, $\exists\delta > 0$ and $N_0 \in \mathbb{N}$ such that if $n, m \geq N_0$ and $d(x, x_0) < \delta \implies d_Y(f_n(x), f_m(x)) < \epsilon$.

Theorem. $(X, d_x), (Y, d_y)$ metric space, $\{f_n : X \to Y\}$ such that $f_n \to f_0$ point wise on X. Assume that f_n convergence uniformly at x_0 and $\{f_n\}$ is a sequence of continuous function at x_0 This implies f_0 is continuous at x_0.

Theorem. Let $f_n : (a, b) \to \mathbb{R}$ be a sequence of continuous functions that converges point wise to f_0. This implies $\exists x_0 \in (a, b)$ such that f_n converges uniformly at x_0.

Claim: There exists a closed interval $[\alpha_1, \beta_1] \subset (a, b)$ with $\alpha_1 < \beta_1$ and $N_1 \in \mathbb{N}$ such that if $n, m \geq N$, and $x \in [\alpha_1, \beta_1]$. Then $|f_n(x) - f_m(x)| \leq 1$.

Inductively, we can construct a sequence $\{[\alpha_k, \beta_k]\}$ with $(a, b) \supset [\alpha_1, \beta_1] \supset (\alpha_1, \beta_1) \supset [\alpha_2, \beta_2] \supset (\alpha_2, \beta_2) \supset \cdots$ and a sequence $N_1 < N_2 < N_3 < \cdots$ such that $n, m \geq N_k$ and $x \in [\alpha_k, \beta_k]$. This implies $|f_n(x) - f_m(x)| \leq \frac{1}{k}$. Let $x_0 \in \bigcap_{k=1}^{\infty}[\alpha_k, \beta_k]$. Given $\epsilon > 0$, if $\frac{1}{k} < \epsilon$, and $n, m \geq N_k$ and $x \in (\alpha_k, \beta_k)$, then $|f_n(x) - f_m(x)| \leq \frac{1}{k} < \epsilon$. Pick $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset (\alpha_k, \beta_k)$. For δ as above, and N_k, the definition of uniform convergence at x_0 is verified.

Corollary. $\{f_n\} \subset C[a, b]$ such that $f_n \to f_0$ point wise on $[a, b]$. This implies $\exists$ a residual set $A \subset [a, b]$ such that f_0 is continuous at each $x \in A$. A^c is first category, i.e. $A^c = \bigcup_{n=1}^{\infty} A_n$, A_n nowhere dense.

$A = \{x \in [a, b] | f_0 \text{ is continuous at } x\}$.

Claim: A is dense in $[a, b]$, i.e. given any $(c, d) \subset [a, b]$, $(c, d) \cap A \neq \emptyset$. Let $(c, d) \subset [a, b]$, then $\exists x_0 \in (c, d)$ such that f_n converges uniformly at x_0. But each f_n is continuous. Then f_0 is continuous at x_0. This implies $x_0 \in A \bigcap (c, d)$. and $A^c = D(f_0)$ is $F_\sigma \implies$ A is G_δ. This implies $A = \bigcap_{n=1}^{\infty} U_n$, U_n open dense $\iff U_n^c$ closed, nowhere dense. i.e. $A^c = \bigcup_{n=1}^{\infty} U_n^c$, i.e., A is residual.

Corollary. Suppose $f(x)$ is differentiable on $\mathbb{R}$. Then $f'(x)$ is continuous for every point in a dense G_δ-subset of $\mathbb{R}$.

$f_n(x) = \frac{f(x+1/n) - f(x)}{1/n}$ Then $f(x)$ pointwise. Apply Corollary.

2.13 Compactness

Definition. An open cover for $A \subseteq X$ is a collection $\{U_\alpha\}_{\alpha \in I}$ of open sets for which $A \subseteq \bigcup_{\alpha \in i} U_\alpha$. Given a cover $\{U_\alpha\}_{\alpha \in I}$ for $A \subseteq X$, a sub cover is a sub collection $\{U_\alpha\}_{\alpha \in I}$, for $J \subseteq I$ such that $A \subseteq \bigcup_{\alpha \in I} U_\alpha$. A sub cover $\{U_\alpha\}_{\alpha \in I}$ is finite if I is finite. We say that $A \subseteq X$ i compact if every open cover of A has a finite sub cover. (X, d) is compact if X is compact. We say that $A \subseteq X$ is sequentially compact if every sequence $\{x_n\} \subseteq A$ has a converging subsequence converging to a point in A. (X, d) is sequentially compact if so is X. We say that X has the Bolzano-Weierstrass property (BWP) if every infinite subset in X has a limit point.

Theorem. (X, d) metric space, TFAE

1. X is sequentially compact
2. X has the BWP

Proof. 1 to 2: X sequentially compact and $S \subseteq X$ infinite. S has a countable infinite subset $\{x_1, x_2, \cdots\}$. This implies $\exists \{x_{n_k}\}$ subsequence of $\{x_n\}$ such that $x_{n_k} \to x_0$. $\forall \epsilon > 0, (B(x_0, \epsilon) \bigcap S) \backslash \{x_0\}$ has infinitely many points. Hence $x_0 \in LIm(S)$.

2 to 1: Assume X has the BWP, and $\{x_n\} \subseteq X$. If $\exists x_0 \in X$ appearing infinitely many times in $\{x_n\}$, then $\{x_n\}$ has a constant, converging subsequence. If such an x_0 doesn't exists, viewed as a subset of X, $\{x_n\}$ is infinite. We can assume the terms of $\{x_n\}$ are distinct. Thus $\exists x_0 \in Lim(\{x_n\})$. This implies $\exists n_1 \in \mathbb{N}$ such that $d(x_0, x_{n_1}) < 1$. Find $n_2 > n_1$ such that $d(x_0, x_{n_2}) < \frac{1}{2}$ If we have $n_1 < n_2 < \cdots < n_k$ such that $d(x_0, x_k) < \frac{1}{k}$. Choose $n_{k+1} > n_k$ such that $d(x_0, x_{n_{k+1}} < \frac{1}{k+1}$ This implies $\{x_{n_k}\} \subseteq \{x_n\}$ such that $x_{n_k} \to x_0$ □

Proposition. (X, d) metric space, $A \subseteq X$.

