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PMATH 450
Lebesgue Integration and Fourier Analysis

Instructor: Nicolaas Spronk
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Contents

1 Introduction

In this section we will review basic integration theory, and discuss some aspects used later in the course.

1.1 Vector-valued Riemann integration

1.1 Definition. A **Banach space** is a real (or complex) vector space $\mathcal{X}$, equipped with a norm $\|\cdot\|$, i.e.

1. $\|x\| \geq 0$ for $x \in \mathcal{X}$ (non-negative).
2. $\|x\| = 0$ if and only if $x = 0$ (non-degenerate).
3. $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in \mathcal{X}$ (sub-additivity).
4. $\|\lambda x\| = |\lambda|\|x\|$ for $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), $x \in \mathcal{X}$ ($|\cdot|$-homogeneity).

such that $(\mathcal{X}, \|\cdot\|)$ is *complete*, that is, any Cauchy[1] sequence $(x_n)_{n=1}^{\infty} \subseteq \mathcal{X}$ admits a limit[2],

$$x = \lim_{n\to\infty} x_n.$$

1.2 Definition. Let $\mathcal{X}$ be a Banach space, $a < b$ in $\mathbb{R}$, and $f : [a, b] \to \mathcal{X}$ be a function. Then a **partition** of $[a, b]$ is a collection of points

$$\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_n = b\}.$$

A **Riemann sum** for f, over $\mathcal{P}$, is any sum of the form

$$S(f, \mathcal{P}) = \sum_{i=1}^{n} \underbrace{f(t_i^*)}_{\text{vector}} \underbrace{(t_i - t_{i-1})}_{\text{scalar}}, \qquad t_i^* \in [t_{i-1}, t_i], i = 1, \ldots, n.$$

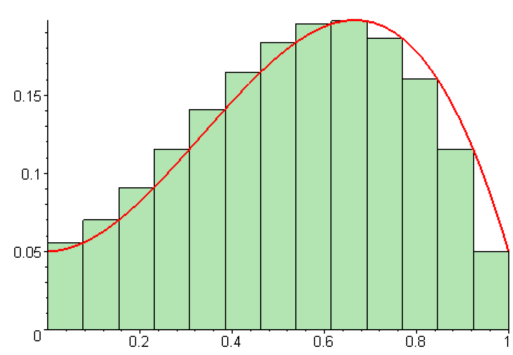
Note that:

1. Riemann sums are not unique but depend on the choice of "tags" t_i^*. However, we will notationally omit the dependence of the sum on the tags.
2. Each Riemann sum is an "average value"

$$\frac{1}{b-a} S(f, \mathcal{P}) = \sum_{i=1}^{n} f(t_i^*) \underbrace{\frac{t_i - t_{i-1}}{b-a}}_{:=\lambda_i}$$

   where $\sum_i \lambda_i = 1$, $\lambda_i \geq 0$ (a convex combination).
3. In the case $\mathcal{X} = \mathbb{R}$ (norm given by the absolute value), and $f \geq 0$ then a Riemann sum

$$S(f, \mathcal{P}) \approx \{\text{area under curve } y = f(x) : a \leq x \leq b\}$$



[1]This means that for every $\epsilon > 0$ there is an N such that $n, m \geq N$ implies $\|x_n - x_m\| < \epsilon$.
[2]This means that for every $\epsilon > 0$, there is N so for $n \geq N$ we have $\|x_n - x\| < \epsilon$.

1.3 Definition. We say $f : [a, b] \to \mathcal{X}$ is **Riemann integrable** if there is $x \in \mathcal{X}$ such that for every $\epsilon > 0$, there is a partition $\mathcal{P}_\epsilon$ of $[a, b]$ such that for every refinement, $\mathcal{P} \supseteq \mathcal{P}_\epsilon$ and every Riemann sum $S(f, \mathcal{P})$ with respect to $\mathcal{P}$ we have that

$$\|x - S(f, \mathcal{P})\| < \epsilon.$$

1.4 Remark. Suppose both $x, y \in \mathcal{X}$ satisfy the definition of Riemann integrability, above. Then $x = y$, for otherwise the definition will never be satisfiable with $\epsilon = \frac{\|x-y\|}{2}$. Hence, if it exists, the point x is unique. We will call this the **Riemann integral** of f over the interval $[a, b]$, and denote it by

$$\int_a^b f = \int_a^b f(t)\, dt.$$

Note that this is a vector quantity (it lies in $\mathcal{X}$).

1.5 Theorem (Cauchy Criterion for Riemann integrability). Let $a < b$ in $\mathbb{R}$, and $\mathcal{X}$ be a Banach space, and $f : [a, b] \to \mathcal{X}$. Then f is Riemann integrable on $[a, b]$ if and only if for every $\epsilon > 0$ there is a partition $\mathcal{Q}_\epsilon$ such that for any pair of refinements $\mathcal{P}, \mathcal{Q} \supseteq \mathcal{Q}_\epsilon$ and any associated Riemann sums,

$$\|S(f, \mathcal{P}) - S(f, \mathcal{Q})\| < \epsilon.$$

Proof. The forward direction is an easy exercise (use $\epsilon/2$). For the reverse direction, proceed as follows. For each n, let $\mathcal{Q}_n$ be a partition of $[a, b]$ such that for refinements $\mathcal{P}, \mathcal{Q} \supseteq \mathcal{Q}_n$ and any associated Riemann sums we have

$$\|S(f, \mathcal{Q}) - S(f, \mathcal{P})\| < \frac{1}{2^n}.$$

We let $\mathcal{P}_1 = \mathcal{Q}_1$, $\mathcal{P}_2 = \mathcal{Q}_1 \cup \mathcal{Q}_2$, ..., $\mathcal{P}_n = \bigcup_{j=1}^n \mathcal{Q}_j$ and we let $x_n = S(f, \mathcal{P}_n)$ be a fixed Riemann sum with respect to $\mathcal{P}_n$. Notice that $\mathcal{P}_n \supseteq \mathcal{Q}_n$, and $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \ldots$. Now if $n > m$ we have

$$\begin{aligned}
\|x_n - x_m\| &= \|x_n - x_{n-1} + x_{n-1} - \ldots - x_{m+1} + x_{m+1} - x_m\| \\
&\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \ldots + \|x_{m+1} - x_m\| \\
&= \|S(f, \mathcal{P}_n) - S(f, \mathcal{P}_{n-1})\| + \ldots + \|S(f, \mathcal{P}_{m+1}) - S(f, \mathcal{P}_m)\| \\
&< \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \ldots + \frac{1}{2^m} = \frac{1}{2^{m-1}} \left(\frac{1}{2^{n-m}} + \ldots + \frac{1}{2} \right) < \frac{1}{2^{m-1}}
\end{aligned}$$

since $\mathcal{P}_n \supseteq \mathcal{P}_{n-1} \supseteq \mathcal{Q}_{n-1}$. If $\epsilon > 0$ is given, choose m so that $\frac{1}{2^{m-1}} < \epsilon$, and we see that $(x_n)_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is a Banach space, we have a limit point

$$x = \lim_{n \to \infty} x_n.$$

It remains to show that $x = \int_a^b f$, i.e. it satisfies the definition of Riemann integrability. Let $\epsilon > 0$, and n be such that

$$\frac{1}{2^{n-1}} < \frac{\epsilon}{2}.$$

If $\mathcal{P}_n$ is as above, and $\mathcal{P} \supseteq \mathcal{P}_n$ then for any Riemann sum $S(f, \mathcal{P})$ we have

$$\begin{aligned}
\|S(f, \mathcal{P}) - x\| &\leq \|S(f, \mathcal{P}) - x_{n+1}\| + \|x_{n+1} - x\| \\
&= \|S(f, \mathcal{P}) - S(f, \mathcal{P}_{n+1})\| + \lim_{m \to \infty} \underbrace{\|x_{n+1} - x_m\|}_{\frac{1}{2^n}, m > n}
\end{aligned}$$

Now $\mathcal{P} \supseteq \mathcal{P}_n \supseteq \mathcal{Q}_n$, so this is strictly less than

$$\frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}} < \epsilon. \qquad \square$$

1.2 Shortcomings

Having discussed the Riemann integral of $f : [a, b] \to \mathcal{X}$ where $\mathcal{X}$ is a Banach space, we now examine some shortcomings of Riemann integration.

1.6 Example. For a subset $S \subseteq \mathbb{R}$, we denote by χ_S the indicator function of S, that is,

$$\chi_S(t) = \begin{cases} 1 & t \in S \\ 0 & t \notin S \end{cases}$$

Let $A = [0,1] \cap \mathbb{Q}$. We observe that the Riemann integral

$$\int_0^1 \chi_A$$

does not exist.

Proof. Let $\mathcal{P}$ be any partition of $[0,1]$, say

$$\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_n = 1\}.$$

Since $\mathbb{Q} \cap [0,1]$ and $[0,1] \setminus \mathbb{Q}$ are each dense in $[0,1]$, we can always find tags $t_1^*, \ldots, t_n^*$, such that $t_{j-1} \leq t_j^* \leq t_j$ with $t_j^* \in \mathbb{Q}$ and likewise we can find tags $t_1^{**}, \ldots, t_n^{**}$, such that $t_{j-1} \leq t_j^{**} \leq t_j$ such that $t_j^{**} \notin \mathbb{Q}$. Consider the Riemann sums

$$S_1(\chi_A, \mathcal{P}) = \sum_{j=1}^n \overbrace{\chi_A(t_j^*)}^{=1}(t_j - t_{j-1}) = 1$$
$$S_2(\chi_A, \mathcal{P}) = \sum_{j=1}^n \underbrace{\chi_A(t_j^{**})}_{=0}(t_j - t_{j-1}) = 0.$$

Thus for $\epsilon = \frac{1}{2}$, no partition $\mathcal{P}_\epsilon$ will satisfy the definition of Riemann integrability. The details are left as an exercise. $\square$

Now, we can enumerate $\mathbb{Q} \cap [0,1]$ as $\{q_1, q_2, \ldots\} = \{q_n\}_{n=1}^\infty$. Let us define

$$f_n = \chi_{\{q_1, \ldots, q_n\}}.$$

Then $f_1 \leq f_2 \leq \ldots$ pointwise, i.e. $f_1(t) \leq f_2(t) \leq \ldots$ for all $t \in [0,1]$. Also, $\{f_n\} \to \chi_A$ pointwise. Yet

$$\int_0^1 f_n = 0 \qquad \text{while} \qquad \int_0^1 \chi_A \text{ fails to exist.}$$

2 Lebesgue measure

2.1 Motivation

We want to develop a new integral (the Lebesgue integral). The idea is as follows. Suppose $\mathcal{X} = \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ is a bounded function which satisfies $f \geq 0$. We now "chop up" the range of the function, such that the range of f is contained in $[y_0, y_n)$.

$$y_0 < y_1 < \ldots < y_n$$

Let $E_i = \{t : f(t) \in [y_{i-1}, y_i]\}$. We estimate "$\displaystyle\int_a^b f$" by sums of the form

$$\sum_{j=1}^n y_{j-1} \lambda(E_j).$$

The first problem is: what is $\lambda(E_j)$? Let us investigate this.

2.2 Lebesgue outer measure

Step 1: We first consider open intervals. Let $a, b \in \mathbb{R}$, $a \leq b$

$$(a,b) = \begin{cases} \{t \in \mathbb{R} : a < t < b\} & \text{if } a < b \\ \varnothing & \text{if } a = b. \end{cases}$$

Declare $\ell((a,b)) = b - a$. Also, $\ell((a,\infty)) = \ell((-\infty,b)) = \infty$.

Step 2: Lebesgue outer measure.

2.1 Definition. If $E \subseteq \mathbb{R}$, a sequence $\{I_n\}_{n=1}^{\infty}$ of open intervals is a **cover** of E if $E \subseteq \bigcup_{n=1}^{\infty} I_n$. In this case we also say the sequence of intervals **covers** E. We define the **outer measure** of E by

$$\lambda^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : \{I_n\}_{n=1}^{\infty} \text{ is a cover of } E \text{ by open intervals} \right\}.$$

Observe that this infimum could be infinite.

2.2 Definition. Let $\mathcal{P}(\mathbb{R}) = \{E \subseteq \mathbb{R}\}$ be the **power set** of $\mathbb{R}$. We may think of outer measure as a function

$$\lambda^* : \mathcal{P}(\mathbb{R}) \to \mathbb{R} \cup \{\infty\}.$$

2.3 Proposition (Properties of outer measure). We have the following:

1. $\lambda^*(\varnothing) = 0$.
2. $\lambda^*(E) \geq 0$ for all $E \subseteq \mathbb{R}$ (nonnegativity).
3. If $E \subseteq F \subseteq \mathbb{R}$, then $\lambda^*(E) \leq \lambda^*(F)$ (increasing).
4. $\lambda^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \lambda^*(E_n)$ for all $E_1, \ldots \in \mathcal{P}(\mathbb{R})$ (σ-subadditivity).

Proof. Parts 1 and 2 are easy. We have:

3. We note that any cover of F, by a sequence of open intervals, is also a cover of E.

$$\lambda^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : \{I_n\}_{n=1}^{\infty} \text{ covers } E \right\} \leq \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : \{I_n\}_{n=1}^{\infty} \text{ covers } F \right\} = \lambda^*(F).$$

4. First, if $\sum_{n=1}^{\infty} \lambda^*(E_n) = \infty$, we are done. So assume otherwise. Given $\epsilon > 0$, let $\{I_{in}\}_{i=1}^{\infty}$ be a cover of E_n by open intervals for which

$$\sum_{i=1}^{\infty} \ell(I_{in}) < \lambda^*(E_n) + \frac{\epsilon}{2^n}.$$

Here we are using the definition of λ^* and of "inf". Now, we simply consider $\{I_{in}\}_{i,n=1}^{\infty}$. Clearly, this is a cover of $\bigcup_{n=1}^{\infty} E_n =: E$. We observe that

$$\begin{aligned} \lambda^*(E) \leq \sum_{n=1}^{\infty} \underbrace{\sum_{i=1}^{\infty} \ell(I_{in})}_{<\lambda^*(E_n)+(\epsilon/2^n)} &\leq \sum_{n=1}^{\infty} \left(\lambda^*(E_n) + \frac{\epsilon}{2^n}\right) = \sum_{n=1}^{\infty} \lambda^*(E_n) + \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} \\ &= \sum_{n=1}^{\infty} \lambda^*(E_n) + \epsilon \end{aligned}$$

and we can be liberal with interchanging sums, because it's a non-negative series (if it's summable, it's absolutely summable). Since $\epsilon > 0$ was arbitrary, we have

$$\lambda^*(E) \leq \sum_{n=1}^{\infty} \lambda^*(E_n).$$

□

2.4 Proposition. Let $a \leq b$ in $\mathbb{R}$ and J be any of the intervals

$$(a,b), [a,b], (a,b], [a,b).$$

Then $\lambda^*(J) = b - a$.

Proof. First, let $\epsilon > 0$. Then $\{(a-\epsilon, b+\epsilon)\}$ is a cover of J, hence

$$\lambda^*(J) \leq \ell((a-\epsilon, b+\epsilon)) = b - a + 2\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we see that $\lambda^*(J) \leq b - a$. We assume $J = [a,b)$. The proof for the others is similar. Let $\epsilon > 0$, with $\epsilon < b - a$. We note that $[a, b-\epsilon] \subseteq [a,b)$ and $[a, b-\epsilon]$ is compact. Let $\{(c_i, d_i)\}_{i=1}^\infty$ be a cover of $J = [a,b)$ by open intervals. Then this is also a cover of $[a, b-\epsilon]$, hence admits a finite subcover $\{(c_i, d_i)\}_{i=1}^n$ by compactness. By reordering indices, and dropping some intervals if necessary, we can arrange that $c_1 < a$, $b - \epsilon < d_n$, and moreover $d_i > c_{i+1}$ $(1 \leq i \leq n-1)$. Then we have

$$\begin{aligned}\sum_{i=1}^\infty \ell((c_i, d_i)) \geq \sum_{i=1}^n \ell((c_i, d_i)) = \sum_{i=1}^n (d_i - c_i) &= d_1 - c_1 + d_2 - c_2 + \ldots + d_n - c_n \\ &= -c_1 + d_1 - c_2 + d_2 - c_3 + \ldots + d_n \\ &> -c_1 + d_n = d_n - c_1 > (b - \epsilon) - a\end{aligned}$$

Thus

$$\sum_{i=1}^\infty \ell((c_i, d_i)) > b - a - \epsilon$$

and since ϵ is arbitrary, as is the cover, $\lambda^*(J) \geq b - a$. $\square$

Recall that our first goal was to describe $\lambda(E_i)$ – the "length" of E_i. The desirable outcome: since $E_i \cap E_j = \varnothing$ for $i \neq j$, we want

$$\lambda([a,b]) = \lambda\left(\bigsqcup_{j=1}^n E_j\right) = \sum_{j=1}^n \lambda(E_j)$$

(note that $\sqcup$ is notation for a disjoint union).

2.3 Measurable sets

Step 3: We wish to define measurable sets.

2.5 Definition. We say that $A \subseteq \mathbb{R}$ is **measurable** (or **Lebesgue measurable**) if for any $E \subseteq \mathbb{R}$, we have

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A).$$

In addition, we introduce the following notation. Let

$$\mathcal{L}(\mathbb{R}) = \{A \subseteq \mathbb{R} : A \text{ is measurable}\}.$$

2.6 Remark. We have the following notes:

1. This is known as *Caratheodory's criterion* for defining measurable sets.
2. The inequality
$$\lambda^*(E) \leq \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$
is always true by virtue of σ-subadditivity. Thus we generally need only verify the "$\geq$" inequality to see that A is measurable.

2.7 Theorem. We have:

1. $\varnothing, \mathbb{R} \in \mathcal{L}(\mathbb{R})$.
2. If $A \in \mathcal{L}(\mathbb{R})$, then $\mathbb{R} \setminus A \in \mathcal{L}(\mathbb{R})$.

3. If $A_1, A_2, \ldots \in \mathcal{L}(\mathbb{R})$ is a sequence (countable) then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{L}(\mathbb{R}).$$

Moreover, if $A_i \cap A_j = \varnothing$ for $i \neq j$ then

$$\lambda^*\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lambda^*(A_i).$$

Proof. We have:

1. If $E \subseteq \mathbb{R}$, then $E \cap \varnothing = \varnothing$ and $E \setminus \varnothing = E$. Therefore

$$\lambda^*(E) = 0 + \lambda^*(E \setminus \varnothing) = \lambda^*(E \cap \varnothing) + \lambda^*(E \setminus \varnothing).$$

Hence $\varnothing \in \mathcal{L}(\mathbb{R})$, i.e. it is Lebesgue measurable. Similar proof shows $\mathbb{R} \in \mathcal{L}(\mathbb{R})$.

2. If $A \in \mathcal{L}(\mathbb{R})$, then for $E \subseteq \mathbb{R}$ we have

$$\lambda^*(E \cap (\mathbb{R} \setminus A)) + \lambda^*(E \setminus (\mathbb{R} \setminus A)) = \lambda^*(E \setminus A) + \lambda^*(E \cap A) = \lambda^*(E)$$

by the measurability of A, and hence $R \setminus A \in \mathcal{L}(\mathbb{R})$.

3. Let $A_1, A_2, \ldots \in \mathcal{L}(\mathbb{R})$ be a sequence of measurable sets and $E \subseteq \mathbb{R}$. We write $A := \bigcup_{i=1}^{\infty} A_i$. Then

$$\begin{aligned} E \cap A = \bigcup_{i=1}^{\infty} (E \cap A_i) &= (E \cap A_1) \cup (E \cap A_2) \cup (E \cap A_3) \cup \ldots \\ &= (E \cap A_1) \cup ((E \setminus A_1) \cap A_2) \cup ((E \setminus (A_1 \cup A_2)) \cap A_3) \cup \ldots \\ &= \bigcup_{i=1}^{\infty} \left[\left(E \setminus \bigcup_{k=1}^{i-1} A_k \right) \cap A_i \right]. \end{aligned}$$

Hence, by σ-subadditivity, we have

$$\lambda^*(E) \leq \lambda^*(E \cap A) + \lambda^*(E \setminus A) \leq \sum_{i=1}^{\infty} \lambda^*\left(\left(E \setminus \bigcup_{k=1}^{i-1} A_k \right) \cap A_i \right) + \lambda^*(E \setminus A). \tag{†}$$

Since each of the A_i is measurable,

$$\begin{aligned} \lambda^*(E) &= \lambda^*(E \cap A_1) + \lambda^*(E \setminus A_1) \\ &= \lambda^*(E \cap A_1) + \lambda^*((E \setminus A_1) \cap A_2) + \underbrace{\lambda^*((E \setminus A_1) \setminus A_2)}_{=E \setminus (A_1 \cup A_2)} \\ &\vdots \\ &= \sum_{i=1}^{n} \lambda^*\left(\left(E \setminus \bigcup_{k=1}^{i-1} A_k \right) \cap A_n \right) + \lambda^*\left(E \setminus \bigcup_{i=1}^{n} A_i \right) \\ &\geq \sum_{i=1}^{n} \lambda^*\left(\left(E \setminus \bigcup_{k=1}^{i-1} A_k \right) \cap A_i \right) + \lambda^*(E \setminus A) \end{aligned}$$

by the increasing condition of λ^*. Now, take $n \to \infty$, and obtain

$$\lambda^*(E) \geq \sum_{i=1}^{\infty} \lambda^*\left(\left(E \setminus \bigcup_{k=1}^{i-1} A_k \right) \cap A_i \right) + \lambda^*(E \setminus A). \tag{††}$$

Combining (†) and (††), we see that

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

and, since E is arbitrary, it follows that $A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{L}(\mathbb{R})$. Now, if $A_i \cap A_j = \varnothing$ for $i \neq j$, then

$$\left(E \setminus \bigcup_{k=1}^{i-1} A_k\right) \cap A_i = E \cap A_i.$$

Hence if we let $E = A$, it follows from (††) that

$$\lambda^*(A) \geq \sum_{i=1}^{\infty} \lambda^*(\underbrace{A \cap A_i}_{=A_i}) + \underbrace{\lambda^*(A \setminus A)}_{=0} = \sum_{i=1}^{\infty} \lambda^*(A_i).$$

The other ($\leq$) inequality follows from σ-subadditivity, so we are done. □

2.4 Lebesgue measure

2.8 Definition. We can regard λ^* as a map $\mathcal{P}(\mathbb{R}) \to [0, \infty]$. We define the **Lebesgue measure** λ by restricting λ^* to measurable sets, that is,

$$\lambda = \lambda^*\Big|_{\mathcal{L}(\mathbb{R})} : \mathcal{L}(\mathbb{R}) \to [0, \infty].$$

That is, the Lebesgue measure is the same as the Lebesgue outer measure but only accepting measurable sets.

2.9 Theorem. The Lebesgue measure λ satisfies:

1. [non-negativity] $\lambda(\varnothing) = 0$ and $\lambda(A) \geq 0$ for $A \in \mathcal{L}(\mathbb{R})$.
2. [increasing] If $A, B \in \mathcal{L}(\mathbb{R})$ and $A \subseteq B$, then $\lambda(A) \leq \lambda(B)$.
3. [σ-additivity] If $A_1, A_2, \ldots \in \mathcal{L}(\mathbb{R})$ are such that $A_i \cap A_j = \varnothing$ for $i \neq j$, then

$$\lambda\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \lambda(A_i).$$

Proof. Collect prior facts about λ^* and $\mathcal{L}(\mathbb{R})$. Notice that #2 follows from #1 and #3, i.e.

$$\lambda(B) = \lambda^*(B) = \lambda^*(\underbrace{B \cap A}_{=A \text{ since } B \supseteq A}) + \underbrace{\lambda^*(B \setminus A)}_{\geq 0} \geq \lambda^*(A) = \lambda(A).$$ □

2.10 Lemma. If $a < b$ in $\mathbb{R}$, then $(a, b) \in \mathcal{L}(\mathbb{R})$.

Proof. We need to establish for $E \subseteq \mathbb{R}$ that

$$\lambda^*(E) \geq \lambda^*(E \cap (a, b)) + \lambda^*(E \setminus (a, b)).$$

If $\lambda^*(E) = \infty$, we are done. Suppose otherwise, that is, say $\lambda^*(E) < \infty$, and let $\epsilon > 0$. Find a cover $\{I_n\}_{n=1}^{\infty}$ of open intervals for E such that

$$\sum_{n=1}^{\infty} \ell(I_n) < \lambda^*(E) + \frac{\epsilon}{2}.$$

For each n, let

$$\begin{aligned} J_n &= I_n \cap (a, b) \\ L_n &= I_n \cap (-\infty, a) \\ R_n &= I_n \cap (b, \infty). \end{aligned}$$

Many of these may be empty, hence of length 0. Then $\{J_n\}_{n=1}^{\infty}$ covers $E \cap (a, b)$, and

$$\{L_n, R_n, (a - \tfrac{\epsilon}{8}, a + \tfrac{\epsilon}{8}), (b - \tfrac{\epsilon}{8}, b + \tfrac{\epsilon}{8})\}$$

covers $E \setminus (a, b)$. Let us relabel the latter collection by $\{K_n\}_{n=1}^{\infty}$. Notice

$$\sum_{n=1}^{\infty} \ell(K_n) = \sum_{n=1}^{\infty} (\ell(L_n) + \ell(R_n)) + \frac{\epsilon}{2}.$$

By definition of λ^* we have

$$\lambda^*(E\cap(a,b))+\lambda^*(E\setminus(a,b))\leq\sum_{n=1}^{\infty}\ell(J_n)+\sum_{n=1}^{\infty}\ell(K_n)=\sum_{n=1}^{\infty}(\ell(J_n)+\ell(L_n)+\ell(R_n))+\frac{\epsilon}{2}$$

$$\underbrace{=}_{\text{check}}\sum_{n=1}^{\infty}\ell(I_n)+\frac{\epsilon}{2}$$

however

$$\sum_{n=1}^{\infty}\ell(I_n)+\frac{\epsilon}{2}<\lambda^*(E)+\frac{\epsilon}{2}+\frac{\epsilon}{2}=\lambda^*(E)+\epsilon.$$

Since $\epsilon>0$ is arbitrary, $\lambda^*(E\cap(a,b))+\lambda^*(E\setminus(a,b))\leq\lambda^*(E)$ as required. □

2.11 Corollary. Let G be open. Then $G\in\mathcal{L}(\mathbb{R})$.

Proof. First note that

$$(a,\infty)=\bigcup_{n=1}^{\infty}(a,n)\in\mathcal{L}(\mathbb{R})$$

where $(a,n)=\varnothing$ if $n\leq a$. Similarly, $(-\infty,b)\in\mathcal{L}(\mathbb{R})$. From Assignment 1 question 4, we have that any open G is of the form

$$G=\bigsqcup_{n=1}^{\infty}(a_n,b_n)$$

for $a_i,b_i\in\mathbb{R}\cup\{\pm\infty\}$ so we are done. □

2.5 Scope of $\mathcal{L}(\mathbb{R})$

2.12 Definition. Let X be a set. An **algebra**[3] (of subsets of X) is any family $\mathcal{M}\subseteq\mathcal{P}(X)$ such that

1. $\varnothing, X\in\mathcal{M}$.
2. If $A\in\mathcal{M}$, then $X\setminus A\in\mathcal{M}$.
3. If $A_1,\dots,A_n\in\mathcal{M}$, then $\bigcup_{i=1}^{n}A_i\in\mathcal{M}$.

We further say $\mathcal{M}$ is a σ-algebra if it satisfies the above, in addition to

4. If $A_1,\dots\in\mathcal{M}$ then $\bigcup_{n=1}^{\infty}A_n\in\mathcal{M}$.

2.13 Example. We have the following examples:

1. We always have the trivial σ-algebra on X, $\mathcal{M}=\{\varnothing,X\}$.
2. We can consider $\mathcal{P}(X)$ itself. This is always a σ-algebra on X.
3. $\mathcal{L}(\mathbb{R})$ is a σ-algebra on $\mathbb{R}$.
4. If $\{\mathcal{M}_\beta\}_{\beta\in B}\subseteq\mathcal{P}(X)$ is a family of σ-algebras, then

   $$\bigcap_{\beta\in B}\mathcal{M}_\beta=\{A\subseteq X:A\in\mathcal{M}_\beta\text{ for all }\beta\in B\}.$$

   is also a σ-algebra.

   Proof. Easy exercise. □

5. Define the **Borel σ-algebra** by

   $$\mathcal{B}(\mathbb{R})=\bigcap\{\mathcal{M}:\mathcal{M}\subseteq\mathcal{P}(\mathbb{R})\text{ is a }\sigma\text{-algebra such that }\mathcal{M}\text{ contains all open sets}\}.$$

   This is the smallest σ-algebra containing all open sets. Then clearly since every open set is Lebesgue measurable, we observe that $\mathcal{B}(\mathbb{R})\subseteq\mathcal{L}(\mathbb{R})$. We call the members of $\mathcal{B}(\mathbb{R})$ **Borel sets**.

[3] This is not related to the notion of an algebra over a ring or field. Wikipedia calls this a **field of sets**, but note that the word "field" here is also not related to field theory.

2.14 Remark (Notation). Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a family of sets such that $\varnothing, X \in \mathcal{A}$. Let

$$\mathcal{A}_\sigma = \left\{ \bigcup_{n=1}^{\infty} A_n : A_1, A_2, \ldots \in \mathcal{A} \right\}, \quad \mathcal{A}_\delta = \left\{ \bigcap_{n=1}^{\infty} A_n : A_1, A_2, \ldots \in \mathcal{A} \right\}.$$

2.15 Proposition. If $\mathcal{M}$ is a σ-algebra and $A_1, A_2, \ldots \in \mathcal{M}$, then $\displaystyle\bigcap_{n=1}^{\infty} A_n \in \mathcal{M}$.

Proof. Suppose $\mathcal{M} \subseteq \mathcal{P}(X)$ is a σ-algebra. Then each $X \setminus A_n \in \mathcal{M}$. Hence

$$\bigcup_{n=1}^{\infty} (X \setminus A_n) \in \mathcal{M}.$$

Then

$$\bigcap_{n=1}^{\infty} A_n = X \setminus \left(X \setminus \bigcap_{n=1}^{\infty} A_n \right) = X \setminus \left(\bigcup_{n=1}^{\infty} (X \setminus A_n) \right) \in \mathcal{M}. \qquad \square$$

Let $\mathcal{G}$ be the family of all open sets in $\mathbb{R}$, and let $\mathcal{F}$ be the family of all closed sets.

2.16 Remark. Note that $\mathcal{G}_\sigma = \mathcal{G}$ and $\mathcal{F}_\delta = \mathcal{F}$. However we also have the so-called G_δ sets: $\mathcal{G}_\delta$ and the F_σ sets: $\mathcal{F}_\sigma$.