1. A compact $\implies$ A is closed and bounded.
2. If A is closed and X is compact, then so is A.
3. If A is sequentially compact. Then A is closed and bounded.

4. A is closed, X is sequentially compact. This implies A is sequentially compact.
5. If X is sequentially compact, then X is complete.

Proof. 1. Bounded pick $x_0 \in A$. This implies $\{B(x_0, n)\}$ is an open cover of A. A compact $\implies$ There exists a finite sub cover $\{B(x_0, n_k)\}$ let $M = \max\{n_j : j = 1, \cdots, k\} \implies A \subset B(x_0, M)$

Closed: Suppose A is not closed $\implies \exists x_0 \in Lim(A)\backslash A$, $U_n = (B[x_0, \frac{1}{n}]^c$. $\{U_n\}$ open cover of A, with no finite sub cover but A compact. Then contradiction.

2. Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover of A. Then $\{U_\alpha\}_{\alpha\in I} \cup \{A^c\}$ is an open cover of X. This implies $\exists \alpha_1, \cdots, \alpha_n$ such that $\{U_{\alpha_1}, \cdots, U_{\alpha_n}\} \cup \{A^c\}$ covers X. Thus $\{U_{\alpha_n}\}$ covers A. A is compact.

3. Bounded: Assume A is not bounded. Choose $x_1 \in A \implies \exists x_2 \in A, d(x_1, x_2) > 1$. Therefore, $\exists x_3 \in A$ such that $d(x_i, x_3) > 1, i = 1, 2$. Recursively, we define $\{x_n\}$ such that $d(x_n, x_m) > 1$, if $n \neq m$. Therefore, $\{x_n\}$ cannot have a convergent subsequence. Contradiction.

Closed: Assume A is not closed. This means $\exists \{x_n\} \subseteq A$ such that $x_n \to x_0$ but $x_0 \notin A$. $\implies \{x_n\}$ has no convergent subsequence in A. Contradiction.

□

Examples:

- $A \subseteq \mathbb{R}$, A is sequentially compact $\iff$ A is closed and bounded.
- $A \subseteq \mathbb{R}^n$, works too.
- $A \subseteq \mathbb{R}^n$, A compact $\iff$ A is closed and bounded.

Theorem. Heine-Borel Theorem: $A \subseteq \mathbb{R}^n$ is compact if and only if A is closed and bounded.

Notation:
A closed cell in $\mathbb{R}^n$ is a set $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$.

Proof. 1. A is closed and bounded. Assume A is not compact. Let $F_1 = A$, J_1 be a closed cell such that $A \subseteq J_1$. Bisect each of the intervals $[a_i, b_i]$ of J_1. This implies we obtain 2^n closed cells $\{J_{11}, J_{12}, \cdots, J_{12^n}\}$. Exists some open cover $\{U_\alpha\}_{\alpha\in I}$ such that it does not have a finite sub cover. One of the subcells, call it J_2, must be such that $F_2 = J_2 \cap A$ does not have a finite sub cover of $\{U_\alpha\}_\alpha$. Recursively, we construct a sequence of closed cells $\{J_n\}$ and closed sets $F_n = J_n \cap A$ such that

   (a) $J_{n+1} \subseteq J_n, \forall n \implies F_{n+1} \subseteq F_n, \forall n$.
   (b) Claim $(J_{n+1}) = \frac{1}{2}diam(J_n) \implies diam(F_{n+1}) \leq \frac{diam(F_n)}{2}$.
   (c) $F_n = J_n \bigcap A$ cannot be covered by finitely many U_α's.

2. By Cantor intersection theorem,

$$\bigcap_{n=1}^{\infty} F_n = \{x_0\} \implies x_0 \in A \implies \exists \alpha_0 | x_0 \in U_{\alpha_0} \implies \exists \epsilon > 0 | B(x_0, \epsilon) \subseteq U_{\alpha_0}$$

Pick n_0 such that $diam F_{n_0} < \epsilon$. Then $F_{n_0} \subseteq B(x_0, \epsilon) \subseteq U_{\alpha_0}$. $\{U_\alpha\}$ covers F_{n_0}. Contradiction.

□

Questions:
$A \subseteq X$ is compact $\iff$ A is closed and bounded?
No, X is infinite set, d is discrete metric space. X is bounded but not compact. But if it is compact, then it is also sequential compact.

Definition. X set, a collection $\{A_\alpha\}_{\alpha\in I}$, $A_\alpha \subseteq X, \forall\alpha$ has finite intersection.

Property: (FIP) if whenever $\{A_\alpha, \cdots, A_{\alpha_n}\}$ is any finite sub collection, we have

$$\bigcap_{i=1}^{n} A_{\alpha_i} \neq \emptyset$$

Theorem. (X, d) metric space, TFAE

1. X is compact
2. If $\{F_\alpha\}_{\alpha\in I}$ is a collection of closed sets of X with the FIP then $\bigcap_{\alpha\in I} F_\alpha \neq \emptyset$.

Corollary. (X, d) compact metric space, $\{F_n\}$ of non-empty, closed sets such that $F_{n+1} \subseteq F_n, \forall n \in \mathbb{N} \implies \bigcap_{n\in\mathbb{N}} F_n \neq \emptyset$.