2.17 Proposition. We have $\mathcal{G} \subseteq \mathcal{F}_\sigma$ and $\mathcal{F} \subseteq \mathcal{G}_\delta$.

Proof. Let $G \in \mathcal{G}$. By Assignment 1, question 4, we can write G as the disjoint union

$$G = \bigsqcup_{n=1}^{\infty} (a_n, b_n).$$

Define, for each k, the set[4]

$$F_k = \bigcup_{n=1}^{k} [a_n + \tfrac{1}{k}, b_n - \tfrac{1}{k}].$$

Then each F_k, being a finite union of closed sets, is closed. Also, $G = \bigcup_{k=1}^{\infty} F_k$. So $\mathcal{G} \subseteq F_\sigma$. On the other hand if $F \in \mathcal{F}$, then $G = \mathbb{R} \setminus F \in \mathcal{G}$, so that $\mathbb{R} \setminus F = \bigcup_{k=1}^{\infty} F_k$ with each $F_k \in \mathcal{F}$. Thus

$$F = \mathbb{R} \setminus (\mathbb{R} \setminus F) = \mathbb{R} \setminus \bigcup_{k=1}^{\infty} F_k = \bigcap_{k=1}^{\infty} \underbrace{(\mathbb{R} \setminus F_k)}_{\in \mathcal{G}} \in \mathcal{G}_\delta. \qquad \square$$

2.18 Remark. The following is true. Let $\mathcal{A}_{\delta\sigma} = (\mathcal{A}_\delta)_\sigma$. Then we have the inclusions

$$\mathcal{G} \subsetneq \mathcal{G}_{\delta\sigma} \subsetneq (\mathcal{G}_{\delta\sigma})_{\delta\sigma} \subsetneq \ldots$$

For this reason we write $\mathcal{G}_0 = \mathcal{G}$, and $\mathcal{G}_{n+1} = (\mathcal{G}_n)_{\delta\sigma}$ for $n \geq 0$. One might hope that

$$\bigcup_{n=1}^{\infty} \mathcal{G}_n = \mathcal{B}(\mathbb{R}).$$

However, this is false. We have to index over all countable *ordinals*. Any finite number is an ordinal. Then there is a first infinite (limit ordinal), say ω. Then we can take $\omega + 1$ and $\omega + 2$ and so on, until we get to $\omega + \omega = 2\omega$.

[4] We follow the usual convention that if the closed interval makes no sense, we just declare it to be the empty set. Also $[a, \infty - \frac{1}{k}] = [a, \infty)$ and similarly on the other side.

2.6 Cantor set

Define

$$
\begin{aligned}
C_0 &= [0,1] \\
C_1 &= [0,\tfrac{1}{3}] \cup [\tfrac{2}{3},1] = C_0 \setminus \overbrace{I_{11}}^{\text{open middle 3rd}} \\
C_2 &= [0,\tfrac{1}{9}] \cup [\tfrac{2}{9},\tfrac{1}{3}] \cup [\tfrac{2}{3},\tfrac{7}{9}] \cup [\tfrac{8}{9},1] = C_1 \setminus (\underbrace{I_{21}}_{\text{open middle of } [0,\frac{1}{3}]} \cup I_{22}) \\
&\vdots \\
C_n &= C_{n-1} \setminus (I_{n,1} \cup \ldots \cup I_{n,2^{n-1}}).
\end{aligned}
$$

2.19 Definition. Let $C = \bigcap_{n=1}^{\infty} C_n$. We call C the **Cantor set**.

We claim that $C \neq \varnothing$. We note each $C_n \neq \varnothing$ and compact,

$$\bigcap_{i=1}^{n} C_i = C_n \neq \varnothing$$

By finite intersection property, $C \neq \varnothing$ and is compact.

2.20 Proposition. We have the following:

1. C is nowhere dense in $\mathbb{R}$.
2. $\lambda(C) = 0$.

Proof. Assignment 2, Question 3. □

2.21 Proposition. $|C| = c$, where $c = |\mathbb{R}|$ is the cardinality of the real line.

Proof. If $x \in [0,1]$, we can write x in ternary expansion

$$x = 0.t_1 t_2 \ldots = \sum_{i=1}^{\infty} \frac{t_i}{3^i}$$

where $t_i \in \{0,1,2\}$. This is not unique: $\frac{1}{3} = 0.1000\ldots = 0.02222\ldots$. We claim that

$$C = \{x \in [0,1] : x \text{ admits a ternary expansion without 1s}\}.$$

Notice that

$$I_{11} = (\tfrac{1}{3},\tfrac{2}{3}) = \{x = 0.1t_2t_3\ldots : t_\ell \neq 2 \text{ for some } \ell \geq 2 \text{ and } t_\ell \neq 0 \text{ for some } \ell \geq 0\}$$

also

$$I_{21} = (\tfrac{1}{9},\tfrac{2}{9}) = \{x = 0.01t_3t_4\ldots : t_\ell \neq 2 \text{ for some } \ell \geq 3 \text{ and } t_\ell \neq 0 \text{ for some } \ell \geq 0\}$$

For $1 \leq k \leq 2^{n-1}$,

$$I_{nk} = \Bigg\{ 0.t_1t_2\ldots t_{n-1}1t_{n+1}t_{n+2}\ldots : t_\ell \neq 0 \text{ for some } \ell \geq n+1, \\ k = 1 + \sum_{\ell=1}^{k} t_\ell 2^{\ell-1}, t_i \neq 1 \text{ for } 1 \leq i < n \Bigg\}.$$

We notice that

$$C = \bigcap_{n=1}^{\infty} C_n = [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^{n-1}} I_{n,k}$$

observing that

$$\bigcup_{k=1}^{2^{n-1}} I_{n,k}$$

is the set of all points necessarily admitting a 1 for t_n. We thus have an obvious bijection $\varphi : C \to \{0,2\}^{\mathbb{N}}$, in other words $\varphi(0.t_1t_2\ldots) = (t_i)_{i=1}^{\infty}$ where each t_ℓ is 0 or 2. By Assignment 1, question 2, $|\{0,2\}|^{\mathbb{N}} = c$. □

2.7 Non-measurable sets

2.22 Definition. If $E \subseteq \mathbb{R}$, we define the **translate** of E by x as follows:

$$x + E = \{x + y : y \in E\}.$$

2.23 Proposition. We have the following:

1. If $E \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, then $\lambda^*(E) = \lambda^*(x + E)$.
2. If $E \in \mathcal{L}(\mathbb{R})$, and $x \in \mathbb{R}$, then $x + E \in \mathcal{L}(\mathbb{R})$.

Thus we conclude that for $E \in \mathcal{L}(\mathbb{R})$, $x \in \mathbb{R}$, $\lambda(x + E) = \lambda(E)$. This property is called **translation invariance** of the Lebesgue measure.

Proof. We have:

1. First, let $G \subseteq \mathbb{R}$ be open. By Assignment 1, question 4, we can write G as a disjoint union $G = \bigsqcup_{n=1}^{\infty} (a_n, b_n)$ with

$$\lambda(G) = \sum_{n=1}^{\infty} \lambda((a_n, b_n)) = \sum_{n=1}^{\infty} \lambda^*((a_n, b_n)) = \sum_{n=1}^{\infty} (b_n - a_n).$$

This is a series of nonnegative terms. Then for $x \in \mathbb{R}$, we obtain the disjoint union

$$x + G = \bigsqcup_{n=1}^{\infty} (x + a_n, x + b_n) \implies \lambda(x + G) = \sum_{n=1}^{\infty} ((x + b_n) - (x + a_n)) = \sum_{n=1}^{\infty} (b_n - a_n) = \lambda(G).$$

Now if $E \subseteq \mathbb{R}$, $x \in \mathbb{R}$ we have for open $G \subseteq \mathbb{R}$ that $E \subseteq G$ exactly when $x + E \subseteq x + G$. Hence

$$\lambda^*(E) = \inf\{\lambda(G) : E \subseteq G,\ G \text{ open}\} = \inf\{\lambda(x + G) : x + E \subseteq x + G,\ G \text{ open}\} = \lambda^*(x + E).$$

2. If $E \in \mathcal{L}(\mathbb{R})$ and $A \subseteq \mathbb{R}$,

$$\lambda^*(A \cap (x + E)) + \lambda^*(A \setminus (x + E)) = \lambda^*(x + [(-x + A) \cap E]) + \lambda^*(x + [(-x + A) \setminus E])$$

which we see is equal to

$$\lambda^*((-x + A) \cap E) + \lambda^*((-x + A) \setminus E) = \lambda^*(-x + A)$$

since E is measurable, which is equal to $\lambda^*(A)$ by part 1. $\square$

2.24 Theorem. There exists a subset $E \subseteq \mathbb{R}$ such that E is not measurable.

Proof. Fix $a > 0$. On $(-a, a)$ define a relation by $x \sim y$ if and only if $x - y \in \mathbb{Q}$. This is an equivalence relation. To see reflexivity, note that $x \sim x$ since $x - x = 0 \in \mathbb{Q}$. To see symmetry, note that $x \sim y$ implies $y \sim x$ since $-(x - y) = y - x \in \mathbb{Q}$. To see transitivity, note $x \sim y$ and $y \sim z$ implies $x \sim z$ since

$$z - x = (z - y) + (y - x) \in \mathbb{Q}.$$

For each $x \in (-a, a)$ we let its equivalence class

$$[x] = \{y \in (-a, a) : x \sim y\} = \{y \in (-a, a) : x - y \in \mathbb{Q}\} = \{y \in (-a, a) : y - x \in \mathbb{Q}\}$$

which is to say that $y \in x + \mathbb{Q}$. Hence we see $[x] = (x + \mathbb{Q}) \cap (-a, a)$. Let E be a subset of $(-a, a)$ such that

1. If $x, y \in E$, $x \neq y$ then $x \nsim y$.
2. $(-a, a) = \bigcup_{x \in E} [x]$.

Thus, E contains exactly one point from each equivalence class. Such a thing exists due to the Axiom of Choice. We enumerate

$$(-2a, 2a) \cap \mathbb{Q} = \{r_k\}_{k=1}^{\infty}.$$

We claim that

$$(-a, a) \subseteq \bigsqcup_{k=1}^{\infty} (r_k + E) \subseteq (-3a, 3a) \tag{††}$$

and we note that $(r_k+E)\cap(r_\ell+E)=\varnothing$ for $k\neq\ell$, since if $x=r_k+y=r_\ell+z$ for $y\neq z$ in E it would imply that $y-z=r_\ell-r_k\in\mathbb{Q}$ which is impossible by definition of E. To see the first inclusion, note that if $x\in(-a,a)$ then $x\in[y]$ for some $y\in E$, so $x-y\in\mathbb{Q}$ and $|x-y|<2a$ so $x-y=r_k$ for some k. Hence $x=r_k+y\subseteq r_k+E$. To see the second inclusion, we have that $|r_n+x|<3a$ for any $x\in(-a,a)$ and $r_k\in(-2a,2a)$.

We now show that $E\notin\mathcal{L}(\mathbb{R})$. Assume otherwise. Then either $\lambda(E)=0$ or $\lambda(E)=\alpha>0$. If $\lambda(E)=0$, then $\lambda(r_k+E)=0$ for all k, but then by the increasing and σ-additivity properties, we would find

$$2a=\lambda((-a,a))\leq\lambda\left(\bigcup_{k=1}^{\infty}(r_k+E)\right)=\sum_{k=1}^{\infty}\underbrace{\lambda(r_k+E)}_{0}=0.$$

where the union is disjoint, but this is absurd. Hence $\lambda(E)=\alpha>0$. But then for any $n\in\mathbb{N}$, using the increasing, σ-additivity, and translation invariance properties on (††),

$$n\alpha=\sum_{k=1}^{n}\lambda(r_k+E)=\lambda\left(\bigcup_{k=1}^{n}(r_k+E)\right)\leq\lambda\left(\bigcup_{k=1}^{\infty}(r_k+E)\right)\leq\lambda((-3a,3a))=6a.$$

where the first union is disjoint. Clearly this cannot hold for $n>\frac{6a}{\alpha}$. Thus $E\notin\mathcal{L}(\mathbb{R})$. ☐

2.25 Remark. R. M. Solovay, Ann. of Math (2), v. 92, 1970. Shows that if Axiom of Choice is weakened to only allowing countable choice, then we get the surprising consequence that

$$\mathcal{P}(\mathbb{R})=\mathcal{L}(\mathbb{R})=\mathcal{B}(\mathbb{R}).$$

2.26 Remark. We have the following notes: with E as above we have

1. $0<\lambda^*(E)\leq 2a$.
2. $0=\lambda_*(E)$, where λ_* is defined in Assignment 2, question 2.

2.27 Definition. A subset $N\subseteq\mathbb{R}$ is called a **(Lebesgue) null** set if $\lambda^*(N)=0$.

2.28 Proposition. A null set is measurable.

Proof. For any $E\subseteq\mathbb{R}$, we have[5]

$$\lambda^*(\underbrace{E\cap N}_{\subseteq N})+\lambda^*(\underbrace{E\setminus N}_{\subseteq E})\leq\underbrace{\lambda^*(N)}_{0}+\lambda^*(E)=\lambda(E).$$

Thus $N\in\mathcal{L}(\mathbb{R})$. ☐

3 Lebesgue integration

3.1 Measurable functions

Idea: As far as notation is concerned, we let χ_A be the **characteristic** or **indicator function**,

$$\chi_A(x)=\begin{cases}1 & \text{if } x\in A\\ 0 & \text{if } x\notin A\end{cases}.$$

It would be reasonable that

$$\int_{\mathbb{R}}\chi_A=\lambda(A),\qquad\int_{\mathbb{R}}(f+g)=\int_{\mathbb{R}}f+\int_{\mathbb{R}}g.$$

3.1 Definition. A function $f:\mathbb{R}\to\mathbb{R}$ is called **measurable** if

$$f^{-1}((\alpha,\infty))=\{x\in\mathbb{R}:f(x)>\alpha\}$$

is measurable for all $\alpha\in\mathbb{R}$. We say f is **Borel measurable** if $f^{-1}((\alpha,\infty))\in\mathcal{B}(\mathbb{R})$ for all $\alpha\in\mathbb{R}$.

3.2 Example. Let $A\subseteq\mathbb{R}$. We have that χ_A is measurable if and only if $A\in\mathcal{L}(\mathbb{R})$.

[5] Uh, why does this say $\lambda(E)$? E isn't necessarily measurable...

Proof. Note that

$$\chi_A^{-1}((\alpha,\infty)) = \begin{cases} \varnothing & \text{if } \alpha \geq 1 \\ A & \text{if } 0 \leq \alpha < 1 \\ \mathbb{R} & \text{if } \alpha < 0 \end{cases}$$

so $\chi_A^{-1}((\alpha,\infty)) \in \mathcal{L}(\mathbb{R})$ for all $\alpha \in \mathbb{R}$ if and only if $A \in \mathcal{L}(\mathbb{R})$. □

3.3 Proposition. Let $f : \mathbb{R} \to \mathbb{R}$. Then the following are equivalent:

1. f is measurable.
2. $f^{-1}((-\infty, a]) \in \mathcal{L}(\mathbb{R})$ for all $\alpha \in \mathbb{R}$.
3. $f^{-1}((-\infty, a)) \in \mathcal{L}(\mathbb{R})$ for all $\alpha \in \mathbb{R}$.
4. $f^{-1}([\alpha, \infty)) \in \mathcal{L}(\mathbb{R})$ for all $\alpha \in \mathbb{R}$.

Proof. To prove $1 \leftrightarrow 2$:

$$f^{-1}((-\infty,\alpha]) = \{x \in \mathbb{R} : f(x) \leq \alpha\} = \mathbb{R} \setminus \{x \in \mathbb{R} : f(x) > \alpha\} = \mathbb{R} \setminus f^{-1}((\alpha,\infty)).$$

We recall for $A \subseteq \mathbb{R}$, $A \in \mathcal{L}(\mathbb{R})$ if and only if $\mathbb{R} \setminus A \in \mathcal{L}(\mathbb{R})$.

To prove $2 \to 3$: Note that

$$f^{-1}((-\infty,\alpha)) = f^{-1}\left(\bigcup_{n=1}^{\infty} (-\infty, \alpha - \tfrac{1}{n}]\right) = \bigcup_{n=1}^{\infty} f^{-1}((-\infty, \alpha - \tfrac{1}{n}]).$$

As each $f^{-1}((-\infty, \alpha - \frac{1}{n}]) \in \mathcal{L}(\mathbb{R})$, their countable union is as well.

$3 \to 4$ is similar to $1 \to 2$. $4 \to 1$ is similar to $2 \to 3$. □

3.4 Corollary. Let $f : \mathbb{R} \to \mathbb{R}$. Then f is measurable if and only if[6] $f^{-1}(B) \in \mathcal{L}(\mathbb{R})$ for any $B \in \mathcal{B}(\mathbb{R})$.

Proof. Observe that only one direction requires proof. Suppose f is measurable. First, let G be open. Then $G = \bigsqcup_{k=1}^{\infty} (a_k, b_k)$. Hence

$$\begin{aligned} f^{-1}(G) = f^{-1}\left(\bigcup_{k=1}^{\infty} (a_k,b_k)\right) &= \bigcup_{k=1}^{\infty} f^{-1}(\underbrace{(a_k,b_k)}_{(a_k,\infty)\cap(-\infty,b_k)}) \\ &= \bigcup_{k=1}^{\infty} \left[f^{-1}((a_k,\infty)) \cap f^{-1}((-\infty,b_k))\right] \in \mathcal{L}(\mathbb{R}). \end{aligned}$$

Now, let

$$\mathcal{M}_f = \{M \subseteq \mathbb{R} : f^{-1}(M) \in \mathcal{L}(\mathbb{R})\}.$$

We note that $f^{-1}(\mathbb{R}) = \mathbb{R}$, therefore $\mathbb{R} \in \mathcal{M}_f$. Also, if $M_1, M_2, \ldots \in \mathcal{M}_f$ then

$$f^{-1}\left(\bigcup_{i=1}^{\infty} M_i\right) = \bigcup_{i=1}^{\infty} \underbrace{f^{-1}(M_i)}_{\mathcal{L}(\mathbb{R})} \in \mathcal{L}(\mathbb{R})$$

Also, if $M \in \mathcal{M}_f$, then

$$f^{-1}(\mathbb{R} \setminus M) = \mathbb{R} \setminus \underbrace{f^{-1}(M)}_{\mathcal{L}(\mathbb{R})} \in \mathcal{L}(\mathbb{R}).$$

Thus $\mathcal{M}_f$ is a σ-algebra. From above, $\mathcal{G} \subseteq \mathcal{M}_f$, and $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing $\mathcal{G}$. Thus $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}_f$. □

3.5 Proposition. Let $f, g : \mathbb{R} \to \mathbb{R}$ be measurable, $c \in \mathbb{R}$, and $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous. Then

(i) $cf : \mathbb{R} \to \mathbb{R}$, i.e. $(cf)(x) = cf(x)$, is measurable.

(ii) $f + g : \mathbb{R} \to \mathbb{R}$, i.e. $(f+g)(x) = f(x) + g(x)$, is measurable.

[6] Of course, f is measurable if and only if $f^{-1}(U)$ is measurable for any open U, too.

(iii) $\varphi \circ f : \mathbb{R} \to \mathbb{R}$ is measurable.

(iv) $fg : \mathbb{R} \to \mathbb{R}$, i.e. $(fg)(x) = f(x)g(x)$, is measurable.

Proof. We have:

(i) For $\alpha \in \mathbb{R}$, note that

$$(cf)^{-1}((\alpha,\infty)) = \begin{cases} f^{-1}((\frac{\alpha}{c},\infty)) & \text{if } c > 0 \\ \mathbb{R} & \text{if } c = 0, \alpha < 0 \\ \varnothing & \text{if } c = 0, \alpha \geq 0 \\ f^{-1}((-\infty,\frac{\alpha}{c})) & \text{if } c < 0 \end{cases}$$

and note that all these values are in $\mathcal{L}(\mathbb{R})$ by assumption on f.

(ii) Enumerate $\mathbb{Q} = \{r_n\}_{n=1}^{\infty}$. Observe

$$(f+g)^{-1}((\alpha,\infty)) = \{x \in \mathbb{R} : f(x) + g(x) > \alpha\} = \{x \in \mathbb{R} : f(x) > \alpha - g(x)\}$$

However $\overline{\mathbb{Q}} = \mathbb{R}$, so we can consider this as

$$\begin{aligned}\bigcup_{k=1}^{\infty}\{x \in \mathbb{R} : f(x) > r_k \text{ and } r_k > \alpha - g(x)\} &= \bigcup_{k=1}^{\infty}\{x \in \mathbb{R} : f(x) > r_k\} \cap \{x \in \mathbb{R} : g(x) > \alpha - r_k\} \\ &= \bigcup_{k=1}^{\infty}\left(\underbrace{f^{-1}((r_k,\infty))}_{\in\mathcal{L}(\mathbb{R})} \cap \underbrace{g^{-1}((\alpha - r_k,\infty))}_{\in\mathcal{L}(\mathbb{R})}\right) \in \mathcal{L}(\mathbb{R}).\end{aligned}$$

(iii) Let $\alpha \in \mathbb{R}$.

$$(\varphi \circ f)^{-1}((\alpha,\infty)) = f^{-1}(\varphi^{-1}(\underbrace{(\alpha,\infty)}_{\text{open}})) \in \mathcal{L}(\mathbb{R}).$$

(iv) We observe

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$$

and $f+g$ is measurable by (ii), $-g$ is measurable by (i), so $f-g$ is measurable by (ii). Also, $x \mapsto x^2$ is a continuous function, so the squares are measurable. It easily follows that fg is measurable. □

3.6 Remark (Notation). For $f : \mathbb{R} \to \mathbb{R}$ we let

$$\begin{aligned} |f|(x) &= |f(x)| \\ f^+(x) &= \max\{f(x), 0\} \\ f^-(x) &= \max\{-f(x), 0\}. \end{aligned}$$

Observe that

$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^-. \tag{$*$}$$

3.7 Corollary. If $f : \mathbb{R} \to \mathbb{R}$ is measurable, then f^+, f^-, and $|f|$ are all measurable.

Proof. We first note that $x \mapsto |x|$ is continuous so the measurability of $|f|$ follows from (iii) of the proposition. Also, $f^+ = \frac{1}{2}(|f| + f)$ by $(*)$, and $f^- = \frac{1}{2}(|f| - f)$. □

3.8 Remark. For $A \in \mathcal{L}(\mathbb{R})$, we let

$$\mathcal{M}(A) = \{f : A \to \mathbb{R} \mid f \text{ is measurable}\}.$$

We can extend f to $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by putting $\tilde{f}(x) = f(x)$ if $x \in A$, and $\tilde{f}(x) = 0$ otherwise. We say f is measurable if and only if $\tilde{f}$ is.

3.9 Remark. The previous proposition (i), (ii), (iv) shows that $\mathcal{M}(A)$ is an algebra of functions. Further, condition (iii) tells us that "continuous functions operate on $\mathcal{M}(\mathbb{R})$".

3.10 Definition. We define $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty,\infty\} = [-\infty,\infty]$. We call $\overline{\mathbb{R}}$ the set of **extended real numbers**. A function $f : \mathbb{R} \to \overline{\mathbb{R}}$ (or $f : A \to \overline{\mathbb{R}}$) is called **extended real valued**. We say that f is **measurable** if $f^{-1}(B) \in \mathcal{L}(\mathbb{R})$ for each $B \in \mathcal{B}(\mathbb{R})$ and $f^{-1}(\{\pm\infty\}) \in \mathcal{L}(\mathbb{R})$.

3.11 Proposition. Let $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ (usually $f_n : \mathbb{R} \to \mathbb{R}$) be a measurable function for each n. Then the following are measurable:

(i) $\sup_n\{f_n\}$, i.e. $\sup_n f_n(x)$.

(ii) $\inf_n\{f_n\}$, i.e. $\inf_n f_n(x)$.

(iii) $\limsup_{n\to\infty} f_n$, i.e. $\limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left[\sup_{k\geq n} f_k(x)\right]$.

(iv) $\liminf_{n\to\infty} f_n$.

Proof. We have:

(i) Fix $\alpha \in \mathbb{R}$.

$$\left(\sup_n f_n\right)^{-1}([-\infty,\alpha]) = \{x \in \mathbb{R} : \sup_n f_n(x) \leq \alpha\} = \bigcap_{n=1}^{\infty}\{x \in \mathbb{R} : f_n(x) \leq \alpha\}$$
$$= \bigcap_{n=1}^{\infty} f_n^{-1}([-\infty,\alpha]) \in \mathcal{L}(\mathbb{R}).$$

(ii) Similarly show $\left(\inf_n f_n\right)^{-1}([\alpha,\infty]) \in \mathcal{L}(\mathbb{R})$.

(iii) We have

$$\limsup_{n\to\infty} f_n = \lim_{n\to\infty}\sup_{k\geq n} f_k = \inf_n \sup_{k\geq n} f_n$$

and for all n, $\sup_{k\geq n} f_k$ is measurable by (i).

$$\sup_{k\geq n} f_k \geq \sup_{k\geq n+1} f_k.$$

(iv) Same as (iii). ☐

3.12 Corollary. If $f_n : \mathbb{R} \to \overline{\mathbb{R}}$ is measurable for each n and $\lim_{n\to\infty} f_n(x)$ exists for each x (we accept $-\infty, \infty$ as limits) then $\lim f_n$ is measurable.

Proof. In this case we have

$$\lim_{n\to\infty} f_n = \limsup_{n\to\infty} f_n = \liminf_{n\to\infty} f_n.$$ ☐

Rough outline:

- Non-negative measurable simple function – "proto-integral"
- Non-negative extended real-valued measurable functions – approximation from below.
- Integrable functions – differences of non-negative integrable functions.

3.2 Simple functions

3.13 Definition. Let $A \in \mathcal{L}(\mathbb{R})$ (usually $A \subseteq \mathbb{R}$ an interval). A function $f : A \to \mathbb{R}$ is **simple** if

$$f(A) = \{a_1, \ldots, a_n\},$$

in other words f is finite-valued. **Standard form**: suppose $f(A) = \{a_1 < \ldots < a_n\}$. Let

$$E_i = f^{-1}(\{a_i\}), \qquad \forall i\ (1 \leq i \leq n).$$

We write $f = \sum\limits_{i=1}^{n} a_i \chi_{E_i}$.

3.14 Proposition. A simple function $f : A \to \mathbb{R}$ is measurable if and only if when written in standard form $f = \sum\limits_{i=1}^{n} a_i \chi_{E_i}$ for $a_1 < \ldots < a_n$, we have that each E_i is measurable.

Proof. ($\rightarrow$) If f is measurable, we note that each set $\{a_i\}$ is Borel, hence

$$E_i = f^{-1}(\{a_i\}) \in \mathcal{L}(\mathbb{R}).$$

($\leftarrow$) We note that χ_{E_i} is measurable (as a function) if and only if $E_i \in \mathcal{L}(\mathbb{R})$. Linear combinations of measurable functions are measurable. $\square$

Let us now define

$$\begin{aligned} \mathcal{S}(A) &= \{\varphi : A \to \mathbb{R} : \varphi \text{ is simple and measurable}\}. \\ \mathcal{S}^+(A) &= \{\varphi \in \mathcal{S}(A) : \varphi \geq 0 \text{ (pointwise)}\}. \end{aligned}$$

3.3 Proto-integral

3.15 Definition. If $\varphi \in \mathcal{S}^+(A)$ written in standard form, $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$ with $a_i \neq a_j$ if $i \neq j$, $E_i \cap E_j = \varnothing$ if $i \neq j$, then we define

$$I_A(\varphi) = \sum_{i=1}^{n} a_i \lambda(E_i)$$

noting that this quantity may be ∞. We have $0 \cdot \infty = 0$. We call this the **proto-integral** of φ.

3.16 Proposition. If $\varphi, \psi \in \mathcal{S}^+(A)$, $c \geq 0$, then we have

(i) $I_A(c\varphi) = cI_A(\varphi)$.

(ii) $I_A(\varphi + \psi) = I_A(\varphi) + I_A(\psi)$, where we say $\alpha + \infty = \infty = \infty + \alpha$ for $\alpha \geq 0$.

(iii) $\varphi \leq \psi$ implies $I_A(\varphi) \leq I_A(\psi)$.

Proof. We have:

(i) Easy exercise.

(ii) Let $\varphi(A) = \{a_1 < \ldots < a_n\}$ and $\psi(A) = \{b_1 < \ldots < b_m\}$, $E_i = \varphi^{-1}(\{a_i\})$ and $F_j = \psi^{-1}(\{b_j\})$. Let

$$\{a_i + b_j : 1 \leq i \leq n, 1 \leq j \leq m\} = \{c_1 < \ldots < c_\ell\}.$$

For k with $1 \leq k \leq \ell$, define

$$D_k = \bigcup\{E_i \cap F_j : a_i + b_j = c_k\}$$

We write

$$\varphi + \psi = \sum_{i=1}^{n} a_i \chi_{E_i} + \sum_{j=1}^{m} b_j \chi_{F_j}.$$

We observe that[7]

$$\chi_E + \chi_F = \chi_{E \cup F} + \chi_{E \cap F}.$$

So we can rewrite the above as

$$\varphi + \psi = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \chi_{E_i \cap F_j} + \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \chi_{E_i \cap F_j} = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + b_j) \chi_{E_i \cap F_j} = \sum_{k=1}^{\ell} c_k \chi_{D_k}$$

[7] Does anyone know how exactly this gets used?

by definition of D_k. This is in standard form. On the other hand,

$$\begin{aligned}
I_A(\varphi) + I_A(\psi) &= \sum_{i=1}^{n} a_i \lambda(E_i) + \sum_{j=1}^{m} b_j \lambda(F_j) \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \lambda(E_i \cap F_j) + \sum_{j=1}^{m} b_j \sum_{i=1}^{n} \lambda(E_i \cap F_j), \text{ by } \sigma\text{-add.} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + b_j) \lambda(E_i \cap F_j) \\
&= \sum_{k=1}^{\ell} c_k \lambda(D_k), \text{ by } \sigma\text{-add.} \\
&= I_A(\varphi + \psi), \text{ by the above rewriting.}
\end{aligned}$$

(iii) If a_i, b_j, E_i, F_j are as above, we have that $a_i \leq b_j$ whenever $E_i \cap F_j \neq \varnothing$ since $\varphi \leq \psi$. Then

$$I_A(\varphi) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \lambda(E_i \cap F_j) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} b_j \lambda(E_i \cap F_j) = I_A(\psi). \qquad \square$$

3.4 Non-negative integral

We now use proto-integrals to define an integral for non-negative extended real-valued measurable functions.

3.17 Definition. Now, given $A \in \mathcal{L}(\mathbb{R})$, let

$$\overline{\mathcal{M}}^+(A) = \{f : A \to [0, \infty] : f \text{ is measurable}\}.$$

3.18 Remark. If $f, g \in \overline{\mathcal{M}}^+(A)$ then $f + g \in \overline{\mathcal{M}}^+(A)$ makes sense. Also, we can define $cf \in \overline{\mathcal{M}}^+(A)$, for $c \geq 0$, $f \in \overline{\mathcal{M}}^+(A)$. $[0 \cdot f = 0]$. Also if $f = \lim_{n\to\infty} f_n$, $\limsup_{n\to\infty} f_n$, $\sup_{n\in\mathbb{N}} f_n$, where $(f_n)_{n=1}^\infty \subseteq \overline{\mathcal{M}}^+(A)$ then $f \in \overline{\mathcal{M}}^+(A)$. Also $fg \in \overline{\mathcal{M}}^+(A)$ if each $f, g \in \overline{\mathcal{M}}^+(A)$ since we can allow $\infty^2 = \infty$.

3.19 Definition. If $A \in \mathcal{L}(\mathbb{R})$ and $f \in \overline{\mathcal{M}}^+(A)$ we let $\mathcal{S}_f^+(A) = \{\varphi \in \mathcal{S}^+(A) : \varphi \leq f\}$, and define

$$\int_A f = \sup\{I_A(\varphi) : \varphi \in \mathcal{S}_f^+(A)\}.$$

We call this the **Lebesgue integral** of f.

3.20 Proposition. Let $\varnothing \neq A \in \mathcal{L}(\mathbb{R})$ and $f, g \in \overline{\mathcal{M}}^+(A)$.

(i) If $f \leq g$ on A, then $\int_A f \leq \int_A g$.

(ii) If $\varnothing \neq B \subseteq A$ is measurable, then $\int_B f = \int_A f\chi_B$.

(iii) If $\varphi \in \mathcal{S}^+(A)$, then $\int_A \varphi = I_A(\varphi)$.