Corollary. (X, d) compact metric space. Then X has BWP (X is sequentially compact).

Proof. Assume X is compact. Let S be an infinite set. Then exists a sequence $\{x_n\} \subseteq S$ consisting of distinct points. Let $F_n = \{x_n, x_{n+1}, \cdots\} \implies \{F_n\}$ has the FIP. Then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset \implies \exists x_0 \in \bigcap_{n=1}^{\infty} F_n$. For all $\epsilon > 0, B(x_0, \epsilon) \bigcap \{x_n, x_{n+1}, \cdots\} \neq \emptyset, \forall n \in \mathbb{N}$ This implies $B(x_0, \epsilon) \bigcap S \backslash \{x_0\} = \neq \emptyset \implies x_0 \in Lim(S)$. □

Theorem. $(X, d_x), (Y, d_y)$ metric space. Let $f : (X, d_x) \to (Y, d_y)$ contains. If (X, d_x) sequentially compact. this implies $f(X)$ is sequentially compact. Let $\{y_n\} \subseteq f(X), \implies \forall n, \exists x_n$ such that $y_n = f(x_n)$. This implies $\{x_n\} \subseteq X \implies \exists\{x_{n_k}\}$ such that $x_{n_k} \to x_0 \in X$. Hence $f(x_{n_k}) \to f(x_0) \in f(X)$.

Corollary. Extreme Value Theorem:

Let $f : (X, d_x) \implies \mathbb{R}$ be continuous. If (X, d_x) is sequentially compact, then there exists $c, d \in X$ such that $f(c) \leq f(x) \leq f(d), \forall x \in X$.

Definition. Let $\epsilon > 0$. A collection $\{x_\alpha\}_{\alpha\in I} \subseteq X$ is an ϵ-net for X if $X = \bigcup_{\alpha\in I} B(x_\alpha, \epsilon)$. We say that (X, d) is totally bounded if for each $\epsilon > 0$, X has a finite ϵ-net. Given $A \subseteq X$, A is totally bounded if it is totally bounded n the induced metric. $\forall\epsilon > 0, \exists\{x_1, \cdots, x_n\} \subseteq A$ such that $\bigcup_{i=1}^{\infty} B(x, \epsilon) \supseteq A$.

Proposition. If X is sequentially compact, then X is totally bounded. Suppose X is not totally bounded: Then $\exists\epsilon_0 > 0$, with no finite ϵ_0-net. Then $\exists$ sequence $\{x_n\} \subseteq X$ such that $x_i \notin B(x_j, \epsilon_0)$ if $i \neq j$. Then $\{x_n\}$ has no convergent subsequence. Contradiction.

Remarks:

1. $(\mathbb{N}, d)$ discrete metric $(\mathbb{N}, d)$ is bounded but it is not totally bounded. Then there does not exist finite 1/2-net.
2. If $A \subseteq (X, d)$ is totally bounded. Then so is $\bar{A}$. If $\{x_1, \cdots, x_n\}$ is an ϵ-net for A. Then $\{x_1, \cdots, x_n\}$ is an ϵ-net for $\bar{A}$.

Theorem. Lebesgue (X, d) compact metric space, $\{U_\alpha\}_{\alpha\in I}$ open cover of X. Then $\exists\epsilon > 0, \forall x \in X$ and $0 < \delta < \epsilon$. there exists $\alpha_0 \in I$ with $B(x, \delta) \subseteq U_{\alpha_0}\}$.

Proof. If $X = U_\alpha$ for some α, then any $\epsilon > 0$ would work. Assume $X \neq U_\alpha, \forall\alpha$. For each $x \in X$, let $\phi(x) = sup\{r \in \mathbb{R} | B(x, r) \subseteq U_{\alpha_0}$, for some $\alpha_0 \in I\}$. Then $\phi(x) = 0$. Also, $\phi(x) < \infty$: if $\phi(x) = \infty, \exists\{r_n\} \subseteq \mathbb{R}, \{\alpha_n\} \subseteq I | B(x_1, r_n) \subseteq U_{\alpha_n}, r_n \to \infty\}$. But X sequentially compact. This implies X is bounded and $\exists M > 0, B(x, M) = X$. Pick $r_n > M \implies B(x, r_n) = X \subseteq U_{\alpha_n}$ but $X \neq U_{\alpha_n}$. Contradiction.

If ϕ is continuous: if $x, y \in X, \phi(x) \leq \phi(y) + d(x, y)$:

case 1 $\exists\alpha_0$ and $r > 0$ such that $B(x, r) \subseteq U_{\alpha_0}$ and $y \in B(x, r)$. $B(y, r - d(x, y)) \subseteq U_{\alpha_0} \implies \phi(y) \geq r - d(x, y) \implies \phi(x) \leq d(x, y) + \phi(y)$.

case 2 $\forall r$ and α such that $B(x,r) \subset U_\alpha, y \notin B(x,r)$. $r \le d(x,y), \phi(x) \le d(x,y)$ and $\phi(x) \le d(x,y) + \phi(y)$ and $|\phi(x) - \phi(y)| \le d(x,y) \implies \phi$ is continuous. Therefore, by extreme value theorem, $\epsilon > 0$, such that $\phi(x) \ge \epsilon, \forall x \in X$.

□

Theorem. Borel-Lebesgue (X,d) metric space, TFAE

1. X is compact
2. X has the BWP
3. X is sequentially compact.

Proof. 3 to 1: Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover for X. This implies $\{U_\alpha\}$ has a Lebesgue number $\epsilon > 0$. Since X is totally bounded, there exists finite subset $\{x_1, x_2, \cdots, x_n\} \subseteq X$ such that $\bigcup_{i=1}^n B(x_i, \delta) = X$ where $0 < \delta < \epsilon$. But for each $i = 1, 2, \cdots, n$, we can find $\alpha_i \in I$ such that $B(x_i, \delta) \subseteq U_{\alpha_i}$ This implies $\{U_{\alpha_i}\}_{i=1,\cdots,n}$ is a finite sub cover. This implies X is compact. □

Theorem. Heine Borel for metric space: (X,d) metric space TFAE

1. X is compact
2. X is complete and totally bounded.

Proof. 2 to 1 (X is sequentially compact). Let $\{x_n\}$ be a sequence in X. Since X is totally bounded, $\exists y_1, \cdots, y_n \in X$ such that $\bigcup_{i=1}^n B(y_1, 1) = X$. Then there exists y_i such that $B(y_1, 1) = S_1$ contains infinitely many terms of $\{x_n\}$. Since X is totally bounded, $\exists y_1^2, \cdots, y_{n_2}^2$ such that $\bigcup_{i=1}^n B(y_1^2, \frac{1}{2}) = X$ Therefore $\exists y_i^2 | B(y_i^2, 1/2) = S_2$ contains infinitely many terms of $\{x_n\}$ in S_1. Then, we construct sequence of open balls $\{S_k = B(y^k, 1/k)\}$ and each S_{k+1} contains infinitely many terms of $\{x_n\}$ also in $S_1 \bigcap \cdots \bigcap S_k$. In particular, we can choose $n_1 < n_2 < \cdots$ such that $x_{n_k} \in S_1 \bigcap \cdots \bigcap S_k$. But $diam(S_k) \to 0$, this implies $\{x_{n+k}\}$ is cauchy and X is complete. thus $\{x_{n_k}\}$ is convergent.

□

2.14 Compactness and Continuity

Theorem. Let $f : (X, d_x) \to (Y, d_y)$ be continuous. If (X, d_x) is compact. $f(x)$ is compact.

Corollary. Extreme Value Theorem: Let $f : (X, d_x) \to \mathbb{R}$ be continuous. If (X, d_x) is compact. There exists $c, d \in X$ such that $f(x) \le f(x) \le f(d), \forall x \in X$.

Theorem. Sequential characterization of uniform continuity: suppose $f : (X, d_x) \to (Y, d_y)$ function TFAE

1. f is uniformly continuous on X
2. If $\{x_n\}, \{z_n\}$ in X with $\lim_n d(x_n, z_n) = 0 \implies \lim_n d_Y(f(x_n), f(x_n)) = 0$.

Theorem. $f : (X, d_X) \to (Y, d_y)$ continuous if (X, d_x) is compact. This implies $f(x)$ is uniformly continuous. Suppose $f(x)$ is not uniformly continuous. This implies $\exists \epsilon_0 > 0$ and $\{x_n\}, \{y_n\} \subseteq X$ such that $\lim_n d(x_n, z_n) = 0$ but $d_Y(f(x_n), f(z_n)) \ge \epsilon_0, \forall n \ge n_0$. X compact $\implies \exists \{x_{n_k}\}$ subsequence of $\{x_n\}$ such that it converges to x_0. $\exists \{z_{n_k}\}$ subsequence of $\{z_n\}$ such that it converges to x_0.

f is continuous, then $f(x_{n_k}) \to f(x_0)$ and $f(z_{n_k}) \to f(x_0)$. contradiction

Theorem. $(X, d_x), (Y, d_y)$ metric space, X is compact. Then let $\Phi : X \to Y$ be one-to-one, onto and continuous. then Φ^{-1} is also continuous.

If Φ is continuous $\iff$ ($U \subseteq X$ open $\implies \Phi(U) \subseteq Y$ is open). $U \subseteq X$ is open, then $U^c = F \subseteq X$ closed and X is compact. Then F is compact. Therefore, $\Phi(F) \subseteq Y$ compact $\implies \Phi(F) \subseteq Y$ is closed there fore $\Phi(U^C) = (\Phi(U))^C$

3 The Space $(C(X), \|\cdot\|_\infty)$

We assume (X, d) is a compact metric space. Then every continuous function is bounded $(C(X), \|\cdot\|_\infty) = (C_b(X), \|\cdot\|_\infty)$. In $C(X)$, unless otherwise stated, the norm is $\|\cdot\|_\infty$

3.1 Weierstrass Approximation Theorem

Problem: Given $h \in C([a,b])$ and $\epsilon > 0$. Exists $p(x)$ polynomial on $[a,b]$ such that $\|h - p\|_\infty < \epsilon$?
Remarks

1. We can assume that $[a,b] = [0,1]$. Assume $f, g \in C([0,1])$ and $\|f - g\|_\infty < \epsilon$.

   Define $\Phi : [a,b] \to [0,1]$ and $\Phi(x) = \frac{x-a}{b-a}$, Φ is one-to-one, onto. Then $\Phi'[0,1] \to [a,b]$ then $\Phi^{-1}(x) = (b-a)x + a$. Then $f \circ \Phi, g \circ \Phi \in C([a,b])$. In fact, $\|f \circ \Phi - g \circ \Phi\|_\infty = \|f - g\|_\infty$. Then the map $\Gamma(C[0,1], \|\|_\infty) \implies (C[a,b], \|\|_\infty)$. Then $\Gamma(f) = f \circ \Phi$ is an isometric isomorphism with inverse $\Gamma^{-1}(h) = h \circ \Phi^{-1}, \forall h \in C[a,b]$. Also, $\Gamma(p(x))$ is a polynomial if and only if $p(x)$ is a polynomial.