Proof. We have:

(i) We note that $\mathcal{S}_f^+(A) \subseteq \mathcal{S}_g^+(A)$ since $f \leq g$. Hence

$$\int_A f = \sup_{\varphi \in \mathcal{S}_f^+(A)} I_A(\varphi) \leq \sup_{\psi \in \mathcal{S}_g^+(A)} I_A(\psi) = \int_A g.$$

(ii) If $\varphi \in \mathcal{S}_f^+(B)$, define $\tilde{\varphi}$ on A by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in B \\ 0 & \text{if } x \in A \setminus B. \end{cases}$$

Then it is clear that $\tilde{\varphi}$ is simple, and measurable (check!), so $\tilde{\varphi} \in \mathcal{S}_f^+(A)$. We also note that

$$\{\tilde{\varphi} : \varphi \in \mathcal{S}_f^+(B)\} = \mathcal{S}_{f\chi_B}^+(A).$$

Hence

$$\int_A f\chi_B = \sup\{I_A(\varphi) : \varphi \in \mathcal{S}^+_{f\chi_B}(A)\} = \sup\{I_A(\tilde{\varphi}) : \varphi \in \mathcal{S}^+_f(B)\}$$
$$= \sup\{I_B(\varphi) : \varphi \in \mathcal{S}^+_f(B)\} = \int_B f.$$

(iii) First, if $\psi \in \mathcal{S}^+_\varphi(A)$, then $I_A(\psi) \le I_A(\varphi)$ from last class (proposition) since $\psi \le \varphi$. Hence

$$\int_A \varphi = \sup_{\psi \in \mathcal{S}^+_\varphi(A)} I_A(\psi) \le I_A(\varphi)$$

and on the other hand, $\varphi \in \mathcal{S}^+_\varphi(A)$, so that $I_A(\varphi) \le \int_A \varphi$. $\square$

3.21 Lemma. If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ in $\mathcal{L}(\mathbb{R})$, then

$$\lambda\left(\bigcup_{n=1}^\infty A_n\right) = \lim_{n\to\infty} \lambda(A_n).$$

Proof. Let $C_1 = A_1$, and in general $C_n = A_n \setminus A_{n-1}$ for $n \ge 2$. Since $A_1 \subseteq A_2 \subseteq \dots$, we have that $C_n \cap C_m = \varnothing$ if $n \ne m$. We then have

$$\lambda\left(\bigcup_{n=1}^\infty A_n\right) = \lambda\left(\bigsqcup_{n=1}^\infty C_n\right) = \sum_{n=1}^\infty \lambda(C_n) = \lim_{N\to\infty}\sum_{n=1}^N \lambda(C_n) = \lim_{N\to\infty} \lambda\left(\bigsqcup_{n=1}^N C_n\right) = \lim_{N\to\infty}\lambda(A_N). \qquad \square$$

3.5 Monotone Convergence Theorem

3.22 Theorem (Lebesgue Monotone Convergence Theorem). Let $(f_n)_{n=1}^\infty \subseteq \overline{\mathcal{M}}^+(A)$, with $f_1 \le f_2 \le f_3 \le \dots$ pointwise. Let $f = \lim_{n\to\infty} f_n$. Then

$$\int_A f = \lim_{n\to\infty}\int_A f_n.$$

In particular,

$$\sup_{n\in\mathbb{N}} \int_A f_n < \infty \implies \int_A f < \infty.$$

Proof. We first note that since $f_1 \le f_2 \le \dots$, we have $\int_A f_1 \le \int_A f_2 \le \dots$ and hence,

$$\lim_{n\to\infty}\int_A f_n = \sup_{n\in\mathbb{N}}\int_A f_n.$$

Also, we note that $f \in \overline{\mathcal{M}}^+(A)$, by result on measurable functions. Since $f_n \le f$, for each n, we find that

$$\int_A f_n \le \int_A f.$$

Therefore,

$$\lim_{n\to\infty}\int_A f_n \le \int_A f.$$

Thus, it remains to establish that $\lim_{n\to\infty}\int_A f_n \ge \int_A f$. Let $\varphi \in \mathcal{S}^+_f(A)$, and choose $0 < \eta < 1$. We will first show that

$$\lim_{n\to\infty}\int_A f_n \ge \eta\int_A \varphi \tag{†}$$

Let $A_n = \{x \in A : f_n(x) \ge \eta\varphi(x)\}$. We have that

(i) $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$, since if $f_n(x) \ge \eta\varphi(x)$, then $f_{n+1}(x) \ge f_n(x) \ge \eta\varphi(x)$.

(ii) $\bigcup_{i=1}^\infty A_i = A$, since $\lim_{n\to\infty} f_n(x) = f(x)$, and $\eta\varphi(x) < \varphi(x) \le f(x)$.

Now, let $\eta\varphi(A) = \{a_1 < a_2 < \ldots < a_m\}$, $E_i = (\eta\varphi)^{-1}(\{a_i\}) \subseteq A$, for i $(1 \le i \le m)$. We have, for each n,

$$\begin{aligned}\int_A f_n \ge \int_A f_n \chi_{A_n} &= \int_{A_n} f_n \\ &\ge \int_{A_n} \eta\varphi, \text{ by definition of } A_n \\ &= \sum_{i=1}^m a_i \lambda(E_i \cap A_n).\end{aligned}$$

Now, by the lemma, take $n \to \infty$, and since each $E_i = E_i \cap A = \bigcup_{n=1}^{\infty} (E_i \cap A_n)$, we have that the last term above has limit

$$\sum_{i=1}^m a_i \lambda(E_i) = \int_A \eta\varphi = \eta \int_A \varphi.$$

Thus

$$\lim_{n\to\infty} \int_A f_n \ge \eta \int_A \varphi$$

as required in (†). Since this is true for all choices of η $(0 < \eta < 1)$, we then have

$$\lim_{n\to\infty} \int_A f_n \ge \lim_{\eta\to 1} \eta \int_A \varphi = \int_A \varphi.$$

Thus, as we chose $\varphi \in \mathcal{S}_f^+(A)$,

$$\lim_{n\to\infty} \int_A f_n \ge \sup_{\varphi\in\mathcal{S}_f^+(A)} \int_A \varphi = \int_A f.$$

□

3.23 Lemma. Let $f : A \to [0, \infty]$, where $\varnothing \ne A \in \mathcal{L}(\mathbb{R})$. Then

$$f \in \overline{\mathcal{M}}^+(A) \iff \exists \text{ a sequence } (\varphi_n)_{n=1}^{\infty} \subseteq \mathcal{S}^+(A) \text{ s.t. } \lim_{n\to\infty} \varphi_n = f \text{ pointwise.}$$

Moreover, we can arrange $\varphi_1 \le \varphi_2 \le \ldots \le f$ (pointwise).

Proof. $(\leftarrow)$ A limit of a sequence of measurable functions is still measurable.

$(\to)$ For each $k \in \mathbb{N}$, let $F_k = f^{-1}([k, \infty])$ and for each $i = 1, \ldots, k2^k$, let $E_{k,i} = f^{-1}([\frac{i-1}{2^k}, \frac{i}{2^k}])$. Then for each $k \in \mathbb{N}$,

$$A = F_k \sqcup \bigsqcup_{i=1}^{k2^k} E_{k,i}.$$

Let

$$\varphi_k = k\chi_{F_k} + \sum_{i=1}^{k2^k} \tfrac{i-1}{2^k} \chi_{E_{k,i}}.$$

Check $\varphi_1 \le \varphi_2 \le \ldots$ and $\lim_{k\to\infty} \varphi_k = f$. □

We have the following corollary to the MCT and to the lemma, which establishes several familiar properties of the integral like linearity, additivity across sets, and compatibility with infinite sums.

3.24 Corollary. Let $\varnothing \ne A \in \mathcal{L}(\mathbb{R})$. Then we have:

(i) If $f, g \in \overline{\mathcal{M}}^+(A)$, $c \ge 0$, then

$$\int_A cf = c\int_A f \quad \text{and} \quad \int_A f + g = \int_A f + \int_A g.$$

(ii) If $(f_n)_{n=1}^{\infty} \subseteq \overline{\mathcal{M}}^+(A)$, then

$$\int_A \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_A f_n.$$

(iii) If $A_1, A_2, \ldots \subseteq A$ are measurable sets such that $A = \bigsqcup_{i=1}^{\infty} A_i$ and $A_i \cap A_j = \varnothing$ if $i \neq j$, then

$$\int_A f = \sum_{i=1}^{\infty} \int_{A_i} f$$

for $f \in \overline{\mathcal{M}}^+(A)$.

Proof. We have:

(i) Let $(\varphi_n)_{n=1}^{\infty}, (\psi_n)_{n=1}^{\infty} \subseteq \mathcal{S}^+(A)$ such that $\varphi_1 \leq \varphi_2 \leq \ldots$ and $\psi_1 \leq \psi_2 \leq \ldots$ and $\lim_{n\to\infty} \varphi_n = f$, $\lim_{n\to\infty} \psi_n = g$. Then

$$\varphi_1 + \psi_1 \leq \varphi_2 + \psi_2 \leq \ldots$$

and furthermore $\lim_{n\to\infty} \varphi_n + \psi_n = f + g$. Using MCT, and the linearity of proto-integrals,

$$\begin{aligned}\int_A (f+g) = \lim_{n\to\infty} \underbrace{\int_A (\varphi_n + \psi_n)}_{I_A(\varphi_n)+I_A(\psi_n)} &= \lim_{n\to\infty} \left(\int_A \varphi_n + \int_A \psi_n \right) \\ &= \lim_{n\to\infty} \int_A \varphi_n + \lim_{n\to\infty} \int_A \psi_n = \int_A f + \int_A g.\end{aligned}$$

Similarly, using properties of the proto-integral $I_A(\varphi_n)$,

$$\int_A cf = \lim_{n\to\infty} \int_A c\varphi_n = \lim_{n\to\infty} c \int_A \varphi_n = c \lim_{n\to\infty} \int_A \varphi_n = c \int_A f.$$

(ii) Let $g_n = \sum_{k=1}^{n} f_k \in \overline{\mathcal{M}}^+(A)$. We note that $g_1 \leq g_2 \leq \ldots$ and

$$\lim_{n\to\infty} g_n = \sum_{k=1}^{\infty} f_k$$

(by definition). We just use (i) to see that

$$\int_A g_n = \sum_{k=1}^{n} \int_A f_k$$

and use MCT to see that

$$\sum_{k=1}^{\infty} \int_A f_k = \lim_{n\to\infty} \int_A g_n = \int_A \sum_{k=1}^{\infty} f_k.$$

(iii) We let $f_n = f\chi_{A_n}$ and we have $f\chi_{A_n} \in \overline{\mathcal{M}}^+(A)$. Also, $f = \sum_{i=1}^{\infty} f\chi_{A_i}$, so we appeal to (i) to get

$$\int_A f = \sum_{i=1}^{\infty} \int_{A_i} f.$$ □

3.6 Lebesgue integral

Let $\overline{\mathcal{M}}(A) = \{f : A \to \overline{\mathbb{R}} = [-\infty, \infty] : f \text{ is measurable}\}$. For $f \in \overline{\mathcal{M}}(A)$, let $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$ (pointwise). So $f = f^+ - f^-$, and also $|f| = f^+ + f^-$.

3.25 Definition. Let $\varnothing \neq A \in \mathcal{L}(\mathbb{R})$. We say $f : A \to \overline{\mathbb{R}}$ is **(Lebesgue) integrable** if $f \in \overline{\mathcal{M}}(A)$, and $\int_A f^+ - \int_A f^- < \infty$. In this case we define its **(Lebesgue) integral** by

$$\int_A f = \int_A f^+ - \int_A f^-.$$

We denote the set of such functions by $\overline{L}(A)$.

3.26 Aside. Part (i) of the next lemma tells us that $\overline{\mathbb{R}}$-valued integrable functions are finite except on a set of measure zero. Part (ii) incidentally has an application to L_p spaces; in particular, it will give us the nondegeneracy of the norm in the case $p = 1$.

3.27 Lemma. We have:

(i) $f \in \overline{L}(A)$ implies $\lambda\Big(f^{-1}(\{-\infty, \infty\})\Big) = 0$.

(ii) If $f \in \overline{\mathcal{M}}(A)$, then $\int_A |f| = 0$ if and only if $\lambda\Big(f^{-1}([-\infty, 0) \cup (0, \infty])\Big) = 0$.

Proof. We have:

(i) Since $f \in \overline{L}(A)$, we know $\int_A f^+ < \infty$. Define $E^+ = f^{-1}(\{\infty\})$. Then for any n, we have $n\chi_{E^+} \leq f^+$ (because $f^+ = \infty$ on E^+). Thus

$$n\lambda(E^+) = \int_A n\chi_{E^+} \leq \int_A f^+ < \infty$$

which means $\lambda(E^+) \leq \frac{1}{n}\int_A f^+$ for each n, so $\lambda(E^+) = 0$. Similarly, put $E^- = f^{-1}(\{-\infty\})$ and show $\lambda(E^-) = 0$. Finally

$$\lambda(f^{-1}(\{-\infty, \infty\})) = \lambda(f^{-1}(\{\infty\}) \cup f^{-1}(\{-\infty\})) = \lambda(E^+ \cup E^-) = 0.$$

(ii) ($\to$) Suppose $\int_A |f| = 0$. For all $n \in \mathbb{N}$, let

$$E_n = \{x \in A : |f(x)| \geq \tfrac{1}{n}\}.$$

Then $\frac{1}{n}\chi_{E_n} \leq |f|$. So $\frac{1}{n}\lambda(E_n) = \int_A \frac{1}{n}\chi_{E_n} \leq \int_A |f| = 0$. Hence $\lambda(E_n) = 0$. Hence

$$\bigcup_{n=1}^{\infty} E_n = \{x \in A : f(x) \neq 0\} = f^{-1}([-\infty, 0) \cup (0, \infty])$$

and the countable union of null sets is null. By σ-subadditivity,

$$\lambda\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \lambda(E_n), \text{ and } \lambda(E_n) = 0 \text{ so } \lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = 0.$$

($\leftarrow$) Suppose conversely that $\lambda\Big(f^{-1}([-\infty, 0) \cup (0, \infty])\Big) = 0$. Let $\varphi \in \mathcal{S}^+_{|f|}(A)$. Write

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$$

for $a_1 < a_2 < \ldots < a_n$. If at least one $a_i > 0$, then $0 < a_i\chi_{E_i} \leq \varphi \leq |f|$, so that $E_i \subseteq f^{-1}(\overline{\mathbb{R}} \setminus \{0\})$. By the increasing property of λ, we have

$$\lambda(E_i) \leq \lambda\Big(f^{-1}([-\infty, 0) \cup (0, \infty])\Big) = 0$$

which implies $\lambda(E_i) = 0$. Hence for all $\varphi \in \mathcal{S}^+_{|f|}(A)$, we have

$$\int_A \varphi = I_A(\varphi) = 0,$$

and so it follows that

$$\int_A |f| = \sup\{I_A(\varphi) : \varphi \in \mathcal{S}^+_{|f|}(A)\} = 0. \qquad \square$$

3.28 Remark. For $\varnothing \neq A \in \mathcal{L}(\mathbb{R})$,

$$\begin{aligned} \overline{\mathcal{M}}(A) &= \{f : A \to \overline{\mathbb{R}} : f \text{ is measurable}\} \\ \overline{L}(A) &= \{f : A \to \overline{\mathbb{R}} : f \text{ is integrable}\} \\ L(A) &= \{f : A \to \mathbb{R} : f \in \mathcal{M}(A), f \text{ is integrable}\}. \end{aligned}$$

3.29 Corollary (to lemma of last class). Let $f \in \overline{L}(A)$. Then there is $f_0 \in L(A)$ such that

$$f(x) = f_0(x)$$

except for $x \in N \subseteq A$, where $\lambda(N) = 0$. We will simply say that $f = f_0$ **almost everywhere (a.e.)** when this condition holds.

Proof. We saw that since $f \in \overline{L}(A)$, we have $\lambda(f^{-1}(\{-\infty, \infty\})) = 0$. We define $f_0 : A \to \mathbb{R}$ by

$$f_0(x) = \begin{cases} f(x) & x \notin f^{-1}(\{-\infty, \infty\}) \\ 0 & \text{otherwise.} \end{cases}$$

□

3.30 Remark. We will write for a function $f : A \to \mathbb{R}$ and a sequence $(f_n)_{n=1}^{\infty} \subseteq \mathcal{M}(A)$

$$f = \lim_{n\to\infty} f_n \text{ (a.e.)}$$

to mean that there is some set N with $\lambda(N) = 0$ and

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad \forall x \in A \setminus N.$$

Since null sets are measurable, we note that such f, as above, remain measurable i.e. $f \in \mathcal{M}(A)$.

Recall that if $f \in L(A)$ then

$$\int_A f = \int_A f^+ - \int_A f^-$$

is the Lebesgue integral of f over A.

3.31 Theorem (Properties of the integral). If $f, g \in L(A)$ and $c \in \mathbb{R}$ then

(i) $cf \in L(A)$ with $\int_A cf = c \int_A f$.

(ii) $f + g \in L(A)$ with $\int_A (f + g) = \int_A f + \int_A g$.

(iii) $|f| \in L(A)$ and we have $|\int_A f| \leq \int_A |f|$.

In fact, for $f : A \to \mathbb{R}$, we have

$$f \in L(A) \iff |f| \in L(A) \text{ and } f \in \mathcal{M}(A).$$

Proof. We have:

(i) Straightforward.

(ii) First, we note that $f + g = (f + g)^+ - (f + g)^-$, and we have that

$$(f + g)^+ \leq f^+ + g^+, \qquad (f + g)^- \leq f^- + g^-.$$

Hence, using previous results about integrating non-negative functions,

$$\int_A (f + g)^+ \leq \int_A (f^+ + g^+) = \underbrace{\int_A f^+}_{<\infty} + \underbrace{\int_A g^+}_{<\infty} < \infty$$

and similarly,

$$\int_A (f + g)^- < \infty$$

so that $f + g \in L(A)$.

We now claim the following: if $h, k, \varphi, \psi \in L^+(A)$ and $h - k = \varphi - \psi$ then

$$\int_A h - \int_A k = \int_A \varphi - \int_A \psi.$$

To see this, note that we have $h + \psi = \varphi + k$, so by the corollary (to MCT) we obtain

$$\int_A h + \int_A \psi = \int_A (h + \psi) = \int_A (\varphi + k) = \int_A \varphi + \int_A k$$

where each integral is finite. We subtract $\int_A k + \int_A \psi$ from both sides.

Back to the proof of (ii), we observe that

$$(f+g)^+ - (f+g)^- = f + g = f^+ - f^- + g^+ - g^- = (f^+ + g^+) - (f^- + g^-)$$

where $(f+g)^+, \dots, f^- + g^- \in L^+(A)$. Hence, by the claim,

$$\begin{aligned}\int_A (f+g) = \int_A (f+g)^+ - \int_A (f+g)^- &= \int_A (f^+ + g^+) - \int_A (f^- + g^-)\\ &= \int_A f^+ + \int_A g^+ - \left(\int_A f^- + \int_A g^-\right)\\ &= \int_A f^+ - \int_A f^- + \int_A g^+ - \int_A g^- = \int_A f + \int_A g.\end{aligned}$$

(iii) $|f| = f^+ + f^-$. Hence

$$\begin{aligned}\left|\int_A f\right| &= \left|\int_A f^+ - \int_A f^-\right|\\ &\leq \left|\int_A f^+\right| + \left|-\int_A f^-\right|\\ &= \int_A f^+ + \int_A f^-\\ &= \int_A (f^+ + f^-) = \int_A |f|.\end{aligned}$$

Note that $\int_A f^+, \int_A f^- < \infty$ and hence the sum is finite. Finally, we note that if $|f| \in L(A)$ and $f \in \mathcal{M}(A)$, the latter assumption tells us that $f^+, f^- \in \mathcal{M}(A)$. The first assumption gives that

$$\int_A f^+, \int_A f^- \leq \int_A f^+ + \int_A f^- = \int_A (f^+ + f^-) = \int_A |f| < \infty. \qquad \square$$

3.7 Dominated Convergence Theorem

Before introducing the Lebesgue dominated convergence theorem, we will require the following lemma.

3.32 Lemma (Fatou's lemma). If $(f_n)_{n=1}^\infty \subseteq \overline{\mathcal{M}}^+(A)$ then

$$\int_A \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_A f_n$$

Proof. Let $g_n = \inf\{f_k : k \geq n\}$. Then $0 \leq g_1 \leq g_2 \leq \dots$ and $\lim_{n\to\infty} g_n = \liminf_{n\to\infty} f_n$ by definition. Thus by MCT,

$$\int_A \liminf_{n\to\infty} f_n = \lim_{n\to\infty} \int_A g_n. \tag{†}$$

Now, $g_n \leq f_k$ for each $k \geq n$, so we find $\int_A g_n \leq \int_A f_k$ and we have

$$\int_A g_n \leq \liminf_{k\to\infty} \int_A f_k \tag{††}$$

Combining (†) and (††) we find

$$\int_A \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_A f_n. \qquad \square$$

3.33 Example. Let $f_n = n\chi_{(0,\frac{1}{n})}$. Then $\lim_{n\to\infty} f_n = 0$, pointwise, so

$$\liminf_{n\to\infty} f_n = 0.$$

However

$$\int_{\mathbb{R}} f_n = \int_{\mathbb{R}} n\chi_{(0,\frac{1}{n})} = n\lambda((0,\tfrac{1}{n})) = n\frac{1}{n} = 1.$$

Hence

$$\liminf_{n\to\infty} \int_{\mathbb{R}} f_n = 1.$$

Hence strict inequality can hold in Fatou's lemma.

3.34 Remark. Both MCT and Fatou's lemma hold when

$$f = \liminf_{n\to\infty} f_n$$

pointwise is replaced by

$$f = \liminf_{n\to\infty} f_n \text{ (a.e.)}.$$

3.35 Theorem (Lebesgue Dominated Convergence Theorem). If $A \in \mathcal{L}(\mathbb{R})$ with $\lambda(A) > 0$, and $(f_n)_{n=1}^{\infty} \subseteq \mathcal{M}(A)$ and $g \in L^+(A)$ such that

(i) there is $f : A \to \mathbb{R}$ such that

$$f = \lim_{n\to\infty} f_n \text{ (a.e.)}$$

on A.

(ii) $|f_n| \le g$ (a.e.) for each n (we call g an **integrable majorant**).

Then $f \in L(A)$, and we have

$$\int_A f = \lim_{n\to\infty} \int_A f_n.$$

Proof. Let

$$N = \bigcup_{n=1}^{\infty} \{x \in A : |f_n(x)| > g(x)\} \cup \{x \in A : \lim_{n\to\infty} f_n(x) \ne f(x) \text{ or limit DNE}\}$$

so by σ-subadditivity of λ (i.e. of λ^*), $\lambda(N) = 0$. We note that

$$\int_N f_n, \int_N g = 0$$

by the lemma, so we can replace A by $A \setminus N$, and just call the new set A. We note that $f = \lim_{n\to\infty} f_n$ (now pointwise) is measurable. Also

$$|f| = \lim_{n\to\infty} |f_n| \le g$$

so $\int_A |f| \le \int_A g < \infty$ so f is integrable. We consider, first, the functions $g + f_n \ge 0$ by assumption and $g + f = \lim_{n\to\infty}(g + f_n) = \liminf_{n\to\infty}(g + f_n)$ (pointwise). Then, by Fatou's lemma, we get

$$\int_A (g + f) \le \liminf_{n\to\infty} \int_A (g + f_n)$$

thus

$$\int_A g + \int_A f = \int_A (g + f) \le \liminf_{n\to\infty} \int_A (g + f_n) = \liminf_{n\to\infty} \left(\int_A g + \int_A f_n \right) = \int_A g + \liminf_{n\to\infty} \int_A f_n.$$

Thus

$$\int_A f \le \liminf_{n\to\infty} \int_A f_n \tag{$\dagger$}$$

and we also note that $g - f_n \ge 0$ with $g - f = \lim_{n\to\infty}(g - f_n) = \liminf_{n\to\infty}(g - f_n)$. As above, we obtain the following:

$$\int_A g - \int_A f = \int_A (g - f) \le \liminf_{n\to\infty} \int_A (g - f_n) = \liminf_{n\to\infty} \left(\int_A g - \int_A f_n \right) = \int_A g + \liminf_{n\to\infty} \left(- \int_A f_n \right)$$

$$= \int_A g - \limsup_{n\to\infty} \int_A f_n$$

thus we have

$$\int_A f \ge \limsup_{n\to\infty} \int_A f_n \tag{$\dagger\dagger$}$$

combining ($\dagger$) and ($\dagger\dagger$) we have

$$\limsup_{n\to\infty} \int_A f_n \le \int_A f \le \liminf_{n\to\infty} \int_A f_n$$

thus

$$\int_A f = \lim_{n\to\infty} \int_A f_n.$$

$\square$

4 L_p spaces

4.1 $p = 1$ case

We first treat L_p spaces where $p = 1$. Let $A \in \mathcal{L}(\mathbb{R})$, $\lambda(A) > 0$. (Usually, $A = [a, b]$, $[a, \infty)$, $\mathbb{R}$.) Recall

$$L(A) = \{f : A \to \mathbb{R} : f \text{ is measurable and integrable}\}.$$

4.1 Proposition. Define, for $f \in L(A)$, $\|f\|_1 = \int_A |f|$. Then

(i) $\|cf\|_1 = |c|\|f\|_1$ for $c \in \mathbb{R}$ ($|\cdot|$-homogeneity)

(ii) $\|f + g\|_1 \le \|f\|_1 + \|g\|_1$ for another $g \in L(A)$ (subadditivity)

Hence $\|\cdot\|_1$ is a seminorm. We are lacking nondegeneracy, so it is not a norm.

Proof. We have:

(i) Straightforward.

(ii) $\|f + g\|_1 = \displaystyle\int_A |f + g| \le \int_A (|f| + |g|) = \int_A |f| + \int_A |g| = \|f\|_1 + \|g\|_1.$ □

4.2 Definition. We define an equivalence relation on $L(A)$ by

$$f \sim g \iff f = g \text{ (a.e.)}$$

Check that this is an equivalence relation. We observe, from an earlier lemma[8], that

$$f \sim g \iff \int_A |f - g| = 0.$$

We define L_1 **space** on A by

$$L_1(A) = L(A)/\sim.$$

We note that $\sim$ is a linear equivalence:

$$f \sim f_1, g \sim g_1, c \in \mathbb{R} \implies f + cg \sim f_1 + cg_1$$

Hence $L_1(A)$ is a vector space. Also $f \sim f_1$ implies $|f| \sim |f_1|$ so $\|f\|_1 = \|f_1\|_1$ and hence $\|\cdot\|_1$ is well-defined on $L_1(A)$. Moreover, for $f \in L(A)$, $\|f\|_1 = 0$ if and only if $\int_A |f| = 0$ if and only if $f \sim 0$. On L_1 we have

(iii) $\|f\|_1 = 0$ if and only if $f = 0$ (in $L_1(A)$).

We think of elements of $L_1(A)$ as integrable functions with the agreement that $f = f_1$ in $L_1(A)$ if and only if $f = f_1$ (a.e.).

4.3 Remark (warning). For all $x \in A$, $\lambda(\{x\}) = 0$. Hence for any $c \in \mathbb{R}$, $f = f + c\chi_{\{x\}}$ in $L_1(A)$. Hence for $f \in L_1(A)$, we cannot make sense of "$f(x)$". However, we can make sense of "$f(x)$ for almost every x".

$$\int_A f = \int_A \underbrace{f(x)\,dx}_{\text{dealing with } x \text{ "in the large"}}$$

Note:

$$L(A) = \{f : A \to \mathbb{R} : \text{measurable and integrable}\}$$

is a set of functions. On the other hand, $L_1(A) = L(A)/\sim$ is a set of a.e.-equivalence classes.

4.4 Definition (L_1 convergence). We say that

$$f = \lim_{n\to\infty} f_n$$

in L_1 if $\lim_{n\to\infty} \|f - f_n\|_1 = 0$.

[8] Which lemma?

4.5 Remark. If $f, (f_n)_{n=1}^{\infty}$ in $L(A)$ and $f = \lim_{n\to\infty} f_n$ (a.e.) and there is $g \in L^+(A)$ such that $|f_n| \le g$ (a.e.) then (by LDCT) $f = \lim_{n\to\infty} f_n$ (in L_1). Indeed,

$$|f - f_n| \le |f| + |f_n| \le 2g$$

$$\lim_{n\to\infty} |f - f_n| = 0 \text{ (a.e.)}.$$

Thus,

$$\lim_{n\to\infty} \|f - f_n\|_1 = \int_A |f - f_n| \xrightarrow{n\to\infty} \int_A 0 = 0.$$

Question: Is it true that for $f, (f_n)_{n=1}^{\infty}$ in $L(A)$ that $f = \lim_{n\to\infty} f_n$ (in L_1) implies $f = \lim_{n\to\infty} f_n$ almost everywhere?

4.2 $1 < p < \infty$ case

4.6 Definition. Let $A \in \mathcal{L}(\mathbb{R})$, $\lambda(A) > 0$. (Usually, $A = [a,b]$, $[a,\infty)$, $\mathbb{R}$.) Define

$$L_p(A) = \left\{ f \in \mathcal{M}(A) : \int_A |f|^p < \infty \right\} \Big/ \sim$$

where $f \sim g$ if and only if $f = g$ (a.e.). For $f \in L_p(A)$, let

$$\|f\|_p = \left(\int_A |f|^p \right)^{1/p}.$$

We wish to show that $L_p(A)$ is a linear space, and that $\|\cdot\|_p$ is a norm on $L_p(A)$.

4.7 Definition. If $1 < p < \infty$ is fixed, we let q be defined by the expression

$$\frac{1}{p} + \frac{1}{q} = 1.$$

That is, $q = \frac{p}{p-1}$ and we call q the **conjugate** (or **dual**[9]) **index** to p.

4.8 Lemma (Young's Inequality). If $1 < p < \infty$ and q is the conjugate index, then for any $a, b \ge 0$ we have

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

with equality if and only if $a^p = b^q$.

Proof. Let $0 < \alpha < 1$ and define $\varphi : [0,\infty) \to \mathbb{R}$ by

$$\varphi(t) = \alpha t - t^{\alpha}.$$

We have

$$\varphi'(t) = \alpha - \alpha t^{\alpha-1} = \alpha\left(1 - \frac{1}{t^{1-\alpha}}\right)$$

and we have $\varphi'(t) < 0$ for $0 < t < 1$ and $\varphi'(t) > 0$ for $t > 1$. Thus, by MVT we have that

$$\alpha t - t^{\alpha-1} = \varphi(t) \ge \varphi(1) = \alpha - 1$$

with equality exactly when $t = 1$. Thus, for $t \ge 0$,

$$t^{\alpha} \le \alpha t - (1 - \alpha)$$

with equality only for $t = 1$. Assume $b \ne 0$ since for $b = 0$ the desired inequality is obvious. Let $t = a^p/b^q$. We get

$$\frac{a^{p\alpha}}{b^{q\alpha}} \le \alpha \frac{a^p}{b^q} - (1 - \alpha)$$

and hence

$$a^{p\alpha} b^{q(1-\alpha)} \le \alpha a^p + (1 - \alpha) b^q.$$

Now we let $\alpha = \frac{1}{p}$ so $1 - \alpha = \frac{1}{q}$, and we're done. □

[9]This terminology is no mistake: the continuous dual of L_p is isomorphic to L_q, where q is the dual index to p (we consider 1 and ∞ as a dual pair of indices). Observe that $p = 2$ is self-dual.