2. We can assume $f(0) = 0, f(1) = 0$. If $f \in C[0,1]$, let $g(x) = f(x) - [(f(1) - f(0))x + f(0)]$. Then $g(x) \in C[0,1], g(0) = 0 = g(1)$. if we approximate $g(x)$ uniformly with error at most ϵ by a polynomial, the n we can do so for $f(x)$. $\epsilon > |g(x) - p(x)| = |f(x) - \{[(f(1) - f(0))x - f(0)] + p(x)\}| = |f(x) - p_1(x)|$

Lemma. If $n \in \mathbb{N}, (1-x^2)^n \geq 1 - nx^2, \forall x \in [0,1]$. Let $f(x) = (1-x^2)^n - (1 - nx^2)$. $f(0) = 0$, $f'(x) = \cdots > 0$ on $(0,1)$. Then the inequality follows.

Theorem. Weierstrass Approximation Theorem: let $f \in C[a,b]$. Then there exists a sequence $\{p_n(x)\}$ of polynomials such that

$$p_n(x) \to f(x) \text{ uniformly on } [a,b]$$

Proof. Assume that $[a,b] = [0,1]$ and $f(0) = 0 = f(1)$. We can extend $f(x)$ to a uniformly continuous function on $\mathbb{R}$ by setting $f(x) = 0$ if x in $(-\infty, 0) \cup (1, \infty)$. Note that $\int_{-1}^1 (1-x^2)^n dx \neq 0, \forall n$. Pick c_n such that $\int_{-1}^1 c_n(1-x^2)^n dx = 1$. Let $Q_n(x) = c_n(1-x^2)^n$. Since $(1-x^2)^n \geq 1 - nx^2, \forall x \in [0,1]$.

$$\int_{-1}^1 (1-x^2)^n dx = 2\int_0^1 (1-x^2)^n dx \geq 2\int_0^{1/\sqrt{n}} 1 - nx^2 dx = \frac{4}{3\sqrt{n}} \geq 1/\sqrt{n}$$

Then $c_n > \sqrt{n}$. If $0 < \delta < 1 \implies \forall x \in [-1, \delta] \cup [\delta, 1]$,

$$c_n(1-x^2)^n \geq \sqrt{n}(1-\delta^2)^n$$

Let $p_n(x) = \int_{-1}^1 f(x+t)Q_n(t)dt = \int_{-x}^{1-x} f(x+t)Q_n(t)dt \begin{cases} t < -x \\ t + x < 0 \\ f(t+x) = 0 \end{cases} = \int_0^1 f(u)Q_n(u-x)du \begin{cases} u = x + t \\ du = dt \end{cases}$

$$p_n(x) = \int_0^1 f(u)Q_n(u-x)du$$

$$\frac{d^{2n+1}p(x)}{dx^{2n+1}} =_{\text{leibnizs rule}} \int_0^1 f(u)\frac{d^{2n+1}Q_n(u-x)}{dx^{2n+1}} = 0$$

$p_n(x)$ is a polynomial of degree $2n + 14$ or less. Let $M = \|f\|_\infty \neq 0$. Let $\epsilon > 0$, choose $0 < \delta < 1$ so that if $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{2}$. Since $\int_{-1}^1 Q_n(t)dt = 1$, this implies $f(x) = \int_{-1}^1 f(x)Q_n(t)dt$. If

$x \in [0,1]$,

$$|p_n(x) - f(x)| = |\int_{-1}^{1} f(x+t)Q_n(t)dt - \int_{-1}^{1} f(x)Q_n(t)dt|$$
$$= |\int_{-1}^{1} (f(x+t) - f(x))Q_n(t)dt| \leq \int_{-1}^{1} |f(x+t) - f(x))|Q_n(t)dt$$
$$= \int_{-1}^{-\delta} |f(x+t) - f(x)|Q_n(t)dt + \int_{-\delta}^{\delta} |f(x+t) - f(x)|Q_n(t)dt + \int_{\delta}^{1} |f(x+t) - f(x)|Q_n(t)dt$$
$$\leq 2\sqrt{n}(1-\delta^2)^{n+1}\|f\|_\infty + \frac{\epsilon}{2} + 2\sqrt{n}(1-\delta^2)^{n+1}\|f\|_\infty$$

$$|P_n(x) - f(x)| \leq 4M\sqrt{n}(1-\delta^2)^{n+1} + \frac{\epsilon}{2}$$

Choose n large enough so that

$$4M\sqrt{n}(1-\delta 62)^{n+1} < \frac{\epsilon}{2} \implies \|p_n - f\|_\infty < \epsilon$$

□

Corollary. Let $f(x) \in C[0,1]$ such that $\int_0^1 f(t)dt = 0, \int_0^1 f(t)t^n dt = 0, \forall n$. This implies $f(x) = 0, \forall x \in [0,1]$.

Corollary. $(C[a,b], \|\cdot\|_\infty)$ is separable. $\forall n \in \mathbb{N}$,

$$P_n = \{a_0 + a_1 x + \cdots + a_n x^n | a_i \in \mathbb{R}\}$$

$$Q_n = \{r_0 + r_1 x + \cdots + r_n x^n | r_1 \in \mathbb{Q}\} \implies \bar{Q}_n = P_n$$

but also

$$\overline{\bigcup_{n=1}^{\infty} P_n} = C[a,b] \implies \overline{\bigcup Q_n} = C[a,b]$$

. Q_n is countable.

3.2 Stone-Weierstrass Theorem

(X,d) compact metric space:

Definition. (X,d) compact metric space, $\Phi \subseteq C(X)$ and Φ is a point separating if whenever $x, y \in X$ and $x \neq y$, there exists $f \in \Phi$ such that $f(x) \neq f(y)$.

Remarks

1. $a, b \in X, a \neq b$. $f(x) = d(x,a) \implies f(x) \in C(X)$ and $f(a) \neq f(b)$ Then $C(X)$ is point separating.
2. Suppose X has at least 2 points and $\Phi \subseteq C(X)$. Suppose $f(x) = f(y), \forall f \in \Phi, \forall x, y \in X \implies g(x) = g(y), \forall g \in \bar{\Phi}, \forall x, y \in X$. Then if Φ is dense in $C(X)$; Φ must be point separating.