4.9 Theorem (Hölder's inequality). Let $1 < p < \infty$, $A \in \mathcal{L}(\mathbb{R})$, $\lambda(A) > 0$. Let q be the conjugate index. If $f \in L_p(A)$, and $g \in L_q(A)$, then $fg \in L_1(A)$ and

$$\|fg\|_1 \le \|f\|_p \|g\|_q$$

with equality holding only if $\|g\|_q^q |f|^p = \|f\|_p^p |g|^q$.

Proof. If either $\|f\|_p = 0$ or $\|g\|_q = 0$ then $fg = 0$ (a.e.) and the (in)equality is trivial. We assume that $\|f\|_p \|g\|_q > 0$. Let for almost every $x \in A$

$$a(x) = \frac{|f(x)|}{\|f\|_p}, b(x) = \frac{|g(x)|}{\|g\|_q}$$

we have for almost every $x \in A$ that

$$\frac{|f(x)g(x)|}{\|f\|_p\|g\|_q} = a(x)b(x) \le \frac{a(x)^p}{p} + \frac{g(x)^q}{q} = \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|}{q\|g\|_q^q} \tag{$*$}$$

with equality holding if and only if $a(x)^p = b(x)^q$, i.e.

$$\frac{|f(x)|^p}{\|f\|_p^p} = \frac{|g(x)|^q}{\|g\|_q^q}$$

we note that fg defines a measurable function and hence $|fg|$ is measurable. Integrating $(*)$ we find

$$\frac{1}{\|f\|_p\|g\|_q} \int_A |fg| \le \int_A \left(\frac{|f|^p}{p\|f\|_p^p} + \frac{|g|^q}{q\|g\|_q} \right) = \frac{1}{p\|f\|_p^p} \underbrace{\int_A |f|^p}_{\|f\|_p^p} + \frac{1}{q\|g\|_q^q} \underbrace{\int_A |g|^q}_{\|g\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1.$$

Hence

$$\|fg\|_1 = \int_A |fg| \le \|f\|_p \|g\|_q.$$

We note that equality holds only when equality holds in $(*)$ for almost every x. ☐

4.10 Theorem (Minkowski's inequality). Let $1 < p < \infty$, $A \in \mathcal{L}(\mathbb{R})$, $\lambda(A) > 0$. If $f, g \in L_p(A)$ then $f + g \in L_p(A)$ and moreover

$$\|f + g\|_p \le \|f\|_p + \|g\|_p$$

with equality holding if and only if there are constants $c_1, c_2 \ge 0$ such that $c_1 + c_2 > 0$ and $c_1 f = c_2 g$ (a.e.).

Proof. First, note that pointwise almost everywhere,

$$|f + g|^p \le (2 \max\{|f|, |g|\})^p = 2^p \max\{|f|^p, |g|^p\} \le 2^p(|f|^p + |g|^p)$$

and hence

$$\int_A |f + g|^p \le 2^p \left(\int_A |f|^p \int_A |g|^p \right) < \infty.$$

So $f + g \in L_p(A)$. If $\|f + g\|_p = 0$, then the inequality is trivial. Let's assume $\|f + g\|_p > 0$. Now, we have

$$|f + g|^p = |f + g||f + g|^{p-1} \le |f||f + g|^{p-1} + |g||f + g|^{p-1} \text{ (a.e.)} \tag{$\dagger$}$$

so note that $|f + g|^{p-1} \in L_q(A)$, since $q = \frac{p}{p-1}$. Thus, by Hölder's inequality we have

$$\int_A |f||f + g|^{p-1} \le \||f|\|_p \||f + g|^{p-1}\|_q = \left(\int_A |f|^p \right)^{1/p} \left(\int_A |f + g|^{(p-1)q} \right)^{1/q} = \|f\|_p \|f + g\|_p^{p/q} \tag{$\dagger\dagger$}$$

and similarly

$$\int_A |g||f + g|^{p-1} \le \|g\|_p \|f + g\|_p^{p/q}.$$

Combining $(\dagger)$ and $(\dagger\dagger)$ we see that

$$\begin{aligned} \|f + g\|_p^p &= \int_A |f + g|^p \\ &\le \int_A |f||f + g|^{p-1} + \int_A |g||f + g|^{p-1} && \text{(A)} \\ &\le \left(\|f\|_p + \|g\|_p \right) \|f + g\|_p^{p/q} && \text{(B)} \end{aligned}$$

and we note that

$$p - \frac{p}{q} = p(1 - \frac{1}{q}) = p\frac{1}{p} = 1$$

so we have

$$\|f + g\|_p \le \|f\|_p + \|g\|_p$$

Equality at (A) requires $|f + g| = |f| + |g|$ (a.e.) and at (B) is the equality condition from Hölder. $\square$

4.3 Completeness

We will see that $L_p(A)$ is actually not just a normed linear space, but a Banach space (it is complete).

$$L_p(A) = \left\{ f \in \mathcal{M}(A) : \int_A |f|^p < \infty \right\} /\sim$$

where $A \in \mathcal{L}(\mathbb{R})$, $\lambda(A) > 0$.

4.11 Lemma. Let $(\mathcal{X}, \|\cdot\|)$ be a normed vector space. Then $\mathcal{X}$ is complete if and only if for every sequence $(x_n)_{n=1}^{\infty} \subseteq \mathcal{X}$ with

$$\sum_{n=1}^{\infty} \|x_n\| < \infty$$

then we have

$$\sum_{n=1}^{\infty} x_n = \lim_{N\to\infty} \sum_{n=1}^{N} x_n$$

converges in $\mathcal{X}$.

Proof. $(\to)$ (Abstract Weierstrass Test) Let $(x_n)_{n=1}^{\infty} \subseteq \mathcal{X}$ with $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Let $s_n = \sum_{k=1}^{n} x_k$. Then if $m < n$, we observe

$$\|s_m - s_n\| = \left\| \sum_{k=n+1}^{m} x_k \right\| \le \sum_{k=n+1}^{m} \|x_k\|$$

Since $\sum_{k=1}^{\infty} \|x_k\| < \infty$, $\|s_m - s_n\|$ can be made small. So $\{s_n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{X}$. As $\mathcal{X}$ is complete, $\{s_n\}_{n\in\mathbb{N}}$ converges, to $s \in \mathcal{X}$.

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} x_k = \sum_{k=1}^{\infty} x_k.$$

$(\leftarrow)$ Let $(x_n)_{n=1}^{\infty} \subseteq \mathcal{X}$ be Cauchy. Pick n_1 such that $n, m \ge n_1$ implies $\|x_n - x_m\| < \frac{1}{2}$. Pick n_2 such that $n_2 \ge n_1$ and $n, m \ge n_2$ implies $\|x_n - x_m\| < \frac{1}{2^2}$. And so on; in general choosing n_k such that $n_k \ge n_{k-1}$ and $n, m \ge n_k$ implies $\|x_n - x_m\| < \frac{1}{2^k}$. So we get a subsequence $(x_{n_k})_{k=1}^{\infty}$. For each k, let $y_k = x_{n_{k+1}} - x_{n_k}$. Then

$$\sum_{j=1}^{k} \|y_j\| = \sum_{j=1}^{k} \|x_{n_j} - x_{n_{j-1}}\| < \sum_{j=1}^{k} \frac{1}{2^j}.$$

So,

$$\sum_{j=1}^{\infty} \|y_j\| = \lim_{N\to\infty} \sum_{j=1}^{N} \|y_j\| \le \lim_{N\to\infty} \sum_{j=1}^{N} \frac{1}{2^j} = 1.$$

By hypothesis,

$$x = \lim_{j\to\infty} \sum_{k=1}^{j} y_k \text{ exists.}$$

We observe that by telescoping,

$$\sum_{k=1}^{j} y_k = \sum_{k=1}^{j} x_{n_k} - x_{n_k-1} = x_{n_{k+1}} - x_{n_1}$$

hence

$$x + x_{n_1} = \lim_{j\to\infty} x_{n_{j+1}}$$

exists. However $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence and we have shown that it has a convergent subsequence. So, $(x_n)_{n=1}^{\infty}$ itself also converges. (Details left as exercise). $\square$

4.12 Theorem. Let $A \in \mathcal{L}(\mathbb{R})$, with $\lambda(A) > 0$. Then $L_p(A)$ is complete.

Proof. We use the lemma. Let $(f_n)_{n=1}^{\infty} \subseteq L_p(A)$. Call

$$M := \sum_{n=1}^{\infty} \|f_n\|_p < \infty.$$

We consider each f_n as a measurable function on A with $\int |f|^p < \infty$. Let

$$g_n = \sum_{k=1}^{n} |f_k|$$

so $g_1 \leq g_2 \leq g_3 \leq \dots$ and for each x, put $g(x) = \lim_{n\to\infty} g_n(x)$ (pointwise). We observe that

$$\|g_n\|_p \leq \sum_{k=1}^{n} \||f_k|\|_p = \sum_{k=1}^{n} \|f_k\|_p \leq \underbrace{\sum_{k=1}^{\infty} \|f_k\|_p}_{M} < \infty.$$

Hence by MCT we find that[10]

$$\int_A g^p = \lim_{n\to\infty} \int g_n^p = \lim_{n\to\infty} \|g_n\|_p^p \leq M^p < \infty.$$

So g^p is integrable, hence by a previous lemma $g(x) < \infty$ almost everywhere on A. Thus g represents an element in $L_p(A)$. We then observe that, for almost everywhere $x \in A$,

$$\sum_{k=1}^{n} |f_k(x)| = g_n(x) \leq g(x)$$

thus

$$\sum_{k=1}^{\infty} |f_k(x)| < \infty$$

for almost everywhere $x \in A$. Hence for such x,

$$f(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x).$$

That is,

$$f = \lim_{n\to\infty} \sum_{k=1}^{n} f_k \text{ (a.e.)}.$$

Observe that

$$|f|^p = \left| \lim_{n\to\infty} \sum_{k=1}^{n} f_k \right|^p \leq \lim_{n\to\infty} \Bigg(\underbrace{\sum_{k=1}^{n} |f_k|}_{g_n} \Bigg)^p = \lim_{n\to\infty} g_n^p = g^p \text{ (a.e.)}$$

So,

$$\int_A |f|^p \leq \int_A g^p < \infty$$

thus f is a representative of an element in $L_p(A)$. It remains to show that

$$\left\| f - \sum_{k=1}^{n} f_k \right\|_p \to 0$$

as $n \to \infty$. We observe that

$$\left| f - \sum_{k=1}^{n} f_k \right|^p \leq \left(|f| + \left| \sum_{k=1}^{n} f_k \right| \right)^p \leq (g+g)^p = 2^p g^p$$

[10] We liberally neglect absolute value signs here, since g and g_k are all non-negative by definition.

where

$$\int 2^p g^p < \infty.$$

Note that

$$\lim_{n\to\infty} \left| f - \sum_{k=1}^{n} f_k \right| = 0 \text{ (a.e.)} \implies \lim_{n\to\infty} \left| f - \sum_{k=1}^{n} f_k \right|^p = 0 \text{ (a.e.)}.$$

Therefore, the Lebesgue dominated convergence theorem allows us to conclude that

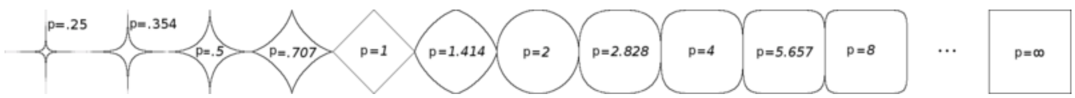
$$\lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} f_k \right\|_p^p = \lim_{n\to\infty} \int_A \left| f - \sum_{k=1}^{n} f_k \right|^p = \int_A 0 = 0.$$

Hence, $\sum_{k=1}^{\infty} f_k = f$ in $(L_p(A), \|\cdot\|_p)$. Therefore by the lemma $L_p(A)$ is complete. □

4.13 Remark (analogy). Take $\mathbb{R}^2$, $1 \le p < \infty$.

$$\left\|(x_1, x_2)\right\|_p = (|x_1|^p + |x_2|^p)^{1/p}.$$

Unit ball $B_p = \{(x_1, x_2) \in \mathbb{R}^2 : \|(x_1, x_2)\|_p \le 1\}$.



4.4 $p = \infty$ case

4.14 Definition. If $f \in \mathcal{M}(A)$, $A \in \mathcal{L}(\mathbb{R})$, $\lambda(A) > 0$, we define

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in A} |f(x)| = \inf\{C > 0 : \lambda(\{x \in A : |f(x)| > C\}) = 0\}.$$

If $\|f\|_\infty < \infty$ we say f is **essentially bounded**. Let

$$L_\infty(A) = \{f \in \mathcal{M}(A) : \|f\|_\infty < \infty\}/\sim$$

Hence $L_\infty(A)$ consists of (equivalence classes of) essentially bounded and measurable functions. We agree that $f = g$ in $L_\infty(A)$ if $f = g$ almost everywhere.

4.15 Proposition. $\|\cdot\|_\infty$ is a norm on $L_\infty(A)$.

Proof. First, if $f \in L_\infty(A)$ then $\|f\|_\infty \ge 0$, by definition. If $\|f\|_\infty = 0$ then

$$\lambda(\{x \in A : |f(x)| > \tfrac{1}{n}\}) = 0$$

hence $\{x \in A : f(x) \ne 0\} = \{x \in A : |f(x)| > 0\} = \bigcup_{n=1}^{\infty} \{x \in A : |f(x)| > \frac{1}{n}\}$ and a countable union of null sets is null. Check that $\|cf\|_\infty = |c|\|f\|_\infty$, $c \in \mathbb{R}$, $f \in L_\infty(A)$. Now let $f, g \in L_\infty(A)$. First, note that

$$\{x \in A : |f(x)| > \|f\|_\infty\} = \bigcup_{n=1}^{\infty} \underbrace{\{x \in A : |f(x)| > \|f\|_\infty + \tfrac{1}{n}\}}_{\text{null set by definition of ess sup and inf}}$$

so that

$$\lambda(\{x \in A : |f(x)| > \|f\|_\infty\}) = 0.$$

We may assume $\|f\|_\infty + \|g\|_\infty > 0$ otherwise the proof is trivial. Consider

$$\begin{aligned} \{x \in A : |f(x) + g(x)| > \|f\|_\infty + \|g\|_\infty\} &\subseteq \{x \in A : |f(x)| + |g(x)| > \|f\|_\infty + \|g\|_\infty\} \\ &\subseteq \{x \in A : |f(x)| > \|f\|_\infty\} \cup \{x \in A : |g(x)| > \|g\|_\infty\} \end{aligned}$$

but these are both null sets, and the union of two null sets is null. Hence it follows that

$$\|f + g\|_\infty = \inf\{C > 0 : \lambda(\{x \in A : |f(x) + g(x)| > C\}) = 0\}.$$ □

4.16 Theorem. $(L_\infty(A), \|\cdot\|_\infty)$ is complete and hence a Banach space.

Proof. We take $(f_k)_{k=1}^\infty \subseteq L_\infty(A)$, and suppose that

$$\sum_{k=1}^\infty \|f_k\|_\infty < \infty.$$

We need to show that $\sum_{k=1}^\infty f_k$ defines an element of $L_\infty(A)$. Let

$$E_k = \{x \in A : |f_k(x)| > \|f_k\|_\infty\},$$

which is null. Hence, if we put $E = \bigcup_{k=1}^\infty E_k$ then this is null as well. Now, if $x \in A \setminus E$, then we have for each $n \in \mathbb{N}$, that

$$\left|\sum_{k=1}^n f_k(x)\right| \le \sum_{k=1}^n \underbrace{|f_k(x)|}_{\le \|f_k\|_\infty \text{ since } x \notin E_k} \le \sum_{k=1}^n \|f_k\|_\infty \le \sum_{k=1}^\infty \|f_k\|_\infty < \infty.$$

Thus for $x \in A \setminus E$ we have by absolute convergence,

$$\left|\sum_{k=1}^\infty f_k(x)\right| \le \sum_{k=1}^\infty \|f_k\|_\infty$$

and thus $\sum_{k=1}^\infty f_k$ (pointwise almost everywhere) defines an element of $L_\infty(A)$. $\square$

4.5 Modes of convergence

Suppose $(f_k)_{k=1}^\infty$, f in $\mathcal{M}(A)$. We have, already, notions of

$$\lim_{k\to\infty} f_k = f \text{ (pointwise)},$$

$$\lim_{k\to\infty} f_k = f \text{ (a.e.)},$$

$$\lim_{k\to\infty} f_k = f \in L_p.$$

By the latter, we mean to say that each f_k is in $L_p(A)$ and

$$\lim_{k\to\infty} \|f_k - f\|_p = \lim_{k\to\infty} \left(\int_A |f_k - f|^p\right)^{1/p} = 0$$

(for $1 \le p < \infty$). In L_∞ case,

$$\lim_{k\to\infty} \|f_k - f\|_\infty = \lim_{k\to\infty} \operatorname*{ess\,sup}_{x\in A} |f_k(x) - f(x)| = 0.$$

4.17 Example. Let

$$f_1 = \chi_{[0,1]},$$

$$f_2 = \chi_{[0,\frac{1}{2}]}, f_3 = \chi_{[\frac{1}{2},1]}$$

$$f_4 = \chi_{[0,\frac{1}{3}]}, f_5 = \chi_{[\frac{1}{2},\frac{2}{3}]}, f_6 = \chi_{[\frac{2}{3},1]}$$

$$f_7 = \chi_{[0,\frac{1}{4}]}, \dots$$

$$f_8 = \chi_{[0,\frac{1}{5}]}, \dots$$

First, note that for $x \in [0,1]$

$$\lim_{k\to\infty} f_k(x) \text{ does not exist.}$$

Indeed, $\limsup f_k(x) = 1$ and $\liminf f_k(x) = 0$ for all x. Hence,

$$\lim_{k\to\infty} f_k \text{ D.N.E. (a.e.)}$$

$$\|f_k - 0\|_p = \left(\int_{[0,1]} |f_k|^p \right)^{1/p}$$

but $|f_k|^p$ is the indicator function of an interval length $1/n_k$, so the above is

$$\left(\frac{1}{n_k} \right)^{1/p} \xrightarrow[n_k \to \infty \text{ as } k \to \infty]{k \to \infty} 0^{1/p} = 0$$

(Likely, there is $c > 0$, $n_k = c \log k$.) Note, for every $f_k = \chi_{[a_k, b_k]}$, $a_k < b_k$, we have

$$\|f_k - 0\|_\infty = \|f_k\|_\infty = 1.$$

Each $c > 1$ is an essential bound, but no $c < 1$ is an essential bound as $\lambda([a_k, b_k]) = b_k - a_k > 0$.

4.6 Inclusion relations

4.18 Theorem. Let $[a, b]$ be a compact interval with $b > a$ (i.e. $A \in \mathcal{L}(\mathbb{R})$ such that $0 < \lambda(A) < \infty$). Then for $1 \le p < r < \infty$ we have that

$$L_r[a, b] \subseteq L_p[a, b]$$

and for $f \in L_r[a, b]$,

$$\|f\|_p \le (b - a)^{\frac{r-p}{pr}} \|f\|_r$$

where the coefficient

$$C = (b - a)^{\frac{r-p}{pr}}$$

is just a constant[11] C which depends on $[a, b]$, p and r.

Proof. Let $f \in L_r[a, b]$. Then

$$|f|^p \in L_{r/p}[a, b]$$

i.e.

$$\int_{[a,b]} (|f|^p)^{r/p} = \int_{[a,b]} |f|^r < \infty,$$

by assumption. Let q be the conjugate index to r/p, i.e.

$$\frac{1}{q} + \frac{1}{r/p} = 1 \implies \frac{1}{q} = 1 - \frac{p}{r} = \frac{r-p}{r} \implies q = \frac{r}{r-p}.$$

By Hölder's inequality we have

$$\begin{aligned} \int_{[a,b]} |f|^p = \int_{[a,b]} |f|^p \cdot 1 &\le \left(\int_{[a,b]} (|f|^p)^{r/p} \right)^{p/r} \left(\int_{[a,b]} |1|^q \right)^{1/q} \\ &= \left[\left(\int_{[a,b]} |f|^r \right)^{1/r} \right]^p (b-a)^{1/q} \\ &= \|f\|_r^p (b-a)^{1/q}. \end{aligned}$$

Hence

$$\|f\|_p \le \|f\|_r (b-a)^{\frac{1}{pq}}$$

but $\frac{1}{pq} = \frac{r-p}{pr}$. ∎

4.19 Remark. We have the following:

1. It is an easy exercise to show that $L_\infty[a, b] \subseteq L_p[a, b]$, for $1 \le p < \infty$, and there is $k > 0$ (depending on $[a, b]$ and p) such that $\|f\|_p \le k \|f\|_\infty$.

$$C[a, b] \subsetneq L_\infty[a, b] \subsetneq L_p[a, b] \subsetneq L_1[a, b]$$

with the first two coming from A4, and the last being shown below.

[11] There is no need to memorize this constant.

2. If $1 \le p < r < \infty$, then $L_p[a,b] \nsubseteq L_r[a,b]$.

   Proof. Let $[a,b] = [0,1]$. Let
   $$f(x) = \frac{1}{x^{1/r}} \text{ (a.e.)}$$
   Compute (where we have applied A3Q4 in the second step)
   $$\int_{[0,1]} |f(x)|^p \, dx = \int_{[0,1]} \frac{1}{x^{p/r}} \, dx = \lim_{a \to 0^+} \int_a^1 x^{-p/r} \, dx = \lim_{a \to 0^+} \frac{1}{1 - \frac{p}{r}} x^{1-\frac{p}{r}} \Big|_a^1$$
   noting that $p < r$ so $p/r < 1$ so $1 - \frac{p}{r} > 0$ we get that the above is
   $$\frac{r}{r-p} < \infty$$
   so $f \in L_p([0,1])$. It is easy to check that
   $$\int_{[0,1]} |f|^r = \infty.$$
   □

Are there any containment relations for $L_p(\mathbb{R})$ and $L_r(\mathbb{R})$ with $1 \le p < r < \infty$? No. This is proved below.

4.20 Theorem. $L_p(\mathbb{R}) \nsubseteq L_r(\mathbb{R})$.

Proof. We have for each $s \ge 1$, an embedding
$$L_s([0,1]) \hookrightarrow L_s(\mathbb{R}),$$
if $f \in L_s([0,1])$ we define
$$\tilde{f} = \begin{cases} f & \text{on } [0,1] \text{ (a.e.)} \\ 0 & \text{off } [0,1]. \end{cases}$$
Then
$$\underbrace{\|f\|_s}_{\in L_s([0,1])} = \underbrace{\|\tilde{f}\|_s}_{\in L_s(\mathbb{R})} .$$
We pick our favourite $f \in L_p([0,1]) \setminus L_r([0,1])$ and then $\tilde{f} \in L_p(\mathbb{R}) \setminus L_r(\mathbb{R})$. □

4.21 Theorem. $L_r(\mathbb{R}) \nsubseteq L_p(\mathbb{R})$.

Proof. Define
$$f(x) = \begin{cases} \frac{1}{x^{1/p}} & \text{for a.e. } x \ge 1 \\ 0 & \text{for a.e. } x < 1. \end{cases}$$
Check that $f \in L_r(\mathbb{R}) \setminus L_p(\mathbb{R})$. □

4.22 Theorem (separability). If $a < b$ in $\mathbb{R}$, and $1 \le p < \infty$, then $L_p[a,b]$ is separable (that is, we can find a countable dense subset).

Proof. First, by A4, we note $C[a,b] \subseteq L_p[a,b]$ with $\|f\|_p \le k\|f\|_\infty$ for $f \in C[a,b]$ with a fixed constant $k > 0$. And $C[a,b]$ is dense in $L_p[a,b]$ (with respect to the p-norm). We "recall" that $(C[a,b], \|\cdot\|_\infty)$ is separable. First, let $\mathbb{R}[x]$ denote the space of polynomials on $[a,b]$. By the Stone-Weierstrass theorem, $\overline{\mathbb{R}[x]}^{\|\cdot\|_\infty} = C[a,b]$. Now we have $\mathbb{Q}[x]$ is countable, call this set $\{d_n\}_{n=1}^\infty$. For each $p \in \mathbb{R}[x]$ and $\epsilon > 0$ there is a polynomial $d \in \mathbb{Q}[x]$ such that
$$\|p - d\|_\infty < \epsilon.$$
Now, if $f \in L_p[a,b]$, and $\epsilon > 0$, we first find $h \in C[a,b]$ such that
$$\|f - h\|_p < \frac{\epsilon}{2} \quad \text{(A4)}.$$
Then, find a $p \in \mathbb{R}[x]$ such that
$$\|h - p\|_\infty < \frac{\epsilon}{4k}$$
and then $d_n \in \mathbb{Q}[x]$ such that
$$\|p - d_n\|_\infty < \frac{\epsilon}{4k}$$

We have

$$\|f - d_n\|_p \le \|f - h\|_p + \|h - d_n\|_p < \frac{\epsilon}{2} + k\|h - d_n\|_\infty < \epsilon.$$

This is because $\|h - d_n\|_\infty \le \frac{\epsilon}{2k}$, by choices above. $\square$

4.23 Theorem. $L_\infty[0,1]$ is not separable.

Proof. For each binary sequence $a = \{a_1, a_2, \dots\} \subseteq \{0,1\}$, that is, $a \in \{0,1\}^{\mathbb{N}}$ we let

$$f_a = \sum_{n=1}^{\infty} a_n \chi_{[\frac{1}{n+1}, \frac{1}{n}]}$$

We observe that if $a, b \in \{0,1\}^{\mathbb{N}}$, then

$$\|f_a - f_b\|_\infty = \left\| \underbrace{\sum_{n=1}^{\infty} (a_n - b_n)}_{\text{pointwise a.e.}} \chi_{[\frac{1}{n+1}, \frac{1}{n}]} \right\|_\infty = \sup_{n \in \mathbb{N}} |a_n - b_n|.$$

Thus, if $a \ne b$, $\|f_a - f_b\|_\infty = 1$. If there were a dense subset $\{d_n\}_{n=1}^{\infty}$ of $L_\infty[0,1]$ then for each $a \in \{0,1\}^{\mathbb{N}}$ there would be a $n = n(a)$ such that $\|f_a - d_{n(a)}\|_\infty < \frac{1}{2}$. We note that $n(a) \ne n(b)$ for $a \ne b$, for otherwise we have

$$\|f_a - f_b\|_\infty \le \|f_a - \underbrace{d_{n(a)} + d_{n(b)}}_{0} - f_b\|_\infty \le \|f_a - d_{n(a)}\|_\infty + \|d_{n(b)} - f_b\|_\infty < 1$$

which contradicts $(*)$. Thus $a \mapsto n(a) : \{0,1\}^{\mathbb{N}} \to \mathbb{N}$ is injective, which implies that $|\{0,1\}^{\mathbb{N}}| \le |\mathbb{N}|$ which is absurd. $\square$

4.24 Remark. We note that for $a < b$ in $\mathbb{R}$, if $f \in L_\infty[a,b]$, then

$$\lim_{p \to \infty} \|f\|_p = \|f\|_\infty.$$

Proof outline. One might prove

(a) $f \in \mathcal{S}[a,b]$, then $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$.

(b) If $f \in L_\infty[a,b]$ and $\epsilon > 0$ then there is a $g \in \mathcal{S}[a,b]$ with $\|f - g\|_\infty < \epsilon$.

Combine (a) and (b) carefully to finish. $\square$

5 Fourier analysis

We need to talk about some functional analysis on our L_p spaces.

5.1 Bounded operators

5.1 Definition. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. A linear transformation $T : \mathcal{X} \to \mathcal{Y}$ is **bounded** provided

$$\|\!|T|\!\| = \sup\{\|Tx\|_{\mathcal{Y}} : x \in \mathcal{X}, \|x\|_{\mathcal{X}} < 1\} < \infty.$$

If $\mathcal{Y} = \mathbb{R}$, we call a linear map $\Gamma : \mathcal{X} \to \mathbb{R}$ a **linear functional**. We will write $\|\Gamma\|_* = \|\!|\Gamma|\!\|$.

5.2 Proposition. Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, and let $T : \mathcal{X} \to \mathcal{Y}$ be a linear operator. Then the following are equivalent:

(i) T is continuous.

(ii) T is bounded.

(iii) T is Lipschitz, in fact,

$$\|Tx - Tx'\|_{\mathcal{Y}} \le \|\!|T|\!\| \cdot \|x' - x\|_{\mathcal{X}}$$

and moreover $\|\!|T|\!\|$ is the smallest[12] $C > 0$ such that $\|Tx\|_{\mathcal{Y}} \le C\|x\|_{\mathcal{X}}$ for all $x \in \mathcal{X}$.

[12] In other words, $\|\!|T|\!\|$ is the "best Lipschitz constant" for T.

Of course, this holds for a functional $\Gamma : \mathcal{X} \to \mathbb{R}$ as well.

Proof. (i) → (ii): Let $B_1(\mathcal{Y}) = \{y \in \mathcal{Y} : \|y\|_{\mathcal{Y}} < 1\}$ which is an open neighbourhood of $0_{\mathcal{Y}}$. Since T is continuous and $T0_{\mathcal{X}} = 0_{\mathcal{Y}}$ we have that there is $\delta > 0$ such that if

$$\|x - 0_{\mathcal{X}}\|_{\mathcal{X}} < \delta$$

then $\|Tx - 0_{\mathcal{Y}}\|_{\mathcal{Y}} < 1$, i.e. $\|x\|_{\mathcal{X}} < \delta$ implies $\|Tx\|_{\mathcal{Y}} < 1$. Suppose $x \in \mathcal{X}$, $\|x\|_{\mathcal{X}} < 1$. Then

$$\|\delta x\|_{\mathcal{X}} = \delta \|x\|_{\mathcal{X}} < \delta \cdot 1 = \delta.$$

Hence

$$\delta \|Tx\|_{\mathcal{Y}} = \|T(\delta x)\|_{\mathcal{Y}} < 1$$

and thus $\|Tx\|_{\mathcal{Y}} < \frac{1}{\delta}$, so

$$|\!|\!|T|\!|\!| = \sup\{\|Tx\|_{\mathcal{Y}} : x \in \mathcal{X}, \|x\|_{\mathcal{X}} < 1\} \leq \frac{1}{\delta}.$$

(ii) → (iii): We have for $x \in \mathcal{X}$, $\epsilon > 0$ that

$$\left\| \frac{1}{\|x\|_{\mathcal{X}} + \epsilon} x \right\|_{\mathcal{X}} = \frac{1}{\|x\|_{\mathcal{X}} + \epsilon} \|x\|_{\mathcal{X}} < 1$$

and hence

$$\frac{1}{\|x\|_{\mathcal{X}} + \epsilon} \|Tx\|_{\mathcal{Y}} = \left\| T \left(\frac{1}{\|x\|_{\mathcal{X}} + \epsilon} x \right) \right\|_{\mathcal{Y}} \leq |\!|\!|T|\!|\!|.$$

Thus $\|Tx\|_{\mathcal{Y}} \leq |\!|\!|T|\!|\!| (\|x\|_{\mathcal{X}} + \epsilon)$. Letting $\epsilon \to 0^+$ we have $\|Tx\|_{\mathcal{Y}} \leq |\!|\!|T|\!|\!| \|x\|_{\mathcal{X}}$. If $x, x' \in \mathcal{X}$, we have

$$\|Tx - Tx'\|_{\mathcal{Y}} = \|T(x - x')\|_{\mathcal{Y}} \leq |\!|\!|T|\!|\!| \|x - x'\|_{\mathcal{X}}.$$

Finally, if $0 < C < |\!|\!|T|\!|\!|$ then since $|\!|\!|T|\!|\!|$ is the supremum, there is $x \in B_1(\mathcal{X})$ such that

$$\|Tx\|_{\mathcal{Y}} > C > C\|x\|_{\mathcal{X}}$$

i.e. $\|Tx - T0\|_{\mathcal{Y}} > C\|x - 0\|_{\mathcal{X}}$, so C is not a Lipschitz estimate.