Definition. A linear subspace $\Phi \subseteq C(X)$ is a lattice if $\forall f, g \in \Phi$ then $(f \vee g)(x) = \max\{f(x), g(x)\} \in \Phi$ and $(f \wedge g)(x) = \min\{f(x), g(x)\} \in \Phi$.

Remarks

Let $f, g \in C(X)$, $(f \vee g)(x) = \frac{(f(x)+g(x))+|f(x)-g(x)|}{2}$ and $(f \wedge g)(x) = -(f \vee g)(x) \implies f \vee g, f \wedge g \in C(X)$ Then $C(X)$ is a lattice.

If $\Phi \subseteq C(X)$, Φ is a linear subspace. Then Φ is a lattice if $f \vee g \in \Phi, \forall f, g \in \Phi$.

Examples

$f : [a,b] \to \mathbb{R}$ is piecewise linear if there exists a partition $\mathcal{P} = \{a = t_0 < \cdots < t_n = b\}$ such that $f_{[t_{i-1},t_i]} = m_i + d_i, \forall i = 1, \cdots, n$.

$f : [a,b] \to \mathbb{R}$ is piecewise polynomial if $\exists \mathcal{P} = \{a = t_0 < \cdots < t_n = b\}$ such that $f_{[t_{i-1},t_i]} = c_{0,i} + c_{1,i}x + \cdots + c_{n,i}x^n$

Theorem. Stone-Weierstrass Theorem (Lattice version): (X,d) is compact metric space, $\Phi \subseteq (C(X), \|\cdots\|_\infty)$ linear subspace such that

1. the constant function $1 \in \Phi$
2. Φ separates points.
3. If $f, g \in \Phi \implies (f \vee g) \in \Phi$

Hence, Φ is dense in $C(X)$.

Note that if $\alpha, \beta \in \mathbb{R}$, and $x \neq y \in X$, then there exists $g \in \Phi$ such that $g(x) = \alpha$ and $g(y) = \beta$. Let $h \in \Phi$ such that $h(x) \neq h(y)$. Let $g(t) = \alpha + (\beta - \alpha)\frac{h(t)-h(x)}{h(y)-h(x)} \implies g \in \Phi$. Let $f \in C(X)$ and $\epsilon > 0$.

Step 1 Fix $x \in X$. For each $y \in X$, $\exists h_{x,y}(t) \in \Phi$ and $h_{x,y}(x) = f(x), h_{x,y}(y) = f(y)$. Since $h_{x,y}(y) - f(y) = 0, \forall y$, we can find $\delta_y > 0$ such that $t \in B(y, \delta_y)$ and $-\epsilon < h_{x,y}(t) - f(t) < \epsilon$. $\{B(y,\delta_y)\}$ open cover of X $\implies \exists$ points $y_1, y_2, \cdots, y_n$ such that $\{B(y_i, \delta_{y_i})\}$ cover X.

$$h_x(t) = h_{x,y_1} \vee \cdots \vee h_{x,y_n}$$

Now if $z \in X$, $\exists i$ such that $z \in B(y_i, \delta_{y_i})$. $f(z) - \epsilon < h_{x,y_i}(z) \leq h_x(t)$.

Step 2 For each $x \in X$, $h_x(x) - f(x) = 0$. For each $x \in X$, $\exists \delta_x > 0$ such that $t \in B(x, \delta_x)$, then $-\epsilon < h_x(t) - f(t) < \epsilon$. As we did before, we can find $\{x_1, x_2, \cdots, x_k\}$ such that $\{B(x_j, \delta_{x_j}\}$ is a cover for X. Let $h(t) = h_{x_1} \wedge \cdots \wedge h_{x_k} \in \Phi$. Then if $z \in X$, then $f(z) - \epsilon < h(z) < f(z) + \epsilon$.

Corollary. Let $\Phi_1 = \{f \in C[a,b] | f \text{ is piecewise linear}\}$ and $\Phi_2 = \{f \in C[a,b] | f \text{ is piecewise polynomial}\}$. Then Φ_i is dense in $C(X)$, $i = 1, 2, \cdots$.

Definition. A subspace $\Phi \subseteq C(X)$ is said to be a sub algebra if $f \cdot g \in \Phi$, for every $f, g \in \Phi$.

Example: If P is the collection of all polynomials on $[a,b]$, P is a sub algebra of $C([a,b])$.
Remark:
If $\Phi \subseteq C(X)$ is a sub algebra, then so is Φ. Let $\{f_n\}, \{g_n\} \subseteq \Phi | f_n \to f, g_n \to g$. Note that $fg \in C(X)$ Note also $\{g_n\}$ is bounded.

$$\|f_n g_n - fg\|_\infty = \|(f_n g_n - f g_n) + (f g_n - fg)\|_\infty \leq \|g_n\|_\infty \|f_n - f\|_\infty + \|f\|_\infty \|g_n - g\|_\infty \to 0$$

Theorem. Subalgebra version) Stone-Weierstrass: (X,d) compact metric space. Let Φ be a linear subspace of $(C(X), \|\|_\infty)$ such that

1. $1 \in \Phi$.
2. Φ separates points
3. Φ is a subalgebra

Then Φ is dense in $C(X)$.

Proof. **Step 1** If $f \in \Phi$, then $|f| \in \bar{\Phi}$. Fix $\epsilon > 0$, since X is compact, $\exists M > 0$ such that $|f(x)| < M, \forall x \in X$. We consider the function $g(t) = |t|$ on $[-M, M]$. By W.A Theorem, $\exists p(t) = c_0 + c_1 t + \cdots + c_n t^n$ such that

$$|g(t) - p(t)| = ||t| - p(t)| < \epsilon, \forall t \in [-M, M]$$

but $pf = c_0 1 + c_1 f + c_2 f^2 + \cdots + c_n f^n \in \Phi$. If $x \in X, f(x) \in [-M, M]$ and then $||f(x)| - p(f(x))| < \epsilon, \forall x \in X$. This implies $|f| \in \bar{\Phi}$.

Step 2 $hg \in \Phi \implies h \vee g \in \bar{\Phi}$. Then $g \vee h(x) = \frac{(g(x)+h(x))-|g(x)-h(x)|}{2} \in \bar{\Phi}$. Then

1. $1 \in \bar{\Phi}$
2. $\bar{\Phi}$ separates points
3. $\bar{\Phi}$ is a lattice.

Therefore, $\bar{\bar{\Phi}} = C(X) = \bar{\Phi}$.

□

3.3 Complex Version

$C(X,\mathbb{C}) = \{f : X \to \mathbb{C} | f(x)$ is continuous on X$\}$. $\|f\|_\infty = sup\{|f(x)| x \in X\}$ A subspace $\Phi \subseteq C(X,\mathbb{C})$ is self-adjoint if $f \in \Phi$ implies that $\bar{f} \in \Phi$.

Theorem. Stone-Weirstrass $\mathbb{C}$-version (X,d) compact metric space. If $\Phi \subset C(X,\mathbb{C}$ is a self-adjoint linear subspace such that

1. $1 \in \Phi$
2. Φ separates points
3. Φ is a subalgebra

This implies $\bar{\Phi} = C(X,\mathbb{C})$.

Example

Let $\pi = \{\lambda \in \mathbb{C} ||\lambda| = 1\}$. Let $\phi : \pi \to [0, 2\pi), e^{i\Theta} \to \Theta$. On $[0, 2\pi)$ we consider the metric $d_*(\Theta_1, \Theta_2) =$ the shortest at-length between $e^{i\Theta_1}$ and $e^{i\Theta_2}$. Thus ϕ is a homeomorphism. This implies $([0, 2\pi), d_*)$ is compact. $C(\pi) \approx \{f \in C([0, 2\pi)) | f(0) = f(2\pi)\}$. A trigonometric polynomial is an element of

$$Trig_{\mathbb{C}}([0, 2\pi)) = span\{f(\theta) = e^{in\theta} | n \in \mathbb{Z}\}$$

. This implies $\overline{Trig_{\mathbb{C}}([0, 2\pi))} = C([0, 2\pi))$.

Example:

$\Psi = \{F(x,y) \in C([0,1] \times [0,1]) | F(x,y) = \sum_{i=1}^{k} f_1(x) g_i(y)$ for $f_i, g_i \in C[0,1]\}$. Then to prove that $\bar{\Psi} = C([0,1] \times [0,1])$

3.4 Compactness in $(C(X), \|\cdot\|_\infty)$ and the Ascoli-Arzela Theorem

Definition. (X,d) metric space. $A \subseteq X$ is relatively compact if $\bar{A}$ is compact. Remark: Assume (X,d) is complete. Recall; if A is totally bounded, then $\bar{A}$ is totally bounded. Then $A \subset X$ is relatively compact $\iff$ A is totally bounded.

Theorem. Arzela-Ascoli Theorem: Let (X,d) be a compact metric space. Let $\mathcal{F} \subseteq (C(X), \|\cdot\|_\infty)$. Then, TFAE:

1. $\mathcal{F}$ is relative compact
2. $\mathcal{F}$ is equicontinuous and pointwise bounded.

Proof. 1 to 2: $\mathcal{F}$ is relative compact. This implies that $\mathcal{F}$ is bounded. This implies $\mathcal{F}$ is point wise bounded. Fix $\epsilon > 0$. $\mathcal{F}$ is relative compact. This implies $\mathcal{F}$ is totally bounded. This implies there exists an $\frac{\epsilon}{3}$-net $\{f_1, \cdots, f_n\} \subseteq \mathcal{F}$. Since $\{f_1, \cdots, f_n\}$ is finite, it's equicontinuous. Given $\frac{\epsilon}{3}$, there exists $\delta > 0$ such that $d(x,y) < \delta$. This implies $|f_i(x) - f_i(y)| < \frac{\epsilon}{3}, \forall i = 1, 2, \cdots, n$. Let $f \in \mathcal{F}$ and $x, y \in X$ such that $d(x,y) < \delta$. This implies $\exists i_0 \in \{1, \cdots, n\}$ such that $\|f_{i_0} - f\| < \frac{\epsilon}{3}$. Then $|f(x) - f(y)| \leq |f(x) - f_{i_0}(x0| + |f_{i_0}(x) - f_{i_0}(y)| + |f_{i_0}(y) - f(y)| < \frac{\epsilon}{3} \times 3 = \epsilon$. □

Definition. Compact operators $\Gamma : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ linear map is compact if $\Gamma(\{x \in X | \|x\|_X \leq 1\})$ is relatively compact.

Remark:

$\Gamma$is compact $\implies$ Γ is continuous.

Example: $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y) = (C([a,b], \|\cdot\|_\infty)$. Let $K : [a,b] \times [a,b] \implies [a,b]$ continuous. If $f \in C([a,b])$. $\Gamma(f)(x) = \int_a^b k(x,y)f(y)dy$. Clearly, Γ is linear.