(iii) → (i): Lipschitz implies uniformly continuous implies continuous. □

5.2 Linear functionals

Fix $A \in \mathcal{L}(\mathbb{R})$, with $\lambda(A) > 0$.

5.3 Theorem (functionals on L_p, for $1 < p < \infty$). Let $1 < p < \infty$, and let q be the conjugate index. If $g \in L_q(A)$, then the functional

$$\Gamma_g : L_p(A) \to \mathbb{R}, \qquad f \mapsto \int_A gf$$

is a bounded linear functional with $\|\Gamma_g\|_* = \|g\|_q$.

We have the following remark (*whose proof is relegated to PMATH 451*), which tells us that the correspondence $g \mapsto \Gamma_g$ described above is actually a surjection from $L_q(A) \to L_p(A)^*$, where $*$ denotes the continuous dual space. This correspondence is indeed an isomorphism, thereby justifying our use of the term "dual index".

5.4 Remark. If $\Gamma : L_p(A) \to \mathbb{R}$ is a bounded linear functional then there is $g \in L_q(A)$ such that $\Gamma = \Gamma_g$. (This is the stuff of PMATH 451 – Radon-Nikodym theorem).

Proof of theorem. First, if $g \in L_q(A)$ and $f \in L_p(A)$ then by Hölder's inequality, $gf \in L_1(A)$ and we have

$$|\Gamma_g(f)| = \left| \int_A gf \right| \leq \int_A |gf| = \|gf\|_1 \leq \|g\|_q \|f\|_p.$$

We saw that $\|\Gamma_g\|_*$ is the smallest $C > 0$ such that

$$|\Gamma_g(f)| \leq C\|f\|_p,$$

and thus $\|\Gamma_g\|_* \leq \|g\|_q$ (it is easy to verify that Γ_g is linear).

To gain the converse inequality, let us take a cue (i.e. hint) from the "equality" case of Hölder's inequality. We have that

$$\int_A |fg| = \|f\|_p \|g\|_q$$

provided $|f|^p = C|g|^q$. We let $\operatorname{sgn} : \mathbb{R} \to \{-1, 1\}$ be given by

$$\operatorname{sgn}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0. \end{cases}$$

This is called the **signum** function. Notice that sgn is Borel measurable: $\operatorname{sgn}^{-1}((\alpha, \infty)) \in \mathcal{B}(\mathbb{R})$ for any $\alpha \in \mathbb{R}$. Exercise:

(i) $\operatorname{sgn}^{-1}(B) \in \mathcal{B}(\mathbb{R})$ for any $B \in \mathcal{B}(\mathbb{R})$.

(ii) If $g \in \mathcal{M}(A)$, then $\operatorname{sgn} \circ g \in \mathcal{M}(A)$.

We define

$$f = C|g|^{q/p} \operatorname{sgn} \circ g$$

where C, to be defined later, is set so $\|f\|_p = 1$. We check

$$\int_A |f|^p = \int_A \left| C|g|^{q/p} \operatorname{sgn} \circ g \right|^p = C^p \int_A \left(|g|^{q/p}\right)^p \underbrace{|\operatorname{sgn} \circ g|^p}_{=1} = C^p \int_A |g|^q = C^p \|g\|_q^q.$$

That is to say,

$$\|f\|_p^p = C^p \|g\|_q^q \implies \|f\|_p = C \|g\|_q^{q/p}.$$

So we want

$$C = \frac{1}{\|g\|_q^{q/p}} \quad \text{to obtain} \quad \|f\|_p = 1.$$

We compute

$$\begin{aligned} \|\Gamma_g\|_* &= \sup\{|\Gamma_g(f)| : f \in L_p(A), \|f\|_p \leq 1\} \\ &\geq \left| \Gamma_g \left(\frac{1}{\|g\|_q^{q/p}} |g|^{q/p} \operatorname{sgn} \circ g \right) \right| = \left| \int_A g \frac{1}{\|g\|_q^{q/p}} |g|^{q/p} \operatorname{sgn} \circ g \right| \end{aligned}$$

and note that $g \operatorname{sgn} \circ g = |g|$, so the above is

$$\frac{1}{\|g\|_q^{q/p}} \underbrace{\int_A |g|^{\frac{q}{p}+1}}_{q(\frac{1}{p}+\frac{1}{q})=q} = \frac{1}{\|g\|_q^{q/p}} \int_A |g|^q = \underbrace{\|g\|_q^{q-\frac{q}{p}}}_{q(1-\frac{1}{p})=1} = \|g\|_q$$

thus $\|\Gamma_g\|_* = \|g\|_q$. ☐

5.5 Theorem (functionals on L_1). If $\varphi \in L_\infty(A)$, define $\Gamma_\varphi : L_1(A) \to \mathbb{R}$ by putting

$$\Gamma_\varphi(f) = \int_A \varphi f.$$

Then Γ_φ is a bounded linear functional with

$$\|\Gamma_\varphi\|_* = \|\varphi\|_\infty.$$

Proof. Let us first observe that for $f \in L_1(A)$, we have $|\varphi f| \leq \|\varphi\|_\infty |f|$ almost everywhere (recall $\|\cdot\|_\infty$ is the essential supremum). Thus,

$$\int_A |\varphi f| \leq \|\varphi\|_\infty \int_A |f| = \|\varphi\|_\infty \|f\|_1.$$

Hence $\varphi f \in L_1(A)$ and we have the "1-∞ Hölder inequality"

$$|\Gamma_\varphi(f)| = \left| \int_A \varphi f \right| \leq \int_A |\varphi f| \leq \|\varphi\|_\infty \|f\|_1$$

so $\|\Gamma_\varphi\|_* \le \|\varphi\|_\infty$. (Of course, we see that Γ_φ is linear on $L_1(A)$). It remains to verify $\|\Gamma_\varphi\|_* \ge \|\varphi\|_\infty$. Let, for $\epsilon > 0$,

$$A_\epsilon = \{x \in A : \|\varphi\|_\infty - \epsilon \le |\varphi(x)|\}.$$

Then $\lambda(A_\epsilon) > 0$ by definition of $\|\varphi\|_\infty$. It may be that $\lambda(A_\epsilon) = \infty$; if this is the case, simply replace A_ϵ with any subset $S \subseteq A_\epsilon$ satisfying $0 < \lambda(S) < \infty$. So we can assume $0 < \lambda(A_\epsilon) < \infty$. Let

$$f_\epsilon = \frac{1}{\lambda(A_\epsilon)} \chi_{A_\epsilon} \operatorname{sgn} \circ \varphi$$

so that

$$\|f_\epsilon\|_1 = \int_A \left| \frac{1}{\lambda(A_\epsilon)} \chi_{A_\epsilon} \right| = \frac{1}{\lambda(A_\epsilon)} \int_A \chi_{A_\epsilon} = \frac{\lambda(A_\epsilon)}{\lambda(A_\epsilon)} = 1.$$

We have

$$\|\Gamma_\varphi\|_* \ge |\Gamma_\varphi(f_\epsilon)| = \left| \int_A \varphi \frac{1}{\lambda(A_\epsilon)} \chi_{A_\epsilon} \operatorname{sgn} \circ \varphi \right| = \frac{1}{\lambda(A_\epsilon)} \int_A \underbrace{|\varphi| \chi_{A_\epsilon}}_{\ge (\|\varphi\|_\infty - \epsilon)\chi_{A_\epsilon}} \ge \frac{\|\varphi\|_\infty - \epsilon}{\lambda(A_\epsilon)} \int_A \chi_{A_\epsilon}$$
$$= \|\varphi\|_\infty - \epsilon.$$

Thus $\|\Gamma_\varphi\|_* \ge \|\varphi\|_\infty - \epsilon$. Taking $\epsilon \to 0$ we obtain $\|\Gamma_\varphi\|_* = \|\varphi\|_\infty$. ☐

5.6 Theorem (functionals on L_∞ and C). Let $a < b$ in $\mathbb{R}$.

(a) If $f \in L_1[a, b]$ then the functional $\Gamma_f : L_\infty[a, b] \to \mathbb{R}$ given by $\Gamma_f(\varphi) = \int_{[a,b]} f\varphi$, is linear and bounded with $\|\Gamma_f\|_* = \|f\|_1$.

(b) Furthermore, we consider

$$\Gamma_f : C[a, b] \to \mathbb{R}$$

(Recall, for A4Q1, this "is" a closed subspace[13]). Then

$$\|\Gamma_f\|_* = \sup\{|\Gamma_f(h)| : h \in C[a, b], \|h\|_\infty \le 1\} = \|f\|_1.$$

Proof. We have:

(a) We recall the 1-∞ version of Hölder's inequality

$$\int_{[a,b]} |\varphi f| \le \|\varphi\|_\infty \|f\|_1$$

which tells us that $\|\Gamma_f\|_* \le \|f\|_1$. (It is clear that Γ_f is linear). Consider $\varphi = \operatorname{sgn} f$, so $\|\varphi\|_\infty \le 1$. We have that

$$\|\Gamma_f\|_* = \sup\left\{ \left| \int_{[a,b]} f\varphi \right| : \varphi \in L_\infty[a, b], \|\varphi\|_\infty \le 1 \right\} \ge \left| \int_{[a,b]} \underbrace{f \operatorname{sgn} \circ f}_{|f|} \right| = \int_{[a,b]} |f| = \|f\|_1.$$

Hence $\|\Gamma_f\| = \|f\|_1$.

(b) From the proof of A4Q1, we have that there exists a sequence $(h_n)_{n=1}^\infty \subseteq C[a, b]$, such that

- $\|h_n\|_\infty \le 1$, i.e. $|h_n| \le 1$.
- $\lim_{n\to\infty} h_n = \operatorname{sgn} \circ f$ a.e.

We note that $|fh_n| \le |f||h_n| \le |f|$, so $|f|$ is an integrable majorant of $(fh_n)_{n=1}^\infty$. Thus

$$\left| \int_{[a,b]} fh_n \right| \xrightarrow{n\to\infty} \left| \int_{[a,b]} f \operatorname{sgn} \circ f \right| = \int_{[a,b]} |f| = \|f\|_1.$$

Hence, as a functional on $C[a, b]$,

$$\|\Gamma_f\|_* \ge \sup_{n\in\mathbb{N}} \left| \int_{[a,b]} fh_n \right| \ge \lim_{n\to\infty} \left| \int_{[a,b]} fh_n \right| = \|f\|_1$$

whereas

$$\sup\left\{ \left| \int_{[a,b]} fh \right| : h \in C[a, b], \|h\|_\infty \le 1 \right\} \le \sup\left\{ \left| \int_{[a,b]} f\varphi \right| : \varphi \in L_\infty[a, b], \|\varphi\|_\infty \le 1 \right\} = \|f\|_1.$$ ☐

[13] There is an embedding, that is. Recall that things in L_p are equivalence classes, not "actual" functions.

5.3 Fourier series

Motivation: heat equation on the disc. [diagram: unit circle in complex plane]. $z = x + iy = |z|e^{i\theta} = re^{i\theta}$ (polar coordinates). $u = u(r, \theta)$ – temperature on the disc.

$$0 = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \text{ on interior,} \qquad T(z) = T(e^{i\theta}) = f(\theta) = u(1, \theta) \text{ (boundary condition).}$$

This is a PDE with boundary condition. Some candidate solutions:

$$\begin{aligned} u_0(r,\theta) &= a_0 \qquad \text{(const.)} \\ u_n(r,\theta) &= a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta), \qquad n \in \mathbb{N} \\ &= a_n r^n \frac{e^{in\theta} + e^{-in\theta}}{2} + b_n r^n \frac{e^{in\theta} - e^{-in\theta}}{2i} \\ &= r^n (c_n e^{in\theta} + c_{-n} e^{-in\theta}), \qquad c_n = \frac{a_n - ib_n}{2}, c_{-n} = \frac{a_n + ib_n}{2} \end{aligned}$$

Boundary condition (Fourier):

$$f(\theta) = u(1, \theta) = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{in\theta} + c_{-n} e^{-in\theta} \right) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}.$$

Question: What do we mean by

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}?$$

Pointwise convergence? Uniform convergence? L_p-convergence ($1 \le p < \infty$)? Etc.?

We now discuss measurability of complex-valued functions.

5.7 Definition (complex-valued functions). A function $f : [a, b] \to \mathbb{C}$ is **measurable** provided the real and imaginary parts

$$\operatorname{Re} f, \operatorname{Im} f : [a, b] \to \mathbb{R}$$

are measurable. If $\operatorname{Re} f, \operatorname{Im} f$ are both integrable we define

$$\int_a^b f = \int_a^b \operatorname{Re} f + i \int_a^b \operatorname{Im} f.$$

It is a tedious exercise to verify that

$$\int_a^b (f + \alpha g) = \int_a^b f + \alpha \int_a^b g$$

for integrable $f, g : [a, b] \to \mathbb{C}$, $\alpha \in \mathbb{C}$.

5.8 Remark. LDCT, Hölder and Minkowski inequalities, all hold in this setting. However, MCT and Fatou's lemma are theorems for non-negative real-valued functions only.

5.9 Remark. From now on, we let $\mathcal{M}_{\mathbb{C}}[a, b] = \{f : [a, b] \to \mathbb{C} : f \text{ is measurable}\}$. We then put

$$\begin{aligned} L[a,b] &= \left\{ f \in \mathcal{M}_{\mathbb{C}}[a,b] : \int_a^b |f| < \infty \right\} \\ L_p[a,b] &= \left\{ f \in \mathcal{M}_{\mathbb{C}}[a,b] : \int_a^b |f|^p < \infty \right\} /\!\sim \qquad 1 \le p < \infty \\ L_\infty[a,b] &= \{ f \in \mathcal{M}_{\mathbb{C}}[a,b] : \operatorname*{ess\,sup}_{x \in [a,b]} |f(x)| < \infty \} /\!\sim \\ C[a,b] &= \{ f : [a,b] \to \mathbb{C} : f \text{ is continuous} \}. \end{aligned}$$

Notice $\theta \mapsto e^{in\theta}$ is 2π-periodic. That is, $e^{in(\theta + 2\pi)} = e^{in\theta}$.

5.10 Definition (Spaces of 2π-periodic functions). Define

$$C(\mathbb{T}) = \{f : \mathbb{R} \to \mathbb{C} : f \text{ is continuous and } 2\pi\text{-periodic}\} \cong \{f \in C[-\pi, \pi] : f(-\pi) = f(\pi)\}$$

and for $1 \le p \le \infty$,

$$L_p(\mathbb{T}) = \{f \in \mathcal{M}_{\mathbb{C}}(\mathbb{R}) : f \text{ is a.e. } 2\pi\text{-periodic and } f\Big|_{[-\pi,\pi]} \text{ “}\in\text{” } L_p[-\pi, \pi]\}/\sim$$

For $1 \le p < \infty$, we equip $L_p(\mathbb{T})$ with the norm

$$\|f\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^p\right)^{1/p}$$

noting the modification factor of $\frac{1}{2\pi}$.

5.11 Remark. We set up some notation.

1. For $n \in \mathbb{Z}$, let $\mathbf{e}^n(t) := e^{int}$. Note that each $\mathbf{e}^n$ is $\frac{2\pi}{n}$ periodic.
2. Let
$$\mathrm{Trig}(\mathbb{T}) := \mathrm{span}_{\mathbb{C}}\{\mathbf{e}^n : n \in \mathbb{Z}\} = \left\{\sum_{n=-N}^{N} c_n\mathbf{e}^n : N \in \mathbb{N}, c_n \in \mathbb{C}\right\}$$
denote the set of **trigonometric polynomials**.
3. Let formal series of the form
$$\sum_{n=-\infty}^{\infty} c_n\mathbf{e}^n \quad (c_n \in \mathbb{C})$$
be called **Fourier series**.

Goal: Let $f \in L(\mathbb{T})$, i.e. f is an a.e. 2π periodic, complex-valued, measurable function, which is Lebesgue integrable on $[-\pi, \pi]$ (note we may view as $L[-\pi, \pi]$, with the understanding that f repeats outside of $[-\pi, \pi]$). (Note that since $L_p[a, b] \subset L_1[a, b]$ for $p \ge 1$, we may view $L(\mathbb{T})$ as “containing” all spaces $L_p(\mathbb{T})$ for $p \ge 1$.) Our goal is to find a Fourier series $\sum_{n=-\infty}^{\infty} c_n(f)\mathbf{e}^n$ (where $c_n(f)$ means $c_n \in \mathbb{C}$ is a function of f) which “represents” f.

5.4 Fourier coefficients

Let us suppose that we may write

$$f(t) = \sum_{n=-\infty}^{\infty} c_n(f)\mathbf{e}^n$$

where “=” is taken to mean pointwise equality. Observe that for every fixed $N \in \mathbb{Z}$, if we permit ourselves the use of a certain questionable operation $(*)$,

$$\begin{aligned}
\int_{-\pi}^{\pi} f\mathbf{e}^{-N} &= \int_{-\pi}^{\pi} f(t)e^{-iNt}\,dt = \int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} c_n(f)e^{i(n-N)t}\,dt \\
&= \sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} c_n(f)e^{i(n-N)t}\,dt \qquad (*) \\
&= \sum_{n=-\infty}^{\infty} c_n(f)\left[\int_{-\pi}^{\pi}\cos((n-N)t)\,dt + i\int_{-\pi}^{\pi}\sin((n-N)t)\,dt\right] \\
&= 2\pi c_N(f)
\end{aligned}$$

since

$$\int_{-\pi}^{\pi}\cos((n-N)t)\,dt = \begin{cases} 2\pi & n = N \\ 0 & n \ne N \end{cases} \quad \text{and} \quad \int_{-\pi}^{\pi}\sin((n-N)t)\,dt = 0.$$

[“If the operation $(*)$ does not make you feel anxious, you will hate the rest of this course.” — N. Spronk.] Hence, we may derive the n**th Fourier coefficient of** f, $c_n(f)$, by

$$c_n(f) = \frac{1}{2\pi}\int_{[-\pi,\pi]} f\mathbf{e}^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt.$$

5.12 Definition. Define the n**th Fourier sum of** f by

$$s_n(f) := \sum_{k=-n}^{n} c_k(f)\mathbf{e}^k, \qquad \text{so that} \qquad s_n(f,t) := s_n(f)(t) = \sum_{k=-n}^{n} c_k(f)e^{ikt}.$$

We notice that if $f = g$ (a.e.) then $2\pi c_k(f) = \int_{-\pi}^{\pi} f\mathbf{e}^{-k} = \int_{-\pi}^{\pi} g\mathbf{e}^{-k} = 2\pi c_k(g)$ and hence their Fourier sums should be equal (*pointwise*).

Let us take another look at $s_n(f)$:

$$\begin{aligned} s_n(f,t) &= \sum_{k=-n}^{n} c_k(f)e^{ikt} = \sum_{k=-n}^{n} \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} f(s)e^{-iks}\,ds\right]e^{ikt} = \sum_{k=-n}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)e^{ik(t-s)}\,ds \\ &\overset{(\dagger)}{=} \sum_{k=-n}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t+s)e^{-iks}\,ds \overset{(\ddagger)}{=} \sum_{k=-n}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)e^{iks}\,ds \overset{(*)}{=} \int_{-\pi}^{\pi}\frac{1}{2\pi}\sum_{k=-n}^{n} e^{iks}f(t-s)\,ds. \end{aligned}$$

where $(\dagger)$ is called **translation invariance of the integral**, and $(\ddagger)$ is called **inversion invariance of the integral** (for periodic functions) and these are problems on Assignment 5. Notice the $(*)$ operation cropping up again here.

5.13 Remark. The last line in the above derivation is usually shortened in applied mathematics to

$$D_n * f(t) := \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(s)f(t-s)\,ds, \qquad \text{where} \qquad D_n := \sum_{k=-n}^{n}\mathbf{e}^k.$$

Here, D_n is called the **Dirichlet kernel of order** n, and $D_n * f$ is called the **convolution product** of D_n and f.

Now that we have in place the notion of a partial sum of the Fourier series and we have "calculated" what we suspect to be the coefficients of the series, it is a good time to ask ourselves again the question we posed earlier: Given $f \in L(\mathbb{T})$ (or $f \in L_p(\mathbb{T})$), how do we understand the statement $f(t) = \sum_{k=-\infty}^{\infty} c_k(f)e^{ikt}$? (Pointwise, a.e., convergence in some L_p norm?)

5.14 Definition. A **homogeneous Banach space over** $\mathbb{T}$ is a subspace $\mathcal{B} \subseteq L_1(\mathbb{T})$, where

$$L_1(\mathbb{T}) = \{f : \mathbb{R} \to \mathbb{C} : f \text{ is measurable, a.e. } 2\pi\text{-periodic } \int_{-\pi}^{\pi}|f| < \infty\}/\sim.$$

together with its own norm $\|\cdot\|_{\mathcal{B}}$, under which $\mathcal{B}$ is a Banach space, and for which

(A) $\mathrm{Trig}(\mathbb{T}) \subseteq \mathcal{B}$.

(B) $s * f \in \mathcal{B}$ for $s \in \mathbb{R}$, $f \in \mathcal{B}$ where $(s * f)(t) := f(t-s)$ for (a.e.) $t \in \mathbb{R}$, and

  (i) $\|s * f\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$ for all $s \in \mathbb{R}$, $f \in \mathcal{B}$.

  (ii) For each $f \in \mathcal{B}$, the map $(s \mapsto s * f) : \mathbb{R} \to \mathcal{B}$ is continuous.

5.15 Example. Here are some examples:

(i) Consider the space

$$C(\mathbb{T}) = \{f : \mathbb{R} \to \mathbb{C} : f \text{ is } 2\pi\text{-periodic and continuous}\}.$$

We observe that since $f(t + 2\pi) = f(t)$ for all t,

$$\sup_{t\in\mathbb{R}}|f(t)| = \sup_{t\in[-\pi,\pi]}|f(t)| < \infty$$

since $[-\pi, \pi]$ is compact. Clearly $\mathrm{Trig}(\mathbb{T}) \subseteq C(\mathbb{T})$. Now we consider translations. If $s \in \mathbb{R}$, $f \in C(\mathbb{T})$, then $s * f \in C(\mathbb{T})$ (exercise). Also $\|s * f\|_\infty = \|f\|_\infty$. We note that $f \in C(\mathbb{T})$ is determined by its values on $[-\pi, \pi]$, and hence on $[-2\pi, 2\pi]$. Let $\epsilon > 0$ be given. Since $[-\pi, \pi]$ is compact, f is *uniformly* continuous on it, so we can take $\delta > 0$ such that for $s, s' \in [-\pi, \pi]$ with $|s - s'| < \delta$ we have

$$|f(s) - f(s')| < \epsilon.$$

If $t \in \mathbb{R}$, find $n \in \mathbb{Z}$ such that

$$t + 2\pi n \in [-\pi, \pi].$$

We observe, by 2π-periodicity, that

$$|(s*f)(t)-(s'*f)(t)|=|f(t-s)-f(t-s')|=|f(t+2\pi n-s)-f(t+2\pi n-s')|<\epsilon$$

since $t+2\pi n-s$ and $t+2\pi n-s'$ both live in $[-2\pi,2\pi]$, and $|(t+2\pi n-s)-(t+2\pi n-s')|=|s-s'|<\delta$. Taking the supremum over all $t\in\mathbb{R}$ we find that

$$\|(s*f)-(s'*f)\|_\infty\le\epsilon$$

and thus $(s\mapsto s*f):[-\pi,\pi]\to C(\mathbb{T})$ is (uniformly) continuous.

(ii) Consider $L_p(\mathbb{T})$ where $1\le p<\infty$:

$$L_p(\mathbb{T})=\{f:\mathbb{R}\to\mathbb{C}: f \text{ is a.e. } 2\pi\text{-periodic, measurable, and } \int_{-\pi}^{\pi}|f|^p<\infty\}/\sim$$

under the norm

$$\|f\|_p=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^p\right)^{1/p}.$$

We saw that $L_p(\mathbb{T})\subseteq L_1(\mathbb{T})$. Also $\mathrm{Trig}(\mathbb{T})\subseteq C(\mathbb{T})\underbrace{\subseteq}_{A4}L_p(\mathbb{T})$. We have that

$$\int_{-\pi}^{\pi}|s*f|^p=\int_{-\pi}^{\pi}s*|f|^p\underbrace{=}_{A5}\int_{-\pi}^{\pi}|f|^p.$$

Before studying the existence of a non-measurable set, we saw a property called the translation invariance of the Lebesgue measure (this is a hint for showing this on A5). Hence,

$$\|s*f\|_p=\|f\|_p$$

and $s*f\in L_p(\mathbb{T})$. Finally, if $f\in L_p(\mathbb{T})$ and $\epsilon>0$, we find $h\in C(\mathbb{T})$ such that $\|h-f\|_p<\epsilon/3$ [practically a question from A4]. By the (uniform) continuity of h, let $\delta>0$ be such that for $s,s'\in\mathbb{R}$ with $|s-s'|<\delta$ we have

$$\|s*h-s'*h\|_\infty<\frac{\epsilon}{3}.$$

We have

$$\begin{aligned}\|s*f-s'*f\|_p&=\|s*f+s*h-s*h+s'*h-s'*h-s'*f\|_p\\&\le\|s*f-s*h\|_p+\|s*h-s'*h\|_p+\|s'*h-s'*f\|_p\\&=\|f-h\|_p+\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|s*h-s'*h|^p\right)^{1/p}+\|h-f\|_p\end{aligned}$$

however by construction of h, the first and third terms are $<\frac{\epsilon}{3}$, and the second is $<\frac{\epsilon}{3}$ since $\|s*h-s'*h\|_\infty<\frac{\epsilon}{3}$. So we conclude

$$\|s*f-s'*f\|_p<\frac{\epsilon}{3}+\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}(\epsilon/3)^p\right)^{1/p}+\frac{\epsilon}{3}=\epsilon.$$

Hence $(s\mapsto s*f):\mathbb{R}\to L_p(\mathbb{T})$ is (uniformly) continuous.

(iii) Consider $L_\infty(\mathbb{T})$ with $\|\cdot\|_\infty$. Most conditions of a homogeneous Banach space are satisfied. However, given $f\in L_\infty(\mathbb{T})$, we may not have continuity of $s\mapsto s*f:\mathbb{R}\to L_\infty(\mathbb{T})$. Consider the "square wave" given by $f=\chi_{[0,\pi]}$ on $[-\pi,\pi]$ and then extending it 2π-periodically to all of $\mathbb{R}$. Now if $0<|s|<\pi$, we have

$$\|s*f-f\|_\infty\ge 1.$$

Indeed,

$$E=\{t\in\mathbb{R}:|f(t)-f(t-s)|\ge 1\}=\bigcup_{n\in\mathbb{Z}}[n\pi,n\pi+s]$$

which is non-null (check this). Thus,

$$\left(\lim_{s\to 0}=\right)\liminf_{s\to 0}\|s*f-\underbrace{0*f}_{f}\|_\infty\ge 1.$$

For homogeneous Banach spaces, we can define something called convolution by continuous functions. Convolution is motivated by the computation with the Dirichlet kernel. We will show the convolution operators are always continuous, and we will compute their norms, at least in the case when we're dealing with L_1 (continuous functions with uniform norm) and study the particular norm applied to the Dirichlet kernel and get surprising results.

5.5 Convolutions

Let $\mathcal{B}$ be a homogeneous Banach space over $\mathbb{T}$ and let $h \in C(\mathbb{T})$ [we may also allow h to be 2π-periodic, bounded and piecewise continuous.]

5.16 Definition. For $f \in \mathcal{B}$, we define the **convolution** of h and f by

$$h * f = \frac{1}{2\pi} \underbrace{\int_{-\pi}^{\pi} h(s) s * f \, ds}_{\text{vector-valued Riemann}} .$$

We observe that our assumptions on h provide that the map $[-\pi, \pi] \to \mathcal{B}$ given by

$$s \mapsto h(s) s * f$$

is continuous (piecewise continuous, bounded).

5.17 Remark. Convolution is commutative, since we have for a.e. $t \in \mathbb{R}$,

$$h * f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s) f(t-s) \, ds \overset{T.I.}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} h(t+s) f(-s) \, ds \overset{I.I}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} h(t-s) f(s) \, ds = f * h(t).$$

Recall the *Dirichlet kernel of order n.*

$$D_n = \sum_{k=-n}^{n} \mathbf{e}^k \in \mathrm{Trig}(\mathbb{T}) \subseteq C(\mathbb{T}).$$

Note that

$$\begin{aligned} D_n * f(t) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(t-s) f(s) \, ds \\ &= \sum_{k=-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik(t-s)} f(s) \, ds \\ &= \sum_{k=-n}^{n} \underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iks} f(s) \, ds}_{c_k(f), \text{the } k\text{th Fourier coefficient}} \cdot e^{ikt} \\ &= \sum_{k=-n}^{n} c_k(f) e^{ikt} = s_n(f,t). \end{aligned}$$

5.18 Proposition. If $\mathcal{B}$ is a homogeneous Banach space over $\mathbb{T}$ and $h \in C(\mathbb{T})$ [or, piecewise continuous, bounded, 2π-periodic] then the convolution operator $C(h) : \mathcal{B} \to \mathcal{B}$ given by

$$C(h) f := h * f$$

is linear and bounded, with

$$\underbrace{\|C(h)\|_{\mathcal{B}}}_{\substack{\text{Lipschitz} \\ \text{constant of } C(h)}} \leq \|h\|_1.$$

Proof. The linearity of $C(h)$ is a consequence of the linearity of Riemann integration. Also we have for $f \in \mathcal{B}$,

$$\|C(h)f\|_{\mathcal{B}} = \|h * f\|_{\mathcal{B}} = \left\| \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s) s * f \, ds \right\|_{\mathcal{B}} \overset{A1}{\leq} \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(s)| \|s * f\|_{\mathcal{B}} \, ds \leq \|h\|_1 \|f\|_{\mathcal{B}}.$$ $\square$

5.19 Theorem (norms of convolution operators). Let $h \in C(\mathbb{T})$. Then

(i) $\|C(h)\|_{C(\mathbb{T})} = \|h\|_1$.

(ii) $\|C(h)\|_{L_1(\mathbb{T})} = \|h\|_1$.

Proof. We have:

(i) Let $f \in C(\mathbb{T})$. Then

$$h * f(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s) f(0-s)\, ds$$

$[f \mapsto f(0)$ is a linear functional with norm 1]. By inversion invariance,

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{h(-s)}_{=:\check{h}(s)} f(s)\, ds = \Gamma_{\check{h}}(f).$$

Hence we have

$$\|C(h)f\|_\infty \geq |h * f(0)| = |\Gamma_{\check{h}}(f)|.$$

Thus,

$$\begin{aligned}
\|C(h)\|_{C(\mathbb{T})} &= \sup\{\|C(h)f\|_\infty : f \in C(\mathbb{T}), \|f\|_\infty \leq 1\} \\
&\geq \sup\{|\Gamma_{\check{h}}(f)| : f \in C(\mathbb{T}), \|f\|_\infty \leq 1\} \\
&= \|\Gamma_{\check{h}}\|_* \text{ by definition} \\
&= \|\check{h}\|_1 \quad \text{by earlier theorem (5.6)} \\
&= \|h\|_1.
\end{aligned}$$

The last proposition showed that $\|C(h)\|_{C(\mathbb{T})} \leq \|h\|_1$.

(ii) As above, we need only establish that $\|C(h)\|_{L_1(\mathbb{T})} \geq \|h\|_1$. Let for $n \in \mathbb{N}$,

$$f_n = \pi n \chi_{[-1/n, 1/n]}.$$

Then

$$\|f_n\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f_n| = 1.$$

Now for a.e. t we have

$$h * f_n(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s) f_n(t-s)\, ds \overset{T.I}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} h(s+t) \underbrace{f_n(-s)}_{f_n(s)}\, ds = \frac{n}{2} \int_{-1/n}^{1/n} h(s+t)\, ds.$$

Given $\epsilon > 0$ there is $\delta > 0$ so that $|h(t) - h(s+t)| < \epsilon$ for $|s - 0| < \delta$. Then for n such that $\frac{1}{n} < \delta$, we have

$$\|h - h * f_n\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(t) - h * f_n(t)|\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| h(t) - \frac{n}{2} \int_{-1/n}^{1/n} h(s+t)\, ds \right| dt.$$

Now

$$h(t) = \frac{n}{2} \int_{-1/n}^{1/n} h(t)\, ds$$

so the above is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{n}{2} \left| \int_{-1/n}^{1/n} (h(t) - h(t+s))\, ds \right| dt \overset{A1}{\leq} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{n}{2} \int_{-1/n}^{1/n} \underbrace{|h(t) - h(t+s)|}_{\leq \epsilon}\, ds\, dt \leq \epsilon.$$

Hence

$$\lim_{n \to \infty} \|h - h * f_n\|_1 = 0$$

and thus due to the reverse triangle inequality,

$$\lim_{n \to \infty} \left| \|h\|_1 - \|h * f_n\|_1 \right| \leq \lim_{n \to \infty} \|h - h * f_n\|_1 = 0.$$

Thus

$$\begin{aligned}
\|C(h)\|_{L_1(\mathbb{T})} &= \sup\{\|C(h)f\|_1 : f \in L_1(\mathbb{T}), \|f\|_1 \leq 1\} \\
&\geq \sup_{n \in \mathbb{N}}\{\|C(h)f_n\|_1\} \\
&\geq \lim_{n \to \infty} \|\underbrace{C(h)f_n}_{h * f_n}\|_1 = \|h\|_1.
\end{aligned}$$

□

Consequence: if $f \in C(\mathbb{T})$ [or $f \in L_1(\mathbb{T})$]

$$\|s_n(f)\|_{\mathcal{B}} = \|D_n * f\|_{\mathcal{B}} = \|C(D_n)f\|_{\mathcal{B}}.$$

If we can understand the sequence of operators $C(D_n)$ acting on $\mathcal{B}$, then we may be able to understand

$$s_n(f) = \sum_{k=-n}^{n} c_k(f)\mathbf{e}^k.$$

5.20 Theorem (Properties of the Dirichlet kernel). The Dirichlet kernel of order n, D_n, satisfies:

(i) D_n is $\mathbb{R}$-valued, 2π-periodic and even.

(ii) $\displaystyle\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n = 1.$

(iii) For $t \in [-\pi, \pi]$ we have

$$D_n(t) = \begin{cases} \dfrac{\sin((n+\frac{1}{2})t)}{\sin(\frac{1}{2}t)} & \text{if } t \neq 0 \\ 2n+1 & \text{if } t = 0. \end{cases}$$

(iv) $\displaystyle\lim_{n\to\infty} \|D_n\|_1 = \infty.$

Note that we often call $\|D_n\|_1 =: L_n$ the **nth Lebesgue constant**.

Proof. We have:

(i) D_n is 2π-periodic, because

$$D_n = \sum_{j=-n}^{n} \mathbf{e}^j \in \mathrm{Trig}(\mathbb{T}).$$

Also, it is even, because

$$D_n(-s) = \sum_{j=-n}^{n} e^{ij(-s)} = \sum_{j=-n}^{n} e^{i(-j)s} = \sum_{j=-n}^{n} e^{ijs} = D_n(s).$$

We shall see from (iii) that D_n is $\mathbb{R}$-valued.

(ii) We have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n = \frac{1}{2\pi}\sum_{k=-n}^{n}\int_{-\pi}^{\pi} \mathbf{e}^k$$

but

$$\int_{-\pi}^{\pi} \mathbf{e}^k = \begin{cases} 0 & \text{if } k \neq 0 \\ 2\pi & \text{if } k = 0 \end{cases}$$

therefore the above is simply equal to 1.

(iii) Suppose, first, that $t \neq 0$. We have

$$\begin{aligned} D_n(t)(e^{-i\frac{1}{2}t} - e^{i\frac{1}{2}t}) &= (e^{-int} + e^{-i(n-1)t} + \ldots + e^{i(n-1)t} + e^{int})(e^{-i\frac{1}{2}t} - e^{i\frac{1}{2}t}) \\ &= (e^{-i(n+\frac{1}{2})t} + \ldots + e^{i(n-\frac{1}{2})t}) - (e^{-i(n-\frac{1}{2})t} + \ldots + e^{i(n+\frac{1}{2})t}) \\ &= e^{-i(n+\frac{1}{2})t} - e^{i(n+\frac{1}{2})t} \end{aligned}$$

and we have

$$D_n(t) = \frac{e^{-i(n+\frac{1}{2})t} - e^{i(n+\frac{1}{2})t}}{e^{-i\frac{1}{2}t} - e^{i\frac{1}{2}t}} = \frac{-2i\sin(n+\frac{1}{2})t}{-2i\sin(\frac{1}{2}t)}.$$

If $t = 0$,

$$D_n(0) = \sum_{k=-n}^{n} \underbrace{e^{ik0}}_{1} = 2n+1$$

(iv) We have, due to evenness of the Dirichlet kernel, that

$$L_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n| = \frac{1}{\pi}\int_0^{\pi} |D_n|.$$

This is

$$\frac{1}{\pi}\int_0^{\pi} \left|\frac{\sin(n+\frac{1}{2})t}{\sin(\frac{1}{2}t)}\right| \, dt \geq \frac{1}{\pi}\int_0^{\pi} \frac{|\sin(n+\frac{1}{2})t|}{\frac{1}{2}t} \, dt$$

because $|\sin(\frac{1}{2}t)| \leq \frac{1}{2}t$.

$$= \frac{2}{\pi}\int_0^{(n+\frac{1}{2})\pi} \frac{|\sin s|}{\frac{s}{n+\frac{1}{2}}} \frac{1}{n+\frac{1}{2}} \, ds,$$

by substituting $s = (n+\frac{1}{2})t$ (change of variables in Riemann integral).

$$\geq \frac{2}{\pi}\int_0^{n\pi} \frac{|\sin s|}{s} \, ds = \frac{2}{\pi}\sum_{j=1}^{n}\int_{(j-1)\pi}^{j\pi} \frac{|\sin s|}{s} \, ds \geq \frac{2}{\pi}\sum_{j=1}^{n}\int_{(j-1)\pi}^{j\pi} \frac{|\sin s|}{j\pi} \, ds$$

and so

$$= \frac{2}{\pi^2}\sum_{j=1}^{n}\frac{1}{j}\overbrace{\int_{(j-1)\pi}^{j\pi} |\sin s| \, ds}^{=2} = \frac{4}{\pi^2}\sum_{j=1}^{n}\frac{1}{j}.$$

Hence,

$$L_n \geq C\sum_{j=1}^{n}\frac{1}{j} \xrightarrow{n\to\infty} \infty.$$

for some fixed constant C. □

5.21 Remark (Baire Category terminology). Let X be a metric space. A set $F \subseteq X$ is of **first category** (or is **meager**) if

$$F \subseteq \bigcup_{n=1}^{\infty} F_n$$

where each F_n is closed and nowhere dense $F_n^{\circ} = \varnothing$ where S° is the interior of S. We will say a set $U \subseteq X$ is of **second category** (or is **non-meager**) if it is not meager.

Recall the Baire Category Theorem.

5.22 Theorem (Baire Category). If X is a complete metric space, then it is non-meager.

This is often presented in a dual manner using open sets.

5.23 Theorem (Banach-Steinhaus Theorem). Let $\mathcal{B}$ and $\mathcal{X}$ be Banach spaces (usually $\mathcal{X} = \mathcal{B}$ or $\mathbb{C}$) and $\mathcal{F}$ be a family of bounded linear maps from $\mathcal{B}$ to $\mathcal{X}$. Then if

$$\sup\{\|Tf\|_{\mathcal{X}} : T \in \mathcal{F}\} < \infty$$

for each f in a non-meager set $U \subseteq \mathcal{B}$, then

$$\sup\{\|T\| : T \in \mathcal{F}\} < \infty.$$

Proof. Let, for each $n \in \mathbb{N}$,

$$\begin{aligned} F_n &= \{f \in \mathcal{B} : \|Tf\|_{\mathcal{X}} \leq n \text{ for all } T \in \mathcal{F}\} \\ &= \bigcap_{T\in\mathcal{F}} \underbrace{\{f \in \mathcal{B} : \|Tf\|_{\mathcal{X}} \leq n\}}_{\text{closed}} \end{aligned}$$

If $g_T(f) = \|Tf\|_{\mathcal{X}}$ then the set above is merely

$$g_T^{-1}(\overbrace{\{z \in \mathbb{C} : |z| \leq n\}}^{\text{closed}}).$$

Then, for our specified non-meager $U \subseteq \mathcal{B}$ we have that, by our hypothesis,

$$U \subseteq \bigcup_{n=1}^{\infty} F_n$$

and hence at least set $F_{n_0}^{\circ} \neq 0$. Hence there is $f_0 \in \mathcal{B}$ and $r > 0$ such that

$$B_r(f_0) = \{f \in \mathcal{B} : \|f - f_0\|_{\mathcal{B}} < r\} \subseteq F_{n_0}.$$

Notice that if $f \in B_r(f_0) \subseteq F_{n_0}$, then $\|T_f\|_{\mathcal{X}} \leq n_0$ for $T \in \mathcal{F}$. Now fix $f \in \mathcal{B}$ with $\|f\|_{\mathcal{B}} \leq 1$ and we note that

$$f_0 + \frac{r}{2}f,\ f_0 - \frac{r}{2}f \in B_r(f_0).$$

Thus if $T \in \mathcal{F}$, we have that

$$\begin{aligned}
\|Tf\|_{\mathcal{X}} = \left\| T\left(\frac{1}{r}\left[f_0 + \frac{r}{2}f - (f_0 - \frac{r}{2}f)\right]\right)\right\|_{\mathcal{X}} &= \frac{1}{r}\left\|T(f_0 + \tfrac{r}{2}f) - T(f_0 - \tfrac{r}{2}f)\right\|_{\mathcal{X}} \\
&\leq \frac{1}{r}\left(\underbrace{\|T(f_0 + \tfrac{r}{2}f)\|_{\mathcal{X}}}_{\leq n_0} + \underbrace{\|T(f_0 - \tfrac{r}{2}f)\|_{\mathcal{X}}}_{\leq n_0}\right) \\
&\leq \frac{2n_0}{r} < \infty.
\end{aligned}$$

Hence $|||T||| = \sup\{\|Tf\|_{\mathcal{X}} : f \in \mathcal{B}, \|f\|_{\mathcal{B}} \leq 1\} \leq \frac{2n_0}{r} < \infty$. This is for all $T \in \mathcal{F}$. □

5.24 Corollary. Let $\mathcal{B}, \mathcal{X}$ be Banach spaces, and for $n \in \mathbb{N}$, let $T_n : \mathcal{B} \to \mathcal{X}$ be a bounded linear map and suppose

$$\sup_{n\in\mathbb{N}} |||T_n||| = \infty. \tag{†}$$

Then, there is a set $U \subseteq \mathcal{B}$ with meager complement such that

$$\sup_{n\in\mathbb{N}} \|T_n f\|_{\mathcal{X}} = \infty$$

for all $f \in U$.

Proof. Let

$$F = \left\{f \in \mathcal{B} : \sup_{n\in\mathbb{N}} \|T_n f\|_{\mathcal{X}} < \infty\right\}.$$

If it were the case that F were non-meager, the Banach-Steinhaus theorem would show that $\sup_{n\in\mathbb{N}} |||T_n||| < \infty$, violating (†). Hence F is meager. Thus $U = \mathcal{B} \setminus F$. □

Note that the set U above is necessarily non-meager (indeed it is the complement of a meager set). [Exercise, since $\mathcal{B}$ is complete].

5.25 Theorem. We have:

(i) The set of $f \in C(\mathbb{T})$ for which $\sup_{n\in\mathbb{N}} \|s_n(f)\|_\infty < \infty$ (in particular for which $\lim_{n\to\infty} \|s_n(f) - f\|_\infty = 0$) is a meager subset of $C(\mathbb{T})$.

(ii) The set of $f \in L_1(\mathbb{T})$ for which $\sup_{n\in\mathbb{N}} \|s_n(f)\|_1 < \infty$ (in particular for which $\lim_{n\to\infty} \|s_n(f) - f\|_1 = 0$) is a meager subset of $L_1(\mathbb{T})$.

Proof. We have:

(i) We saw the following facts:

- $s_n(f) = D_n * f = C(D_n)f$ (computation).
- $|||C(D_n)|||_{C(\mathbb{T})} = \|D_n\|_1$ (theorem).
- $\|D_n\|_1 = L_n \to \infty$ as $n \to \infty$, where L_n is called the Lebesgue constant (theorem).

Hence by the Banach-Steinhaus theorem (corollary) we see that

$$\sup_{n\in\mathbb{N}} \|s_n(f)\|_\infty = \infty$$

for all $f \in C(\mathbb{T}) \setminus F$ where F is meager.

(ii) Similar. □

5.6 Averaging to the rescue

We introduce a technique of Cesàro averages.

5.26 Definition. If $(x_n)_{n=1}^{\infty}$ is a sequence in a Banach space $\mathcal{X}$, we shall call the term

$$\sigma_n = \frac{1}{n}(x_1 + \ldots + x_n)$$

the nth **Cesâro mean**.

5.27 Exercise. If $\lim_{n\to\infty} x_n = x_0$ exists, then

$$\lim_{n\to\infty} \sigma_n = x_0.$$

5.28 Definition. Suppose now that $f \in L_1(\mathbb{T})$. We define the nth **Cesâro sum** of f by

$$\sigma_n(f) = \frac{1}{n+1}(s_0(f) + \ldots + s_n(f)).$$

We observe

$$\sigma_n(f) = \underbrace{\frac{1}{n+1}(D_0 + D_1 + \ldots + D_n)}_{=:K_n} * f.$$

We call K_n the **Fejér kernel**. In summary,

$$\sigma_n(f) = K_n * f.$$

5.29 Theorem (Properties of Fejér kernel). We have:

(i) K_n is $\mathbb{R}$-valued, 2π-periodic and even.

(ii) We have the following ugly formula[14]

$$K_n(t) = \begin{cases} \frac{1}{n+1}\left(\frac{\sin(\frac{1}{2}(n+1)t)}{\sin(\frac{1}{2}t)}\right)^2 & t \neq 0 \\ n+1 & t = 0 \end{cases}$$

for $t \in [-\pi, \pi]$. In particular $K_n \geq 0$.

(iii) $\|K_n\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n = 1.$

(iv) If $0 < |t| < \pi$, then

$$0 \leq K_n(t) \leq \frac{\pi^2}{(n+1)t^2}$$

Proof. We have:

(i) $K_n = \frac{1}{n+1}(D_0 + D_1 + \ldots + D_n)$, where each D_j $(0 \leq j \leq n)$ is $\mathbb{R}$-valued, 2π-periodic and even.

(ii) We have

$$\begin{aligned} K_n = \frac{1}{n+1}\sum_{j=0}^{n} D_j &= \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=-j}^{j} \mathbf{e}^k \\ &= \frac{1}{n+1}\sum_{k=-n}^{n}(n+1-|k|)\mathbf{e}^k \\ &= \frac{1}{n+1}(\mathbf{e}^{-n} + 2\mathbf{e}^{-(n-1)} + \ldots + 2\mathbf{e}^{n-1} + \mathbf{e}^n) \end{aligned}$$

[14]"No book I could find actually does this explicitly, and I taught for many many years by saying "just check it". So one day I thought "I should check it", and it sucks! So, now you will suffer with me." – N. Spronk

thus

$$\begin{aligned}(n+1)K_n(t)(e^{-it}-2+e^{it}) &= e^{-i(n+1)t} + 2e^{-int} + 3e^{-i(n-1)t} + \ldots + (n+1)e^{-it} \\ &\quad + n + (n-1)e^{it} + \ldots + e^{i(n-1)t} \\ &\quad - (2e^{-int} + 2\cdot 2e^{i(n-1)t} + \ldots + 2ne^{-it} + 2(n+1) + 2ne^{it} \\ &\quad + \ldots + 2\cdot 2e^{i(n-1)t} + 2e^{it}) - e^{-i(n-1)t} + \ldots + (n-1)e^{-it} + \underbrace{n}_{\text{leaves}-2} + (n+1)e^{it} \\ &\quad + \ldots + 3e^{i(n-1)t} + 2e^{int} + e^{i(n+1)t} \\ &= e^{-i(n+1)t} - 2 + e^{i(n+1)t}\end{aligned}$$

Hence, if $t \neq 0$, we get

$$\begin{aligned}K_n(t) &= \frac{1}{n+1}\frac{e^{-i(n+1)t}-2+e^{i(n+1)t}}{e^{-it}-2+e^{it}} = \frac{1}{n+1}\left(\frac{e^{-i\frac{1}{2}(n+1)t}-e^{i\frac{1}{2}(n+1)t}}{e^{-i\frac{1}{2}t}-e^{i\frac{1}{2}t}}\right)^2 \\ &= \frac{1}{n+1}\left(\frac{\sin(\frac{1}{2}(n+1)t)}{\sin(\frac{1}{2}t)}\right)^2\end{aligned}$$

and for $t = 0$,

$$K_n(0) = \frac{1}{n+1}(D_0)(0) + \ldots + (D_n)(0)) = \frac{1}{n+1}\sum_{j=0}^{n}(2j+1) = n+1.$$

(iii) $\displaystyle \|K_n\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|K_n| = \frac{1}{2\pi}\frac{1}{n+1}\sum_{j=0}^{n}\int_{-\pi}^{\pi}D_j = \frac{2\pi\cdot(n+1)}{2\pi\cdot(n+1)} = 1.$

(iv) First note $\frac{2}{\pi}\theta \le \sin\theta$, $0 \le \theta \le \frac{\pi}{2}$. Since $\sin\theta$ is concave down, that line must lie below it. Hence for $0 < t < \pi$,

$$\frac{1}{\sin(\frac{1}{2}t)} \le \frac{1}{\frac{t}{\pi}} = \frac{\pi}{t}.$$

Hence for $0 < t < \pi$,

$$K_n(t) = \frac{1}{n+1}\left(\frac{\sin(\frac{1}{2}(n+1)t)}{\sin(\frac{1}{2}t)}\right)^2 \le \frac{1}{n+1}\left(\frac{\pi}{t}\right)^2 = \frac{\pi^2}{(n+1)t^2}.$$

We know that the Fejer kernel is even so $K_n(-t) = K_n(t)$. □

5.30 Definition. A **summability kernel** is a sequence $(k_n)_{n=1}^{\infty}$ of 2π-periodic bounded piecewise-continuous functions such that

(i) $\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n = 1$

(ii) $\sup_{n\in\mathbb{N}}\|k_n\|_1 < \infty$.

(iii) For any $0 < \delta \le \pi$,

$$\lim_{n\to\infty}\left(\int_{-\pi}^{-\delta}|k_n| + \int_{\delta}^{\pi}|k_n|\right) = 0$$

(mass concentrates at 0).

5.31 Proposition. The Fejér kernel $(K_n)_{n=1}^{\infty}$ is a summability kernel.

Proof. We saw

(i) $\displaystyle \frac{1}{2\pi}\int_{-\pi}^{\pi}K_n = 1.$

(ii) $\sup_{n\in\mathbb{N}}\|K_n\|_1 = 1$.

(iii) We have for $0 < \delta \le \pi$,

$$0 \le \int_{\delta}^{\pi}|K_n(t)|\,dt \le \int_{\delta}^{\pi}\frac{\pi^2}{(n+1)t^2}\,dt = \frac{\pi^2}{n+1}\left(\frac{1}{\delta}-\frac{1}{\pi}\right) \to 0$$

as $n \to \infty$. So applying squeeze theorem, we know the integral goes to 0 as well. By symmetry, we also get

$$\int_{-\pi}^{-\delta} |K_n| \to 0$$

by symmetry. $\square$

5.32 Example. Here are some other examples:

- $k_n = n\pi\chi_{[-\frac{1}{n},\frac{1}{n}]}$
- $k_n = 2n\pi\chi_{[0,\frac{1}{n}]}$
- We can make these continuous [diagram on camera] where c_n is chosen so that

$$\frac{1}{2\pi}\int_{-2/n}^{2/n} k_n = 1$$

5.33 Theorem (Abstract Summability Kernel Theorem). Let $\mathcal{B}$ be a homogeneous Banach space on $\mathbb{T}$ and $(k_n)_{n=1}^{\infty}$ be a summability kernel. Then for $f \in \mathcal{B}$,

$$\lim_{n\to\infty} \|k_n * f - f\|_{\mathcal{B}} = 0$$

i.e. $\lim_{n\to\infty} k_n * f = f$ in $\mathcal{B}$.

Proof. Fix $f \in \mathcal{B}$. Let $F : \mathbb{R} \to \mathcal{B}$ given by $F(s) = s * f$ (i.e. $s * f(t) = f(t-s)$ for almost every t). The axioms of a homogeneous Banach space tell us that F is continuous, 2π-periodic, and $\|F(s)\|_{\mathcal{B}} = \|s * f\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}$. Also, $F(0) = 0 * f = f$. Let us compute

$$\begin{aligned} k_n * f - f = \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(s)s * f\ ds - f &= \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(s)F(s)\ ds - F(0) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(s)F(s)\ ds - \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(s)F(0)\ ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(s)[F(s) - F(0)]\ ds \end{aligned}$$

Thus we have

$$\|k_n * f - f\|_{\mathcal{B}} = \left\|\frac{1}{2\pi}\int_{-\pi}^{\pi} k_n(s)[F(s) - F(0)]\ ds\right\| \overset{A1}{\leq} \frac{1}{2\pi}\int_{-\pi}^{\pi} |k_n(s)| \cdot \|F(s) - F(0)\|_{\mathcal{B}}\ ds.$$

Given $\epsilon > 0$, find $\delta > 0$ such that $\|F(s) - F(0)\|_{\mathcal{B}} < \frac{\epsilon}{M}$ for $|s| < \delta$, where

$$M = \sup_{n\in\mathbb{N}} \|k_n\|_1 < \infty.$$

Then, choose $n \in \mathbb{N}$, so for $n \geq N$ we have

$$\frac{1}{2\pi}\int_{[-\pi,-\delta]\cup[\delta,\pi]} |k_n| < \frac{\epsilon}{4\cdot\|f\|_{\mathcal{B}}}.$$

(We may assume $\|f\|_{\mathcal{B}} > 0$). Then for $n \geq N$,

$$\|k_n * f - f\|_{\mathcal{B}} \leq \frac{1}{2\pi}\int_{[-\pi,-\delta]\cup[\delta,\pi]} |k_n| \cdot \|F(s) - F(0)\|_{\mathcal{B}} + \frac{1}{2\pi}\int_{[-\delta,\delta]} |k_n(s)| \cdot \underbrace{\|F(s) - F(0)\|}_{\leq \epsilon/2M}\ ds.$$

Now $\|F(s) - F(0)\|_{\mathcal{B}} \leq \|F(s)\| + \|F(0)\| = 2\|f\|_{\mathcal{B}}$ so that

$$\leq 2\|f\|_{\mathcal{B}}\frac{1}{2\pi}\int_{[-\pi,-\delta]\cup[\delta,\pi]} |k_n(s)|\ ds + \frac{\epsilon}{2M}\underbrace{\frac{1}{2\pi}\int_{[-\delta,\delta]} |k_n(s)|\ ds}_{\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|k_n(s)|\ ds = \|f\|_1}$$

$$\leq 2\|f\|_{\mathcal{B}}\frac{\epsilon}{4\|f\|_{\mathcal{B}}} + \frac{\epsilon}{2M}\underbrace{\|k_n\|_1}_{\leq M} = \epsilon. \qquad \square$$

5.34 Corollary. We have:

(i) For $f \in C(\mathbb{T})$ we have
$$\lim_{n\to\infty} \|\sigma_n(f) - f\|_\infty = 0,$$
i.e. $\lim_{n\to\infty} \sigma_n(f) = f$ uniformly.

(ii) If $1 \le p < \infty$, for $f \in L_p(\mathbb{T})$ we have that
$$\lim_{n\to\infty} \|\sigma_n(f) - f\|_p = 0.$$

Proof. We recall that
$$\sigma_n(f) = K_n * f$$
where K_n is the Fejér kernel and $(K_n)_{n=1}^\infty$ is a summability kernel. Hence we use Abstract Summability Kernel Theorem. □

5.35 Corollary. Suppose $f, g \in L_1(\mathbb{T})$ and $c_k(f) = c_k(g)$ for each $k \in \mathbb{Z}$. Then $f = g$ a.e.

Proof. We have
$$\sigma_n(f,t) = \sum_{j=0}^{n} s_j(f,t) = \sum_{j=0}^{n}\sum_{k=-j}^{j} c_k(f)e^{ikt}.$$
If $c_k(f) = c_k(g)$ for all k, so we have that for each n
$$\|f-g\|_1 = \|f \underbrace{-\sigma_n(f) + \sigma_n(g)}_{0} - g\|_1 \le \|f - \sigma_n(f)\|_1 + \|\sigma_n(g) - g\|_1 \to 0$$
as $n \to \infty$. So $\|f-g\|_1 = 0$ which means $f = g$ (a.e.). □

Recall that
$$L(\mathbb{T}) = \left\{ f : \mathbb{R} \to \mathbb{C} : f \text{ is measurable, a.e. } 2\pi\text{-periodic, and } \int_{-\pi}^{\pi} |f| < \infty \right\}$$
and $L_1(\mathbb{T}) = L(\mathbb{T})/\sim_{a.e.}$. For $f \in L(\mathbb{T})$, $s \in \mathbb{R}$ (usually $s \in [-\pi, \pi]$) we let
$$\omega_f(s) = \frac{1}{2}\lim_{h\to 0^+}[f(s-h) + f(s+h)]$$
provided the limit exists. This is called the "average value of f at s".

5.36 Theorem (Fejér's Theorem). We have:

(i) If $f \in L(\mathbb{T})$, and $x \in [-\pi, \pi]$ is such that $\omega_f(x)$ exists, then
$$\lim_{n\to\infty} \sigma_n(f,x) = \omega_f(x).$$

(ii) If I is an open interval on which f is continuous, then for any closed subinterval J of I,
$$\lim_{n\to\infty} \sup_{t\in J} |\sigma_n(f,t) - f(t)| = 0$$
i.e.
$$\lim_{n\to\infty} \sigma_n(f,t) = f(t)$$
uniformly on closed subintervals of I.

Proof. Recall that where K_n is the Fejér kernel, we have
$$\sigma_n(f,x) = K_n * f(x)$$
where convolution is understood in the sense of A5[15]. We also recall

- K_n is even and non-negative $\mathbb{R}$-valued.
- $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} K_n = 1.$

[15] Hint: Use Dominated convergence theorem.

- If $0 \leq |t| \leq \pi$, then
$$K_n(t) \leq \frac{\pi^2}{(n+1)t^2}.$$

Now, we suppose $\omega_f(x)$ is finite [we leave the case $\omega_f(x) = \infty$ to A5]. Then, given $\epsilon > 0$, let $\delta > 0$ be so that $0 < |s| \leq \delta$ we have
$$\left|\omega_f(x) - \frac{1}{2}(f(x-s) + f(x+s))\right| < \epsilon.$$
[Note that if f is continuous on an open interval I, and $J \subseteq I$ is a closed subinterval, then in fact for $x \in J$ we have $\omega_f(x) = f(x)$ due to continuity from both sides, and, moreover, if J' is any closed interval such that $J \subseteq (J')^\circ \subseteq I$, then f is uniformly continuous on J' and we can choose $\delta > 0$ such that
$$|f(x-s) - f(x)| < \epsilon, \qquad \forall x \in J'.]$$
Then we have
$$\begin{aligned}
|\sigma_n(f,x) - \omega_f(x)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(s) f(x-s)\, ds - \omega_f(x)\right| \\
&= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} K_n(s)(f(x-s) - \omega_f(x))\, ds\right| \\
&\leq \frac{1}{2\pi}\left|\int_{-\delta}^{\delta} K_n(s)(f(x-s) - \omega_f(x))\, ds\right| + \\
&\quad \frac{1}{2\pi}\left|\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) K_n(s)(f(x-s) - \omega_f(x))\, ds\right|
\end{aligned}$$
and now for every n, we have
$$\frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(s)(f(x-s) - \omega_f(x))\, ds = \frac{1}{2\pi}\int_{-\delta}^{\delta} \underbrace{K_n(-s)}_{=K_n(s)}(f(x+s) - \omega_f(x))\, ds.$$
Thus
$$\begin{aligned}
\frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(s)(f(x-s) - \omega_f(x))\, ds &= \frac{1}{4\pi}\int_{-\delta}^{\delta} K_n(s)(f(x-s) - \omega_f(x)) + \\
&\quad \frac{1}{4\pi}\int_{-\delta}^{\delta} K_n(s)(f(x+s) - \omega_f(x))\, ds \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(s)\left(\frac{1}{2}(f(x-s) + f(x+s)) - \omega_f(x)\right)\, ds
\end{aligned}$$
Thus, using our choice of δ, we have
$$\begin{aligned}
\frac{1}{2\pi}\left|\int_{-\delta}^{\delta} K_n(s)(f(x-s) - \omega_f(x))\, ds\right| &= \frac{1}{2\pi}\left|\int_{-\delta}^{\delta} K_n(s)\left(\frac{1}{2}(f(x-s) + f(x+s)) - \omega_f(x)\right)\, ds\right| \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(s)\left|\frac{1}{2}(f(x-s) + f(x+s)) - \omega_f(x)\right|\, ds \\
&\leq \frac{\epsilon}{2\pi}\int_{-\delta}^{\delta} K_n(s)\, ds \\
&\leq \epsilon\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(s)\, ds = \epsilon.
\end{aligned}$$

Also,

$$\begin{aligned}\frac{1}{2\pi}\left|\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)K_n(s)(f(x-s)-\omega_f(x))\,ds\right| &\le \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)K_n(s)|f(x-s)-\omega_f(x)|\,ds\\ &\le \frac{\pi^2}{2(n+1)s^2}\le\frac{\pi^2}{2(n+1)\delta^2}\\ &\le \frac{\pi^2}{2(n+1)\delta^2}\frac{1}{2\pi}\left(\int_{-\pi}^{-\delta}+\int_{\delta}^{\pi}\right)\left|\underbrace{f(x-s)}_{\breve{f}(s-x)=x*\breve{f}(s)}-\omega_f(x)\right|\,ds\end{aligned}$$

where $\breve{f}(t)=f(-t)$, so the above is

$$\le \frac{\pi^2}{2(n+1)\delta^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}|\breve{f}(s-x)-\omega_f(x)|\,ds=\frac{\pi^2}{2(n+1)\delta^2}\|x*\breve{f}-\omega_f(x)1\|_1\to 0$$

as $n\to\infty$. Hence, we conclude that

$$\limsup_{n\to\infty}|\sigma_n(f,x)-\omega_f(x)|\le\epsilon.$$

However ϵ was arbitrary, so we conclude that the limit exists and is equal to 0. This concludes the proof of (i).