Claim: $\Gamma(f) \in C([a,b])$. If $f = \theta$, $\Gamma(f) \in C[a,b]$. If $f \neq \theta$, since K is uniformly continuous given $\epsilon > 0, \exists \delta > 0$ such that $\|(x_1, y_1) - (x_2, y_2)\|_2 < \delta \implies |K(x_1, y_1) - K(x_2, y_2)| < \frac{\epsilon}{(b-a)\|f\|_\infty}$. Now if $|x - z| < \delta$, then $|\Gamma(f)(x) - \Gamma(f)(z)| = |\int_a^b (K(x,y) - K(z,y))f(y)dy| \leq \int_a^b |K(x,y) - K(z,y)||f(y)|dy < \frac{\epsilon}{(b-a)\|f\|_\infty}\|f\|_\infty(b-a) = \epsilon$

Claim: $\Gamma(B_x[0,1])$ is uniformly equicontinuous. Fix $\epsilon > 0$. there exists $\delta_1 > 0$ such that $|x - z| < \delta_1 \implies |K(x,y) - K(z,y)| < \frac{\epsilon}{b-a}, \forall y \in [a,b]$.

let $|x - z| < \delta_1$ and $f \in C([a,b])$ such that $\|f\|_\infty \leq 1$. $|\Gamma(f)(x) - \Gamma(f)(z)| \leq \int_a^b |K(x,y) - K(y,z)||f(y)|dy < \epsilon$

Claim: $\Gamma(B_x[\theta,1])$ is uniformly bounded. Let $M > 0$ such that $|K(x,y)| \leq M, \forall (x,y) \in [a,b] \times [a,b]$. Let $f \in C[a,b]$ such that $\|f\|_\infty \leq 1$. $|\Gamma(f)(x)| \leq \int_a^b |K(x,y)||f(y)|dy \leq M\int_a^b dy = M(b-a), \forall x \in [a,b]$. Therefore, for all $f \in [a,b]$ such that $\|f\|_\infty < 1$. This implies $\Gamma(B_x[\theta,1])$ is relatively compact by Arzela Ascoli Theorem,. Therefore, Γ is compact.

Theorem. Peano's Theorem: Let f be continuous on an open subset D of $\mathbb{R}^2$. Let $(x_0 y_0) \in D$. Then the differential equation $y' = f(x,y)$ has a local solution through the point (x_0, y_0). Let R be a closed rectangle, $R \subseteq D$, with $(x_0, y_0) \in int(R)$. f os continuous on R, R compact; then there exists $M \geq 1$ such that $|f(x,y)| \leq M, \forall (x,y) \in R$. Let $W = \{(x,y) \in R | |y - y_0| \leq M|x - x_0|\}$ and $I = [a,b] = \{x | (x,y) \in W$ for some $y\}$. By uniform continuity, given $\epsilon > 0, \exists 0 < \delta < 1$, such that if $(x_1, y_1), (x_2, y_2) \in W, |x_1 - x_2| < \delta$ and $|y_1 - y_2| < \delta \implies |f(x_1, y_1) - f(x_2, y_2)| < \epsilon$. Choose $a = x_0 < x_1 < \cdots < x_n = b$, with $|x_j - x_{j-1}| < \frac{\delta}{M}, \forall j$. On $[x_0, b]$, we define a function $k_\epsilon(x)$: $k_\epsilon(x_0) = y_0$, and on $[x_0, x_1]$, $k_\epsilon(x)$ is linear and has slope $f(x_0, y_0)$. On $[x_1, x_2]$, $k_\epsilon(x)$ is linear and has slope $f(x, k_\epsilon(x_1))$ and proceed like this to define a piecewise linear function $k_\epsilon(x)$ on $[x_0, b]$.

Note: the graph of $k_\epsilon(x)$ is contained in W and $|k_\epsilon(x) - k_\epsilon(\bar{x})| \leq M|x - \bar{x}|, \forall x, \bar{x} \in [x_0, b]$. Let $x \in [x_0, b]$, $x \neq x_j, j = 0, 1, \cdots, n$. This implies there exists j such that $x_{j-1} < x < x_j$.

$$|k_\epsilon(x) - k_\epsilon(x_{j-1})| \leq M|x - x_{j-1}| < M\frac{\delta}{M} = \delta$$

This implies by uniform continuity of f,

$$|f(x_{j-1}, k_\epsilon(x_{j-1}) - f(x, k_\epsilon(x))| < \epsilon$$

but $k_\epsilon'^{+}(x_{j-1}) = f(x_{j-1}, k_\epsilon(x_{j-1}))$ (slope approaching by the right). This implies $|k_\epsilon^{+'}(x_{j-1}) - f(x, k_\epsilon(x)| < \epsilon, \forall x \in [x_0, b]$ such that $x \neq x_1, i = 0, 1, \cdots, n$. Let $K = \{k_\epsilon | \epsilon > 0\}$. K is pointwise bounded: $(k_\epsilon(x) \in W \subseteq R$ compact) K is equicontinuous. (*) By Arzela-Asidli, K is compact. Let $x \in [x_0, b]$, $k_\epsilon(x) = y_0 + \int_{x_0}^x k'\epsilon(t)dt = y_0 + \int_{x_0}^x f(t, k_\epsilon(t)) + [(k_\epsilon'(t) - f(t, k_\epsilon(t))]dt$. Consider the sequence $\{k_{\frac{1}{n}}(x)\}_n \subseteq \bar{K}$. This implies $\exists$ subsequence $\{k_{\frac{1}{n_k}}(x)\}_k$ converging uniformly on $[x_0, b]$ to some $k(x)$. f uniformly continuous on W. This implies $\{f(t, k_{\frac{1}{n_k}}(t)\}$ converges uniformly to $f(t, k(t))$ on $[x_0, b]$. $k_\epsilon(t) = y_0 + \int_{x_0}^x f(t, k(t))dt$. This implies $k(x)$ is a solution to the DE on $[x_0, b]$. Similarly we can find a solution $k^*(x)$ on $[a, x_0]$