To see (ii), notice that all estimates performed were done uniformly over x in J [i.e. choice of δ]. So (ii) follows immediately. □

5.37 Corollary. Suppose $f\in L(\mathbb{T})$, $x\in[-\pi,\pi]$ such that $\omega_f(x)$ exists, and suppose $\lim_{n\to\infty}s_n(f,x)$ exists (s_n are the regular Fourier sums). Then

$$\lim_{n\to\infty}s_n(f,x)=\omega_f(x).$$

Proof. If $\lim_{n\to\infty}s_n(f,x)$ exists, then

$$\lim_{n\to\infty}\sigma_n(f,x)=\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}s_j(f)=\lim_{n\to\infty}s_n(f,x).$$

Hence, $\lim_{n\to\infty}s_n(f,x)=\lim_{n\to\infty}\sigma_n(f,x)=\omega_f(x)$ by Fejér's theorem. □

5.38 Definition. If $f\in L[a,b]$, a point $x\in(a,b)$ is called a **Lebesgue point** of f if

$$\lim_{h\to0^+}\frac{1}{h}\int_0^h\left|\frac{f(x-s)+f(x+s)}{2}-f(x)\right|ds=0.$$

5.39 Remark (fact). Since f is integrable, it is the case that almost every $x\in(a,b)$ is a Lebesgue point.

Proof. PMATH 451, part of Lebesgue Differentiation Theorem. □

5.40 Theorem (Lebesgue-Fejér Theorem). If $x\in[-\pi,\pi]$ is a Lebesgue point for $f\in L(\mathbb{T})$, then

$$f(x)=\lim_{n\to\infty}\sigma_n(f,x). \tag{*}$$

In particular, the above statement (*) occurs almost everywhere.

Proof. Omitted. □

Recall: Given $f\in L_1(\mathbb{T})$ (or $f\in L(\mathbb{T})$), f has a Fourier series given by

$$\sum_{k=-\infty}^{\infty}c_k(f)\mathbf{e}^k.$$

We know that it is not always the case that f is equal to its Fourier series (sometimes this fails catastrophically, as seen in A5), however, it is always the case that

$$\begin{aligned}f=\lim_{n\to\infty}\sigma_n(f)&=\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=-j}^{j}c_k(f)\mathbf{e}^k\\ &=\lim_{n\to\infty}\sum_{k=-n}^{n}\frac{n+1-|k|}{n+1}c_k(f)\mathbf{e}^k.\end{aligned}$$

5.7 On the Fourier Coefficients

Question: If $(c_k)_{k\in\mathbb{Z}}$ is a sequence of $\mathbb{R}$ (or $\mathbb{C}$) numbers, is there $f \in L_1(\mathbb{T})$ such that $f \approx \sum_{k=-\infty}^{\infty} c_k\mathbf{e}^k$ (i.e., $c_k = c_k(f)$ for all $k \in \mathbb{Z}$)?

5.41 Lemma. If $f \in L_1(\mathbb{T})$ then for all $k \in \mathbb{Z}$, $|c_k(f)| \leq \|f\|_1$.

Proof. Observe that

$$|c_k(f)| = \left|\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt\right| \leq \int_{-\pi}^{\pi} |f(t)||e^{-ikt}|\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|\,dt = \|f\|_1.$$ □

5.42 Theorem (Riemann-Lebesgue Lemma). If $f \in L_1(\mathbb{T})$, then

$$\lim_{k\to\infty} |c_k(f)| = 0 \text{ and } \lim_{k\to-\infty} |c_k(f)| = 0.$$

Proof. Let $\epsilon > 0$. We may find $n_0 \in \mathbb{N}$ such that $\|\sigma_{n_0}(f) - f\|_1 < \epsilon$ (by properties of the abstract summability kernel). Let $b_j = \frac{n_0+1-|j|}{n_0+1} \cdot c_j(f)$ so that $\sigma_{n_0}(f) = \sum_{j=-n_0}^{n_0} b_j\mathbf{e}^j$. Thus, if $|k| \geq n_0$, we have that

$$\begin{aligned}
c_k(\sigma_{n_0}(f) - f) &= c_k(\sigma_{n_0}(f)) - c_k(f)\\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi}\sum_{j=-n_0}^{n_0} b_j\mathbf{e}^{j-k}\right] - c_k(f)\\
&= \frac{1}{2\pi}\left[\sum_{j=-n_0}^{n_0} b_j\int_{-\pi}^{\pi}\mathbf{e}^{j-k}\right] - c_k(f)\\
&= -c_k(f)
\end{aligned}$$

since $|k| > n_0$ implies that $j - k \neq 0$ for all j in the above sum, hence every term in the sum is equal to 0. It follows that $|c_k(f)| = |c_k(\sigma_{n_0}(f) - f)| \leq \|\sigma_{n_0}(f) - f\|_1 < \epsilon$, where the inequality is due to the earlier lemma. □

5.43 Corollary. Let $f \in L(\mathbb{T})$. Then

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = 0 \text{ and } \lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = 0$$

Proof. We have $\cos(nt) = \frac{1}{2}(e^{int} + e^{-int}) = \frac{1}{2}(\mathbf{e}^n + \mathbf{e}^{-n})$. Hence,

$$\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\mathbf{e}^n + f\mathbf{e}^{-n}\,dt = c_n(f) + c_{-n}(f) \to 0$$

as $n \to \infty$. We may similarly show the latter claim by using the identity $\sin(nt) = \frac{1}{2i}(e^{int} - e^{-int}) = \frac{1}{2i}(\mathbf{e}^n - \mathbf{e}^{-n})$. □

Hence, if $(c_k)_{k\in\mathbb{Z}}$ is such that $c_k = c_k(f)$ for all $k \in \mathbb{Z}$ for some $f \in L_1(\mathbb{T})$, then the Riemann-Lebesgue Lemma tells us that $(c_k)_{k\in\mathbb{Z}}$ need necessarily satisfy $\lim_{k\to\infty} |c_k| = 0$ and $\lim_{k\to-\infty} |c_k| = 0$, or more concisely, $\lim_{|k|\to\infty} |c_k| = 0$.

Let $\mathsf{c}_0(\mathbb{Z}) = \{(c_k)_{k\in\mathbb{Z}} : c_k \in \mathbb{C}, \lim_{|k|\to\infty} c_k = 0\}$. Recall from PMATH 351 that $\mathsf{c}_0(\mathbb{Z})$ under the ∞-norm ($\|(c_k)_{k\in\mathbb{Z}}\|_\infty = \max_{k\in\mathbb{Z}} |c_k|$) with operations $(c_k)_{k\in\mathbb{Z}} + (d_k)_{k\in\mathbb{Z}} = (c_k + d_k)_{k\in\mathbb{Z}}$ and $\alpha(c_k)_{k\in\mathbb{Z}} = (\alpha c_k)_{k\in\mathbb{Z}}$ for $\alpha \in \mathbb{C}$ is a Banach space (i.e., $\mathsf{c}_0(\mathbb{Z})$ is essentially the space $\mathsf{c}_0 = \mathsf{c}_0(\mathbb{N}) \subset \ell_\infty$ introduced in PMATH 351). Does every $(c_k)_{k\in\mathbb{Z}} \in \mathsf{c}_0(\mathbb{Z})$ correspond to a sequence of Fourier coefficients of some $f \in L_1(\mathbb{T})$?

5.44 Theorem (Open Mapping Theorem (PMATH 753)). Let X, Y be Banach spaces, and let $T : X \to Y$ be a bounded linear map. If T is surjective, then $T(U)$ is open in Y for every open set $U \subset X$.

5.45 Corollary (Inverse Mapping Theorem). Let X, Y be Banach spaces, and let $T : X \to Y$ be linear and bounded. If T is bijective, then $T^{-1} : Y \to X$ is bounded.

5.46 Corollary. There exists $(c_k)_{k\in\mathbb{Z}} \in \mathsf{c}_0(\mathbb{Z})$ such that there is no $f \in L_1(\mathbb{T})$ with $c_k(f) = c_k$ for all $k \in \mathbb{Z}$.

Proof. (Proof taken from PMATH 450 site.) Let $T : L_1(\mathbb{T}) \to \mathsf{c}_0(\mathbb{Z})$ be given by $Tf = (c_k(f))_{k\in\mathbb{Z}}$. Thus T is linear and bounded with $|||T||| \leq 1$ (i.e. $\|Tf\|_\infty = \sup_{k\in\mathbb{Z}} |c_k(f)| \leq \|f\|_1$) and range is in $\mathsf{c}_0(\mathbb{Z})$ by the Riemann-Lebesgue Lemma. Also T is injective (corollary to Abstract Summability Kernel Theorem: $c_k(f) = c_k(g)$ for all $k \in \mathbb{Z}$ implies $f = g$ a.e., i.e. in $L_1(\mathbb{T})$). If T were bijective then we would have bounded $T^{-1} : \mathsf{c}_0(\mathbb{Z}) \to L_1(\mathbb{T})$ (Inverse Mapping Theorem). However, let

$$d_n = (\ldots, 0, \underbrace{1}_{-n}, 1, \ldots, \underbrace{1}_{0}, \ldots, 1, \underbrace{1}_{n}, 0, \ldots) \in \mathsf{c}_0(\mathbb{Z})$$

so that $\|d_n\|_\infty = 1$. Then $T^{-1}(d_n) = D_n$, the Dirichlet kernel of order n. But then

$$|||T^{-1}||| \geq \sup_{n\in\mathbb{N}} \|T^{-1}(d_n)\|_1 = \sup_{n\in\mathbb{Z}} \|D_n\|_1 = \infty,$$

which contradicts the Inverse Mapping Theorem. $\square$

ERRATUM. I blew the proof of the fact that

$$T : L_1(\mathbb{T}) \ni f \mapsto (c_k(f))_{k=-\infty}^{\infty} \in c_0(\mathbb{Z})$$

is not surjective. Please find a correct proof on the website.

5.8 Localisation and Dini's theorem

Recall that if $f \in L(\mathbb{T})$ and $t \in [-\pi, \pi]$ we have

$$\begin{aligned}
\sum_{j=-n}^{n} c_j(f)e^{ijt} = s_n(f,t) = D_n * f(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(s)f(t-s)\,ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\frac{\sin(n+\frac{1}{2})s}{\sin\frac{1}{2}s}}_{\text{even}} f(t-s)\,ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(n+\frac{1}{2})s}{\sin\frac{1}{2}s} f(t+s)\,ds \qquad (\dagger)
\end{aligned}$$

by inversion invariance.

5.47 Lemma. If $f \in L(\mathbb{T})$ with

$$\int_{-\pi}^{\pi} \left|\frac{f(t)}{t}\right| dt < \infty,$$

then

$$\lim_{n\to\infty} s_n(f,0) = 0$$

Proof. Recall $\sin(x+y) = \sin x\cos y + \cos x \sin y$ and hence

$$D_n(s) = \frac{\sin(n+\frac{1}{2})s}{\sin\frac{1}{2}s} = \frac{\sin(ns)\cos\frac{1}{2}s}{\sin\frac{1}{2}s} + \cos(ns).$$

Hence by ($\dagger$),

$$\begin{aligned}
s_n(f,0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(s)f(s)\,ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sin(ns)\cos\frac{1}{2}s}{\sin\frac{1}{2}s} f(s)\,ds + \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(ns)f(s)\,ds
\end{aligned}$$

We note that for $0 \leq |\theta| \leq \frac{\pi}{2}$, we get $|\sin\theta| \geq \frac{2}{\pi}|\theta|$ [DIAGRAM]. So we have

$$|\sin\frac{1}{2}t| \geq \frac{1}{\pi}|t|$$

for $t \in [-\pi, \pi]$. Thus

$$\int_{-\pi}^{\pi} |\cos\tfrac{1}{2}s| \left|\frac{f(s)}{\sin\frac{1}{2}s}\right| ds \leq \int_{-\pi}^{\pi} \frac{|f(s)|}{\frac{1}{\pi}|s|}\,ds = \pi\int_{-\pi}^{\pi}\left|\frac{f(s)}{s}\right| ds < \infty$$

by assumption. So

$$s \mapsto \frac{\cos\frac{1}{2}s f(s)}{\sin\frac{1}{2}s}$$

almost everywhere $s \in [-\pi, \pi]$ extended 2π-periodically, defines an element in $L(\mathbb{T})$. Hence, we can use (Corollary to) the Riemann-Lebesgue lemma to see

$$s_n(f, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(ns) \frac{\cos \frac{1}{2} s f(s)}{\sin \frac{1}{2} s} \, ds + \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(ns) f(s) \, ds \to 0$$

as $n \to \infty$. $\square$

5.48 Theorem (Localisation Principle). If $f \in L(\mathbb{T})$ and I is an open interval on which $f(t) = 0$ for a.e. $t \in I$, then for $t \in I$,

$$\lim_{n \to \infty} s_n(f, t) = 0.$$

5.49 Corollary. If $f, g \in L(\mathbb{T})$ and I is an open interval on which $f(t) = g(t)$ for a.e. $t \in I$, then for $t \in I$

$$\lim_{n \to \infty} s_n(f, t)$$

converges if and only if

$$\lim_{n \to \infty} s_n(g, t)$$

exists, and the two limits coincide.

Proof. Observe that $\lim\limits_{n \to \infty} s_n(f - g, t) = \lim\limits_{n \to \infty} s_n(f, t) - s_n(g, t) = 0$. $\square$

Proof of localisation principle. Let $g \in L(\mathbb{T})$ be given by $g(s) = f(t - s) = \breve{f}(s - t)$, so $g = t * \breve{f}$, when $t \in I$ is fixed. Then $g(s) = 0$ for a.e. s in a neighbourhood of 0, say for $s \in (-\delta, \delta)$. Hence

$$\begin{aligned}
\int_{-\pi}^{\pi} \left| \frac{g(s)}{s} \right| \, ds &= \int_{-\delta}^{\delta} \left| \frac{0}{s} \right| \, ds + \left(\int_{\delta}^{\pi} + \int_{-\pi}^{-\delta} \right) \underbrace{\left| \frac{g(s)}{s} \right|}_{\leq |g(s)|/\delta} \, ds \\
&\leq \int_{-\pi}^{\pi} \frac{1}{\delta} |g(s)| \, ds \\
&= \frac{1}{\delta} \int_{-\pi}^{\pi} |t * \breve{f}(s)| \, ds \\
&= \frac{1}{\delta} 2\pi \|t * \breve{f}\|_1 = \frac{2\pi}{\delta} \|f\|_1 < \infty
\end{aligned}$$

by translation and inversion invariance and hence, by the lemma,

$$\lim_{n \to \infty} s_n(g, 0) = 0.$$

We have

$$\begin{aligned}
s_n(g, 0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s) \underbrace{g(s - 0)}_{t * \breve{f}(s)} \, ds \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s) \overbrace{f(t - s)} \, ds = s_n(f, t)
\end{aligned}$$

thus $\lim_{n \to \infty} s_n(f, t) = \lim_{n \to \infty} s_n(g, t) = 0$. $\square$

5.50 Theorem (Dini's Theorem). If $f \in L(\mathbb{T})$ and f is differentiable at t in $[-\pi, \pi]$, then

$$\lim_{n \to \infty} s_n(f, t) = f(t).$$

Proof. Given $\epsilon > 0$, there is $\delta > 0$ such that $|s| < \delta$ yields

$$\left| \frac{f(t - s) - f(t)}{s} - f'(t) \right| < \epsilon. \tag{*}$$

Thus

$$s \mapsto \frac{f(t - s) - f(t)}{s}$$

is bounded on $(-\delta, \delta)$. Let $g = t * \check{f} - f(t)$, or

$$g(s) = t * \check{f}(s) - f(s) = f(t-s) - f(s).$$

We have

$$\begin{aligned}
\int_{-\pi}^{\pi} \left| \frac{g(s)}{s} \right| \, ds &= \int_{-\delta}^{\delta} \left| \frac{g(s)}{s} \right| \, ds + \int\limits_{[-\pi,-\delta]\cup[\delta,\pi]} \left| \frac{g(s)}{s} \right| \, ds \\
&\leq \int_{-\delta}^{\delta} \underbrace{(|f'(t)| + \epsilon)}_{\text{by } (*)} \, ds + \frac{1}{\delta} \int_{-\pi}^{\pi} |g| \\
&= 2\delta(|f'(t)| + \epsilon) + \frac{1}{\delta} \left\| t * \check{f} - f(t) \right\|_1 < \infty
\end{aligned}$$

Thus, by the lemma, $\lim_{n\to\infty} s_n(g, 0) = 0$. As before

$$s_n(g, 0) = s_n(t * \check{f} - f(t), 0) = s_n(t * \check{f}, 0) - s_n(f(t), 0) = s_n(f, t) - f(t).$$

Of course, we have

$$s_n(f(t), 0) = f(t) s_n(1, 0) = \frac{f(t)}{2\pi} \int_{-\pi}^{\pi} D_n(s) 1 \, ds = f(t).$$ □

5.51 Theorem (Dini's Theorem for Lipschitz functions). Suppose $f \in L(\mathbb{T})$ and f is Lipschitz on an open interval I, that is there is $M > 0$ such that

$$|f(t) - f(s)| \leq M|t - s|,$$

for $s, t \in I$. Then for $t \in I$ we have

$$\lim_{n\to\infty} s_n(f, t) = f(t).$$

Proof. Fix $t \in I$. Then $(t - \delta, t + \delta) \subseteq I$ for some $\delta > 0$, so for $s \in (-\delta, \delta)$ let

$$g(s) = t * \check{f}(s) - f(t) = f(t - s) - f(t)$$

and we see that for $s \in (-\delta, \delta)$ with $s \neq 0$,

$$\left| \frac{g(s)}{s} \right| = \left| \frac{f(t-s) - f(t)}{(t-s) - t} \right| \leq M.$$

As before, we partition

$$\int_{-\pi}^{\pi} = \int_{-\delta}^{\delta} + \int\limits_{[-\pi,-\delta]\cup[\delta,\pi]}$$

to see that

$$\int_{-\pi}^{\pi} \left| \frac{g(s)}{s} \right| \, ds < \infty.$$

Thus,

$$\lim_{n\to\infty} s_n(g, 0) = 0$$

so we can conclude, as above, that

$$\lim_{n\to\infty} s_n(f, t) = f(t).$$ □

6 Inner products and Hilbert spaces

6.1 Definition. Let $\mathcal{X}$ be a $\mathbb{C}$-vector space (or a $\mathbb{R}$-vector space). An **inner product** on $\mathcal{X}$ is a map

$$(\cdot, \cdot) : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$$

such that for $f, g, h \in \mathcal{X}$ we have

(i) $(f, f) \geq 0$ (positivity).

(ii) $(f, f) = 0$ if and only if $f = 0$ (non-degeneracy).

(iii) $(f, g) = \overline{(g, f)}$.

(iv) $(\alpha f, g) = \alpha(f, g)$.

(v) $(f + g, h) = (f, h) + (g, h)$.

The last three properties are known as **sesquilinearity**.

Observe that (iii) combined with (iv) yields that

$$(f, \alpha g) = \overline{\alpha}(f, g).$$

Also (iii) and (v) gives

$$(f, g + h) = (f, g) + (f, h).$$

6.2 Definition. We define, for $f \in \mathcal{X}$,

$$\|f\| = \sqrt{(f, f)}.$$

6.3 Theorem (Cauchy-Schwarz Inequality). For a vector space $\mathcal{X}$ with inner product $(\cdot, \cdot)$, we have

$$|(f, g)| \leq \|f\|\|g\|$$

for $f, g \in \mathcal{X}$, with equality only if $g = tf$ for $t \in \mathbb{R}$ with $t \geq 0$.

Proof. First, replace g by $(f, g)g$ so that

$$(f, (f, g)g) = \overline{(f, g)}(f, g) \geq 0$$

i.e. we will assume that $(f, g) \geq 0$. If $t \in \mathbb{R}$, then $\bar{t} = t$ and we have

$$\begin{aligned} 0 \leq (tf + g, tf + g) &= t^2(f, f) + t(f, g) + \bar{t}\,\overbrace{(g, f)}^{\overline{(f,g)}} + (g, g) \\ &= t^2\|f\|^2 + 2t \cdot \underbrace{\operatorname{Re}(f, g)}_{(f,g)=|(f,g)|} + \|g\|^2 = p(t). \end{aligned}$$

Therefore, we have a quadratic polynomial $p(t) \geq 0$, and hence by quadratic formula we have

$$(2 \cdot \operatorname{Re}(f, g))^2 - 4\|f\|^2\|g\|^2 \leq 0 \tag{*}$$

and that

$$|(f, g)| \leq \|f\|\|g\|.$$

Notice that equality is going to hold only if $(*) = 0$ i.e. $p(t) = 0$ for some t, in which case

$$t = -\frac{\operatorname{Re}(f, g)}{\|f\|^2}$$

and we see that $tf + g = 0$, by non-degeneracy. $\square$

6.4 Proposition. $\|\cdot\|$ is a norm.

Proof. First, $\|\alpha f\| = |\alpha|\|f\|$ is straight forward. Also,

$$\begin{aligned} \|f + g\|^2 = (f + g, f + g) &= \|f\|^2 + 2 \cdot \operatorname{Re}(f, g) + \|g\|^2 \\ &\leq \|f\|^2 + 2 \cdot |\operatorname{Re}(f, g)| + \|g\|^2 \\ &\leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 \\ &= (\|f\| + \|g\|)^2. \end{aligned}$$

$\square$

6.5 Definition. An **inner product space** is a vector space $\mathcal{X}$ with an inner product $(\cdot, \cdot)$. A **Hilbert space** is an inner product space which is complete with respect to the induced norm $\|f\| = (f, f)^{1/2}$.

6.6 Example. We have:

(i) In $\mathbb{C}^n$ we have inner product

$$((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{i=1}^{n} x_i \overline{y_i}$$

so

$$\|(x_1, \ldots, x_n)\| = \sqrt{\sum_{i=1}^{n} (x_i)^2}.$$

This is always complete, hence a Hilbert space.

(ii) Let $A \in \mathcal{L}(\mathbb{R})$, with $\lambda(A) > 0$. $L_2(A)$ has inner product

$$(f, g) = \int_A f\overline{g}.$$

Recall that $f\overline{g} \in L_1(A)$, thanks to Hölder's inequality. It is an exercise to see that this is an inner product. Note that

$$\|f\|_2 = \left(\int_A |f|^2 \right)^{1/2} = \left(\int_A f\overline{f} \right)^{1/2} = \text{inner product norm}.$$

This is complete, hence a Hilbert Space.

(iii) Consider $C[a, b]$. We let for $f, g \in C[a, b]$

$$(f, g) = \int_a^b f\overline{g}$$

(Riemann integral). It's easy to verify that this is an inner product. Note that $C[a, b] \subseteq L_2[a, b]$ and $C[a, b]$ is $\| \cdot \|_2$-dense in $L_2[a, b]$. Hence if $f \in L_2[a, b] \setminus C[a, b]$, for example

$$f = \chi_{[a, \frac{a+b}{2}]}$$

($f \neq h$ for a continuous h) then there are $(f_n)_{n=1}^{\infty} \subseteq C[a, b]$ such that

$$\lim_{n \to \infty} \|f - f_n\|_2 = 0.$$

Thus, $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence with respect to the inner product norm, which converges to no continuous function in this norm. Hence this is a non-complete inner product space.

(iv) $\ell_2 = \left\{ x = (x_1, \ldots) \in \mathbb{C}^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$. The inner product is given by

$$(x, y) = \sum_{i=1}^{\infty} x_i \overline{y_i}.$$

We need to show that this series converges. First, note that if $x_{(n)} = (x_1, \ldots, x_n, 0, 0, \ldots)$ then

$$\|x - x_{(n)}\| = \left(\sum_{i=n+1}^{\infty} |x_i|^2 \right)^{1/2} \xrightarrow{n \to \infty} 0$$

for $x, y \in \ell_2$ with $n > m$,

$$|(x_{(n)} - x_{(m)}, y)| = \left| \sum_{i=m+1}^{n} x_i \overline{y_i} \right| \leq \left(\sum_{i=m+1}^{n} |x_i|^2 \right)^{1/2} \left(\sum_{i=m+1}^{n} |y_i|^2 \right) \xrightarrow[m<n\to\infty]{} 0.$$

In fact,

$$\left| \sum_{i=m+1}^{n} x_i \overline{y_i} \right| \leq \sum_{i=m+1}^{n} \underbrace{|x_i \overline{y_i}|}_{|x_i||y_i|} \leq \left(\sum_{i=m+1}^{n} |x_i|^2 \right) \|y\|$$

so the sum $\sum_{i=1}^{\infty} x_i \overline{y_i}$ is absolutely convergent, hence converging in $\mathbb{C}$.

ERRATUM/exercise. $(\cdot,\cdot)$ an inner product on $\mathcal{X}$.

$$|(f,g)| = \|f\|\|g\| \iff \alpha f = \beta g$$

for some $\alpha, \beta \in \mathbb{C}$. Meanwhile

$$(f,g) = \|f\|\|g\| \iff t_1 f = t_2 g$$

for some $t_1, t_2 \in \mathbb{R}$ with $t_1, t_2 \geq 0$. Also

$$\|f+g\| = \|f\| + \|g\| \iff t_1 f = t_2 g$$

for $t_1, t_2 \in \mathbb{R}$, $t_1, t_2 \geq 0$.

6.7 Definition. Let $(\mathcal{X}, (\cdot,\cdot))$ be an inner product. A set $\{e_i\}_{i\in I} \subseteq \mathcal{X}$ is **orthogonal** if no $e_i = 0$, and $(e_i, e_j) = 0$ for $i \neq j$. Moreover, we say $\{e_i\}_{i\in I}$ is **orthonormal** if

$$(e_i, e_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Notice, in the orthonormal case, $\|e_i\| = (e_i, e_i)^{1/2} = 1$.

6.8 Proposition (Pythagorean property). If $\{f_1, \ldots, f_n\}$ is an orthogonal set in an inner product space $(\mathcal{X}, (\cdot,\cdot))$, then

$$\|f_1 + \ldots + f_n\|^2 = \|f_1\|^2 + \ldots + \|f_n\|^2.$$

Proof. Let $n = 2$. Then

$$\begin{aligned} \|f_1 + f_2\|^2 &= (f_1 + f_2, f_1 + f_2) \\ &= \|f_1\|^2 + 2\mathrm{Re}(f_1, f_2) + \|f_2\|^2 \\ &= \|f_1\|^2 + \|f_2\|^2. \end{aligned}$$

For $n > 2$, use induction, noting that

$$(f_1 + \ldots + f_{n-1}, f_n) = 0.$$

$\square$

6.9 Lemma (Linear approximation lemma). Suppose $\{e_1, \ldots, e_n\}$ is an orthonormal set in an inner product space $(\mathcal{X}, (\cdot,\cdot))$. Let $E = \mathrm{span}\{e_1, \ldots, e_n\}$. Define for $f \in \mathcal{X}$,

$$\mathrm{dist}(f, E) = \inf\{\|f - g\| : g \in E\}.$$

Then

$$\mathrm{dist}(f,E)^2 = \left\| f - \sum_{i=1}^n (f, e_i) e_i \right\|^2 = \|f\|^2 - \sum_{i=1}^n |(f, e_i)|^2.$$

Moreover, $\sum_{i=1}^n (f, e_i) e_i$ is the unique vector $g \in E$ such that $\|f - g\| = \mathrm{dist}(f, E)$.

[Suggestive picture: $\mathbb{R}^2$, usual dot product.] [Suggestive picture: $\mathbb{R}^2$, $\|(x_1, x_2)\|_1 = |x_1| + |x_2|$.]

Proof. Let $g = \sum_{i=1}^{n} \alpha_i e_i$ be an arbitrary element of E. Then

$$\begin{aligned}
\|f-g\|^2 = (f-g, f-g) &= \|f\|^2 - 2\mathrm{Re}(f,g) + \|g\|^2 \\
&= \|f\|^2 - 2\mathrm{Re}\left[\sum_{i=1}^{n} \overline{\alpha_i}(f,e_i)\right] + \underbrace{\sum_{i=1}^{n} |\alpha_i|^2}_{\text{Pythagoras}} \qquad (\dagger) \\
&\geq \|f\|^2 - 2\sum_{i=1}^{n} |\alpha_i||(f,e_i)| + \sum_{i=1}^{n} |\alpha_i|^2 \\
&= \|f\|^2 - \sum_{i=1}^{n} |(f,e_i)|^2 + \sum_{i=1}^{n} |(f,e_i)|^2 - 2\sum_{i=1}^{n} |\alpha_i||(f,e_i)| + \sum_{i=1}^{n} |\alpha_i|^2 \\
&= \|f\|^2 - \sum_{i=1}^{n} |(f,e_i)|^2 + \sum_{i=1}^{n} (|(f,e_i)| - |\alpha_i|)^2 \\
&\geq \|f\|^2 - \sum_{i=1}^{n} |(f,e_i)|^2. \qquad (\dagger\dagger)
\end{aligned}$$

Notice that both inequalities are equalities exactly when $\alpha = (f, e_i)$ for i $(1 \leq i \leq n)$. Moreover, if

$$g = \sum_{i=1}^{n} (f, e_i) e_i$$

then ($\dagger$) turns to exactly ($\dagger\dagger$). Hence this vector g corresponds to exactly

$$\inf\{\|f-h\| : h \in E\} = \|f-g\|. \qquad \square$$

6.10 Proposition. If $(\mathcal{X}, (\cdot,\cdot))$ is an inner product space and $g \in \mathcal{X}$, then $\Gamma_g : \mathcal{X} \to \mathbb{C}$, given by

$$\Gamma_g(f) = (f, g)$$

is linear and bounded with $\|\Gamma_g\|_* = \|g\|$.

Proof. Linearity follows from properties of the inner product. By Cauchy-Schwarz inequality,

$$|\Gamma_g(f)| = |(f,g)| \leq \|f\|\|g\|$$

so that $\|\Gamma_g\|_* \leq \|g\|$. Also, if $g \neq 0$,

$$\Gamma_g\left(\frac{1}{\|g\|}g\right) = \left(\frac{1}{\|g\|}g, g\right) = \frac{1}{\|g\|}(g,g) = \frac{\|g\|^2}{\|g\|} = \|g\|.$$

Therefore, $\|\Gamma_g\|_* \geq \|g\|$. $\square$

6.11 Remark (Riesz representation theorem). If $\mathcal{H}$ is a Hilbert space, then every bounded linear functional $\Gamma : \mathcal{H} \to \mathbb{C}$ is of the form $\Gamma = \Gamma_g$, $g \in \mathcal{H}$.

6.12 Theorem (Orthonormal Basis Theorem). Let $\mathcal{X}$ be an inner product space and $\{e_i\}_{i=1}^{\infty}$ be an orthonormal sequence. The following are equivalent:

(i) $\operatorname{span}\{e_i\}_{i=1}^{\infty} = \left\{\sum_{i=1}^{n} \alpha_i e_i : n \in \mathbb{N}, \alpha_i \in \mathbb{C}\right\}$ is dense in $\mathcal{X}$.

(ii) For every $f \in \mathcal{X}$, we have $\|f\|^2 = \sum_{i=1}^{\infty} |(f,e_i)|^2$ (Bessel's Equality).

(iii) For every $f \in \mathcal{X}$, we have (where limits occur under $\|\cdot\|$)

$$f = \lim_{n\to\infty} \sum_{i=1}^{n} (f,e_i)e_i$$

we write

$$f = \sum_{i=1}^{\infty} (f,e_i)e_i.$$

(iv) For every $f, g \in \mathcal{X}$ we have

$$(f, g) = \sum_{i=1}^{\infty} (f, e_i)(e_i, g)$$

(Parseval's identity).

Note that (iii) justifies calling $\{e_i\}_{i=1}^{\infty}$ an *orthonormal basis*.

Proof. (i) $\leftrightarrow$ (iii): Let $E_n = \operatorname{span}\{e_1, \ldots, e_n\}$. Then $E_n \subseteq E_{n+1}$ for each n. Thus for $f \in \mathcal{X}$,

$$\operatorname{dist}(f, E_n) \geq \operatorname{dist}(f, E_{n+1}).$$

Thus, by the Linear Approximation Lemma we have

$$\operatorname{span}\{e_i\}_{i=1}^{\infty} = \bigcup_{n=1}^{\infty} E_n$$

if and only if for each $f \in \mathcal{X}$,

$$\left\| f - \sum_{i=1}^{n} (f, e_i) e_i \right\| = \operatorname{dist}(f, E_n) \xrightarrow{n \to \infty} 0.$$

We saw (i) $\leftrightarrow$ (iii) last class.

(ii) $\leftrightarrow$ (iii). By the Linear Approximation Lemma,

$$\left\| f - \sum_{i=1}^{n} (f, e_i) e_i \right\|^2 = \|f\|^2 - \sum_{i=1}^{n} |(f, e_i)|^2.$$

Hence,

$$\|f\|^2 = \lim_{n \to \infty} \sum_{i=1}^{n} |(f, e_i)|^2 \iff \lim_{n \to \infty} \left\| f - \sum_{i=1}^{n} (f, e_i) e_i \right\|^2 = 0.$$

(iii) $\to$ (iv). Let $g \in \mathcal{X}$. By an earlier proposition, the function $\Gamma_g : \mathcal{X} \to \mathbb{C}$ given by $\Gamma_g(f) = (f, g)$ is continuous. Hence,

$$\begin{aligned}
(f, g) = \Gamma_g(f) = \Gamma_g\left(\lim_{n \to \infty} \sum_{i=1}^{n} (f, e_i) e_i\right) & \\
= \lim_{n \to \infty} \Gamma_g\left(\sum_{i=1}^{n} (f, e_i) e_i\right) & \qquad \text{(continuity)} \\
= \lim_{n \to \infty} \sum_{i=1}^{n} (f, e_i) \Gamma_g(e_i) & \qquad \text{(linearity)} \\
= \lim_{n \to \infty} \sum_{i=1}^{n} (f, e_i)(e_i, g). &
\end{aligned}$$

(iv) $\to$ (ii). Take $f = g$ and note that $(f, e_i)(e_i, f) = (f, e_i)\overline{(f, e_i)} = |(f, e_i)|^2$. $\square$

Notice that for any orthonormal sequence $\{e_i\}_{i=1}^{\infty}$, $f \in \mathcal{X}$, we have that

$$0 \leq \operatorname{dist}\left(f, \operatorname{span}\{e_i\}_{i=1}^{n}\right)^2 = \|f\|^2 - \sum_{i=1}^{n} |(f, e_i)|^2 \implies \|f\|^2 \geq \sum_{i=1}^{n} |(f, e_i)|^2$$

for every $n \in \mathbb{N}$, by the Linear Approximation Lemma. Thus, if we take $n \to \infty$, we immediately obtain

$$\|f\|^2 \geq \sum_{i=1}^{\infty} |(f, e_i)|^2 \quad (\textbf{Bessel's Inequality}).$$

Equality holds when $f \in \overline{\operatorname{span}\{e_i\}_{i=1}^{\infty}}$ (here we mean the closure).

6.13 Theorem (Abstract Plancherel Theorem). Let $\mathcal{X}$ be an inner product space and let $\{e_i\}_{i=1}^{\infty} \subset \mathcal{X}$ be an orthonormal basis for $\mathcal{X}$ (in the sense of the earlier theorem). Then the operator $U : \mathcal{X} \to \ell_2$ given by $Uf = ((f, e_i))_{i=1}^{\infty}$ is an isometry; i.e. $\underbrace{\|Uf\|_2}_{\text{in } \ell_2} = \underbrace{\|f\|}_{\text{in } \mathcal{X}}$ and $\underbrace{(Uf, Ug)}_{\text{in } \ell_2} = \underbrace{(f, g)}_{\text{in } \mathcal{X}}$.

Proof. By Bessel's Inequality, we have that for any $f \in \mathcal{X}$,

$$\|Uf\|_2^2 = \sum_{i=1}^{\infty} |(f, e_i)|^2 \leq \|f\|^2,$$

so U is indeed a linear map into ℓ_2. Next, we observe that

$$\begin{aligned}(Uf, Ug) &= (((f, e_i))_{i=1}^{\infty}, ((g, e_i))_{i=1}^{\infty}) \\ &= \sum_{i=1}^{\infty} (f, e_i)(e_i, g) && \text{(Parseval's Equality)} \\ &= (f, g).\end{aligned}$$

Note that we applied Parseval's Equality twice, first to go from (Uf, Ug) to the sum, and then to go from the sum to (f, g); the first application may be justified by the fact that

$$U(\{e_i\}_{i=1}^{\infty}) = \{Ue_i : n \in \mathbb{N}\} = \{(0, \dots, 0, \underbrace{1}_{i}, 0, 0, \dots) : i \in \mathbb{N}\}$$

the latter of which we shall soon see is an orthonormal basis for ℓ_2, and hence $\{Ue_i\}_{i=1}^{\infty} \subset \ell_2$ is an orthonormal basis for ℓ_2, furthermore,

$$(Uf, Ue_i)(Ue_i, Ug) = (f, e_i)(e_i, g) \quad \text{since} \quad (Ue_i)_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases};$$

the second application may be justified by the fact that $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{X}$, by assumption.

We may now take $f = g$ and obtain the desired result. ☐

6.14 Example. We have the following examples:

1. $\mathcal{X} = \ell_2$, with $\{e_i\}_{i=1}^{\infty}$ satisfying $(e_i)_j = 0$ if $i \neq j$, and $(e_i)_i = 1$. It is easy to see that $(e_i, e_j) = 0$ if $i \neq j$, and $(e_i, e_i) = 1$. Notice that if $x = (x_1, x_2, \dots) \in \ell_2$ (so that $\sum_{i=1}^{\infty} |x_i|^2 < \infty$), we have that

$$x^{(n)} := (x_1, \dots, x_n, 0, 0, \dots) = \sum_{i=1}^{n} (x, e_i)e_i \in \operatorname{span}\{e_i\}_{i=1}^{n}.$$

   Furthermore, note that

$$\|x - x^{(n)}\|_2 = \left(\sum_{i=n+1}^{\infty} |x_i|^2 \right)^{1/2} \to 0 \quad \text{as} \quad n \to \infty.$$

   Thus, $\operatorname{span}\{e_i\}_{i=1}^{\infty} = \bigcup_{i=1}^{\infty} \operatorname{span}\{e_i\}_{i=1}^{n}$ is dense in ℓ_2 and, therefore, is an orthonormal basis for ℓ_2.

2. Consider $\{\mathbf{e}^k\}_{k\in\mathbb{Z}} \subset L_2(\mathbb{T})$ (since $\{\mathbf{e}^k\}_{k\in\mathbb{Z}} \subset \operatorname{Trig}(\mathbb{T}) \subset C(\mathbb{T}) \subset L_2(\mathbb{T})$). We have

$$(\mathbf{e}^k, \mathbf{e}^\ell) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{e}^k \overline{\mathbf{e}^\ell} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{e}^{k-\ell} = \begin{cases} 1 & k = \ell \\ 0 & k \neq \ell \end{cases}.$$

   Hence, $\{\mathbf{e}^k\}_{k\in\mathbb{Z}}$ is an orthonormal set in $L_2(\mathbb{T})$. We shall show that it is, furthermore, dense in $L_2(\mathbb{T})$.

6.15 Theorem.

1. $\{\mathbf{e}^k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L_2(\mathbb{T})$.
2. $\{\mathbf{e}^k\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $C(\mathbb{T})$ with inner product

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\overline{g}.$$

Proof. Notice these are essentially the same statements since $C(\mathbb{T})$ is dense in $L_2(\mathbb{T})$; nevertheless, we give two distinct proofs to illustrate some techniques for proving orthonormal sets are dense in an inner product space.

(1.) We have already seen that $\{\mathbf{e}^k\}_{k\in\mathbb{Z}}$ is an orthonormal set, thus, we need only verify that $\{\mathbf{e}^k\}_{k\in\mathbb{Z}}$ is dense in $L_2(\mathbb{T})$. We have that $\sigma_n(f) \in \operatorname{span}\{\mathbf{e}^k\}_{k=-n}^n$ and $\operatorname{dist}(f, \operatorname{span}\{\mathbf{e}^k\}_{k=-n}^n) \le \|f - \sigma_n(f)\|_2 \to 0$ as $n \to \infty$ (by the Abstract Summability Kernel Theorem). Hence, condition (3.) of the Orthonormal Basis Theorem is satisfied and $\{\mathbf{e}^k\}_{k\in\mathbb{Z}}$ is therefore an orthonormal basis for $L_2(\mathbb{T})$. [To use (1.) to imply (2.), use estimate of $\|\cdot\|_\infty$ with respect to $\|\cdot\|_2$.]

(2.) Notice that $\operatorname{Trig}(\mathbb{T}) = \operatorname{span}\{\mathbf{e}^k\}_{k\in\mathbb{Z}}$ is an algebra of functions on $\mathbb{T}/\sim_{-\pi=\pi}$ (that is, we identify $-\pi$ with π) which is point separating and is conjugation closed. Thus, by the Stone-Weierstrass Theorem, we have that $\operatorname{Trig}(\mathbb{T})$ is dense with respect to the norm $\|\cdot\|_\infty$ in $C(\mathbb{T})$. Therefore, for any given $\epsilon > 0$ and for every $f \in C(\mathbb{T})$, we may find $h \in \operatorname{Trig}(\mathbb{T})$ such that $\|f - h\|_\infty < \epsilon$ and hence $\|f - h\|_2 \le \|f - h\|_\infty < \epsilon$ so $\operatorname{Trig}(\mathbb{T})$ is dense with respect to the norm $\|\cdot\|_2$ in $C(\mathbb{T})$. □

6.16 Corollary (L_2-convergence of Fourier Series). Let $f \in L_2(\mathbb{T})$. Then

$$\lim_{n\to\infty} \|f - s_n(f)\|_2 = 0.$$

Proof. Note that

$$s_n(f) = \sum_{k=-n}^{n} c_k(f)\mathbf{e}^k = \sum_{k=-n}^{n} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}\,dt\right)\mathbf{e}^k = \sum_{k=-n}^{n} (f, \mathbf{e}^k)\mathbf{e}^k.$$

Since $\{\mathbf{e}^k\}_{k\in\mathbb{Z}}$ is an orthonormal basis, we have by the Orthonormal Basis Theorem that

$$\lim_{n\to\infty} \left\| f - \sum_{k=-n}^{n} (f, \mathbf{e}^k)\mathbf{e}^k \right\|_2 = 0.$$

□

Recall that $s_n(f) = C(D_n)f = D_n * f$ and that

$$|||C(D_n)|||_{L_1(\mathbb{T})} = |||C(D_n)||| = L_n \to \infty$$

as $n \to \infty$ (L_n is the nth Lebesgue constant). In $L_2(\mathbb{T})$, the situation is radically different: we actually have $|||C(D_n)|||_{L_2(\mathbb{T})} = 1$. To see this, notice that

$$\|C(D_n)f\|_2 = \|s_n(f)\|_2 = \left\| \sum_{k=-n}^{n} (f, \mathbf{e}^k)\mathbf{e}^k \right\|_2 \le \|f\|_2$$

by Bessel's Inequality.

6.17 Theorem (Riesz-Fischer Theorem). Let $f \in L_1(\mathbb{T})$. Then

$$f \in L_2(\mathbb{T}) \iff \sum_{k=-\infty}^{\infty} |c_k(f)|^2 < \infty.$$

Proof. ($\rightarrow$) Since $c_k(f) = (f, \mathbf{e}^k)$, we have that

$$\|f\|_2^2 \ge \sum_{k=-n}^{n} |c_k(f)|^2$$

by Bessel's Inequality. Thus,

$$\sum_{k=-\infty}^{\infty} |c_k(f)|^2 = \sup_{n\in\mathbb{N}} \sum_{k=-n}^{n} |c_k(f)| \le \|f\|_2^2 < \infty \quad (f \in L_2(\mathbb{T})).$$

($\leftarrow$) Define $f_n = \sum_{k=-n}^{n} c_k(f)\mathbf{e}^k$ and let $n > m$. We thus have that

$$\begin{aligned} \|f_n - f_m\|_2^2 &= \sum_{k=-n}^{-(m+1)} |c_k(f)|^2 + \sum_{k=m+1}^{n} |c_k(f)|^2 && \text{(Pythagoras' Theorem)} \\ &\to 0 \text{ as } n \to \infty && \text{(tails of convergent series).} \end{aligned}$$

It follows that $(f_n)_{n=1}^\infty$ is Cauchy in $L_2(\mathbb{T})$. Thus, by the completeness of $L_2(\mathbb{T})$, we have that there is $\tilde{f} \in L_2(\mathbb{T})$ such that $\|\tilde{f} - \sum_{k=-n}^n c_k(f)\mathbf{e}^k\| \to 0$ as $n \to \infty$. We note that $c_k(\tilde{f}) = c_k(f)$ by using the function $\Gamma_{\mathbf{e}^k}$. Hence $\tilde{f} = f$ a.e. so $f = \tilde{f}$ in $L_2(\mathbb{T})$. $\square$

Warning: If $f \in C(\mathbb{T})$, then we know:

- $\displaystyle\sum_{n=-\infty}^{\infty} |c_n(f)|^2 < \infty.$
- $\displaystyle\lim_{n\to\infty} \|f - s_n(f)\|_2 = 0.$

We may not have that $\lim_{n\to\infty} s_n(f) = f$ pointwise! There is no known characterization of sequences $(c_n)_{n=-\infty}^\infty$ such that $c_n = c_n(f)$ for some $f \in C(\mathbb{T})$. In A6, we show that if

$$\sum_{n=-\infty}^{\infty} |c_n(f)| < \infty$$

then in fact $f \in A(\mathbb{T})$ (the **Fourier algebra**) and furthermore $\lim_{n\to\infty} \|f - s_n(f)\|_\infty = 0$ – this is the strongest possible conclusion!

6.18 Theorem (Plancherel's Theorem). The map $U : L_2(\mathbb{T}) \to l_2(\mathbb{Z})$ given by

$$f \mapsto (c_n(f))_{n=-\infty}^\infty$$

is a surjective isometry, with $(Uf, Ug) = (f, g)_{L_2}$.

Proof. This is nearly a restatement of the Riesz-Fischer theorem. However, if $(c_n)_{n=-\infty}^\infty \subseteq l_2(\mathbb{Z})$, we need to show that $f \in L_2(\mathbb{T})$, so $c_n(f) = c_n$ for all n. Define

$$f_n = \sum_{k=-n}^{n} c_k \mathbf{e}^k.$$

Verify that $(f_n)_{n=1}^\infty$ is Cauchy in $L_2(\mathbb{T})$ and hence converges to $f \in L_2(\mathbb{T})$. Moreover, $c_n(f) = c_n$ for each n. That U is an isometry which preserves inner product is a result of Bessel's equality and Parseval's identity, from the Orthonormal Basis Theorem. $\square$

6.19 Corollary. $l_2(\mathbb{Z})$ is complete.

Proof. If $((c_k^{(n)})_{k=-\infty}^\infty)_{n=1}^\infty \subseteq l_2(\mathbb{Z})$ is Cauchy, then for each $(c_k^{(n)})_{k=-\infty}^\infty$, there is $f_n \in L_2(\mathbb{T})$ such that $c_k(f_n) = c_k^{(n)}$ for each k, and each n. We have

$$\|f_n - f_m\|_{L_2} = \|Uf_n - Uf_m\|_{l_2} = \|(c_k^{(n)})_{k=-\infty}^\infty - (c_k^{(m)})_{k=-\infty}^\infty\|_{l_2},$$

so that $(f_n)_{n=1}^\infty \subseteq L_2(\mathbb{T})$ is Cauchy. So put $f = \lim_{n\to\infty} f_n$ and so $(c_k(f))_{k=-\infty}^\infty$ is the limit of $(c_k(f_n))_{k=-\infty}^\infty = (c_k^{(n)})_{k=-\infty}^\infty$. $\square$

6.20 Remark. If $f \in L(\mathbb{T})$ satisfies

$$\int_{-\pi}^{\pi} |f|^p < \infty$$

for some $1 < p < \infty$, then is it the case that

$$\lim_{n\to\infty} s_n(f, x) = f(x)$$

for a.e. x? The answer is yes (see Carleson's Theorem from the 1960s for $p = 2$).

6.21 Lemma. Let $\mathcal{X}$ be a Banach space and $(a_k)_{k=-\infty}^\infty \subseteq \mathcal{X}$. Define

$$s_n = \sum_{k=-n}^{n} a_k, \qquad \sigma_n = \frac{1}{n+1}\sum_{j=0}^{n} s_j.$$

If $\lim_{n\to\infty} \sigma_n$ exists and also $\sup_{k\in\mathbb{N}} \|a_k\| < \infty$ then $\lim_{n\to\infty} s_n$ exists and is equal to $\lim_{n\to\infty} \sigma_n$.

Proof. Fix, for the moment, $\lambda > 1$. If $n \in \mathbb{N}$, then $n+1 \leq \lfloor \lambda n \rfloor$ (i.e. we have $\frac{\lambda n}{n+1} > 1$), then

$$\sum_{k=n+1}^{\lfloor \lambda n \rfloor} \frac{1}{k} \leq \log \frac{\lfloor \lambda n \rfloor}{n} \leq \log \lambda.$$

We recall that

$$\sigma_n = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \sigma_k.$$

Hence for n large enough we have $n+1 \leq \lfloor \lambda n \rfloor$,

$$\frac{\lfloor \lambda n \rfloor}{n+1} \sigma_{\lfloor \lambda n \rfloor} - \sigma_n = \frac{\lfloor \lambda n \rfloor + 1}{n+1} \left(\sum_{k=-\lfloor \lambda n \rfloor}^{-(n+1)} + \sum_{k=n+1}^{\lfloor \lambda n \rfloor} \right) \left(\frac{\lfloor \lambda n \rfloor + 1 - |k|}{\lfloor \lambda n \rfloor + 1} \right) a_k +$$
$$\sum_{k=-n}^{n} \left(\frac{\lfloor \lambda n \rfloor + 1}{n+1} - \frac{|k|}{n+1} \right) a_k - \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) a_k.$$

For convenience, denote

$$E_n = \frac{\lfloor \lambda n \rfloor + 1}{n+1} \left(\sum_{k=-\lfloor \lambda n \rfloor}^{-(n+1)} + \sum_{k=n+1}^{\lfloor \lambda n \rfloor} \right) \left(\frac{\lfloor \lambda n \rfloor + 1 - |k|}{\lfloor \lambda n \rfloor + 1} \right) a_k$$

so the above thing is equal to

$$= E_n + \frac{\lfloor \lambda n \rfloor - n}{n+1}.$$

Thus,

$$s_n - \frac{\lfloor \lambda n \rfloor + 1}{\lfloor \lambda n \rfloor - n} \sigma_{\lfloor \lambda n \rfloor} + \frac{n+1}{\lfloor \lambda n \rfloor - n} \sigma_n = \frac{n+1}{\lfloor \lambda n \rfloor - n} E_n.$$

So then we have

$$\left\| \frac{n+1}{\lfloor \lambda n \rfloor - n} E_n \right\| \leq \frac{\lfloor \lambda n \rfloor + 1}{\lfloor \lambda n \rfloor - n} \left(\sum_{k=-\lfloor \lambda n \rfloor}^{-(n+1)} + \sum_{k=n+1}^{\lfloor \lambda n \rfloor} \right) \left(\frac{\lfloor \lambda n \rfloor + 1 - |k|}{\lfloor \lambda n \rfloor + 1} \right) \|a_k\|,$$

which follows from the fact that

$$\frac{n+1}{\lfloor \lambda n \rfloor - n} \frac{\lfloor \lambda n \rfloor + 1}{n+1} \leq \frac{\lfloor \lambda n \rfloor + 1 - (n+1)}{\lfloor \lambda n \rfloor + 1} = \frac{\lfloor \lambda n \rfloor - n}{\lfloor \lambda n \rfloor + 1}.$$

Thus, we get that

$$\left\| \frac{n+1}{\lfloor \lambda n \rfloor - n} E_n \right\| \leq \left(\sum_{k=-\lfloor \lambda n \rfloor}^{-(n+1)} + \sum_{k=n+1}^{\lfloor \lambda n \rfloor} \right) \frac{C}{|K|}$$

where we put

$$C = \sup_k |k| \|a_k\|.$$

Fix $\epsilon > 0$, pick $\lambda > 1$ such that $2C \log \lambda < \epsilon$. Also note that $\lambda n - 1 \leq \lfloor \lambda n \rfloor \leq \lambda n$. This implies that

$$\lim_{n \to \infty} \frac{\lfloor \lambda n \rfloor}{n} = \lambda$$

thus we get

$$\lim_{n \to \infty} H_n = \lim_{n \to \infty} \left(\frac{\frac{\lfloor \lambda n \rfloor}{n} + \frac{1}{n}}{\frac{\lfloor \lambda n \rfloor}{n} - \frac{n}{n}} \sigma_{\lfloor \lambda n \rfloor} - \frac{\frac{n}{n} + \frac{1}{n}}{\frac{\lfloor \lambda n \rfloor}{n} - \frac{n}{n}} \sigma_n \right) = \frac{\lambda}{\lambda - 1} \lim_{n \to \infty} \sigma_n - \frac{1}{\lambda - 1} \lim_{n \to \infty} \sigma_n = \lim_{n \to \infty} \sigma_n$$

so we have for large enough n,

$$\begin{aligned} \|s_n - \lim_{n \to \infty} \sigma_n\| &\leq \|s_n - H_n\| + \|H_n - \lim_{n \to \infty} \sigma_n\| \\ &\leq \left\| \frac{n+1}{\lfloor \lambda n \rfloor - n} E_n \right\| + \frac{\epsilon}{2} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Assignment #6 - may hand in Monday. Assignment #5 - may retrieve from pick-up box by my office today.

6.22 Theorem (Hardy's Tauberian Theorem). We have:

(i) If $f \in L(\mathbb{T})$ and $\sup_k |kc_k(f)| < \infty$ then for any $t \in [-\pi, \pi]$ for which $\lim_{n\to\infty} \sigma_n(f,t)$ exists, we have $\lim_{n\to\infty} s_n(f,t)$ exists as well. In particular, if $\omega_f(t) = \lim_{h\to 0^+} \frac{1}{2}[f(t-s)+f(t+s)]$ exists, and $\sup_k |kc_k(f)| < \infty$, then

$$\lim_{n\to\infty} s_n(f,t) = \omega_f(t).$$

Moreover if I is any open interval on which f is continuous, and $\sup_k |kc_k(f)| < \infty$, then for any closed interval $J \subseteq I$

$$\lim_{n\to\infty} \sup_{t\in J} |s_n(f,t) - f(t)| = 0$$

(ii) If $\mathcal{B}$ is a homogeneous Banach space such that $\|\mathbf{e}^k\|_{\mathcal{B}} \leq C$ (some fixed C) for all k, and $f \in \mathcal{B}$, such that $\sup_k |kc_k(f)| < \infty$ then

$$\lim_{n\to\infty} \|s_n(f) - f\|_{\mathcal{B}}.$$

Proof. We have:

(i) We let, in the context of the last lemma, $\mathcal{X} = \mathbb{C}$. Then we have for $t \in [-\pi, \pi]$, that

$$|k \underbrace{c_k(f)e^{ikt}}_{\mathbb{C}}| = |kc_k(f)|,$$

the supremum of which, over k, is finite. Hence we always get the conditions of the lemma. Now, appeal to Fejér's theorem.

(ii) Let $\mathcal{X} = \mathcal{B}$.

$$\|kc_k(f)\mathbf{e}^k\|_{\mathcal{B}} = |kc_k(f)|\|\mathbf{e}^k\|_{\mathcal{B}} \leq |kc_k(f)|C$$

and we note that $\lim_{n\to\infty} \sigma_n(f) = f$ in $\mathcal{B}$, by virtue of the Abstract Summability Kernel Theorem. $\square$

7 Gibbs phenomenon

7.1 Example (special example). Let

$$F(t) = \frac{1}{2} - \frac{t}{2\pi}$$

for all $t \in (0, 2\pi)$ continued 2π-periodically to $\mathbb{R}$. [diagram].

7.2 Proposition. We have $c_k(F) = \frac{1}{2\pi ik}$ for choices of $k \neq 0$, noting that $c_0(F) = 0$, and

$$s_n(F,t) = \sum_{k=1}^{n} \frac{\sin kt}{\pi k}.$$

In particular,

(i) $\lim_{n\to\infty} s_n(F,0) = 0 = \omega_f(0)$.

(ii) $\lim_{n\to\infty} s_n(F,t) = F(t)$, for $t \in [-\pi,\pi] \setminus \{0\}$ and on intervals of the form $[\delta, 2\pi-\delta]$, $\delta > 0$ we have

$$\lim_{n\to\infty} \sup_{t\in[\delta,2\pi-\delta]} |s_n(F,t) - F(t)| = 0.$$

Sketch of proof. Computations of $c_k(F)$, $s_k(F,t)$ are left as exercise. Point (i) is obvious. Point (ii) is a consequence of Hardy's Tauberian Theorem. $\square$

The following was noticed: [illustration of this using Maple on the web site] [diagram].

7.3 Lemma. Let $F(t) = \frac{1}{2} - \frac{t}{2\pi}$ for almost every $t \in [0, 2\pi]$, continued 2π-periodically to $\mathbb{R}$. Then

$$\lim_{n\to\infty} s_n(F, \tfrac{\pi}{n}) = \frac{1}{\pi}\int_0^\pi \frac{\sin x}{x}\, dx \approx 0.59.$$

Thus, we define

$$G_s := \lim_{n\to\infty} \left[s_n(F, \tfrac{\pi}{n}) - F(\tfrac{\pi}{n})\right] \approx 0.089$$

and call this the **Gibbs constant**.

Proof. Recall

$$s_n(F,t) = \sum_{k=1}^{n} \frac{\sin(kt)}{\pi k}.$$

So

$$s_n(F, \tfrac{\pi}{n}) = \sum_{k=1}^{n} \frac{\sin(\frac{k\pi}{n})}{\frac{\pi k}{n}} \cdot \frac{1}{n} = \frac{1}{\pi} \sum_{k=1}^{n} \frac{\sin(\frac{k\pi}{n})}{\frac{k\pi}{n}} \left[\frac{k\pi}{n} - \frac{(k-1)\pi}{n} \right] \xrightarrow{n\to\infty} \frac{1}{\pi}\int_0^{\pi} \frac{\sin x}{x}\, dx.$$

The rest is numerical estimation. □

> Bonus question: Let $f : [a,b] \to \mathbb{R}$ be Lipschitz. Show that f' exists a.e. on $[a,b]$. Will accept submissions up to April 11.
>
> Also, show that
>
> $$\int_a^b f' = f(b) - f(a).$$

7.4 Theorem (Gibbs). Let $f \in L(\mathbb{T})$ be boundedly piecewise differentiable, i.e. $f'(t)$ exists except at finitely many points, and $|f'(t)| \le M$ where $f'(t)$ exists. Let $s_1, \ldots, s_m \in [-\pi, \pi]$ be the points where differentiability fails. Then if

$$f(s_j^-) := \lim_{h\to 0^+} f(s_j - h), \qquad f(s_j^+) := \lim_{h\to 0^+} f(s_j + h)$$

we have that these limits exist, and if we let

$$\gamma_j = \gamma_f(s_j) = f(s_j^+) - f(s_j^-)$$

(this is the size of the jump). Then

$$\lim_{n\to\infty} \left[s_n(f, s_j \pm \tfrac{\pi}{n}) - f(s_j \pm \tfrac{\pi}{n}) \right] = \pm \gamma_f(s_j) G_s$$

where

$$G_s = \frac{1}{\pi}\int_0^{\pi} \frac{\sin x}{x}\, dx \approx 0.09$$

is the Gibbs constant.

Proof. First, if we fix j ($1 \le j \le m$), then there is $\delta > 0$ such that f is boundedly differentiable on $(s_j - \delta, s_j)$ and on $(s_j, s_j + \delta)$. Thus, f is uniformly continuous on each of these intervals, in fact Lipschitz (like A6Q2c). Hence if $s_j - \delta < t_1 < \ldots < s_j$ with

$$\lim_{\ell\to\infty} t_\ell = s_j$$

then $(f(t_\ell))_{\ell=1}^{\infty}$ is Cauchy, so

$$f(s_j^-) = \lim_{\ell\to\infty} f(t_\ell)$$

exists (this is a PM351 type argument). Similarly $f(s_j^+)$ exists. As usual, we let

$$\omega_f(s_j) = \frac{1}{2}\left[f(s_j^-) + f(s_j^+) \right].$$

We define[16] $g \in L(\mathbb{T})$ for $t \in [-\pi, \pi]$ by

$$g(t) = \begin{cases} f(t) - \sum\limits_{j=1}^{n} \gamma_j F(t - s_j) & \text{if } t \notin \{s_1, \ldots, s_m\} \\ \omega_f(s_j) - \sum\limits_{\substack{i=1 \\ i\ne j}}^{m} \gamma_j F(s_j - s_i) & \text{if } t = s_j. \end{cases}$$

It is straightforward, though tedious, to check that

$$g(s_j^+) = g(s_j) = g(s_j^-).$$

[16]"OK – what's g going to do for a living?"

So g is continuous, i.e. g is differentiable for $t \notin \{s_1, \ldots, s_m\}$. In fact, $g \in D(\mathbb{T})$ [A6Q2c]. So in particular, $g \in A(\mathbb{T})$ and thus the Fourier series converges uniformly:

$$\lim_{n\to\infty} \|s_n(g) - g\|_\infty = 0.$$

Thus, for each j, we have

$$\begin{aligned} |s_n(g, s+j \pm \tfrac{\pi}{n}) - g(s_j)| &\le |s_n(g, s_j \pm \tfrac{\pi}{n}) - g(s_j \pm \tfrac{\pi}{n})| + |g(s_j, \pm\tfrac{\pi}{n}) - g(s_j)| \\ &\le \|s_n(g) - g\|_\infty + |g(s_j \pm \tfrac{\pi}{n}) - g(s_j)| \end{aligned} \tag{\dagger}$$

as $n \to \infty$, the first term above goes to 0, and

$$g(s_j \pm \tfrac{\pi}{n}) \to g(s_j)$$

since g is continuous. Now

$$f = g + \sum_{j=1}^m \gamma_j s_j * F$$

off of $\{s_1, \ldots, s_m\}$ and hence we have for each j

$$\begin{aligned} \lim_{n\to\infty} \left[s_n(f, s_j \pm \tfrac{\pi}{n}) - f(s_j \pm \tfrac{\pi}{n})\right] &= \lim_{n\to\infty} \left[s_n(g, s_j \pm \tfrac{\pi}{n}) + \sum_{i=1}^m \gamma_i \underbrace{s_n(F, s_j - s_i \pm \tfrac{\pi}{n})}_{\text{just like in proof of Dini etc}} - f(s_j \pm \tfrac{\pi}{n}) \right] \\ &= \underbrace{g(s_j)}_{\text{by } (\dagger)} \pm \gamma_j \underbrace{(G_s + \tfrac{1}{2})}_{\text{by Gibbs lemma}} + \sum_{\substack{i=1 \\ i\ne j}}^m \gamma_i \underbrace{F(s_j - s_i)}_{*} - f(s_j^\pm) \end{aligned}$$

and (*) is true by Hardy's Tauberian Theorem applied to an interval on which F is continuous. The above is equal to

$$\begin{aligned} \overbrace{\underbrace{\frac{1}{2}[f(s_j^+) + f(s_j^-)]}_{\omega_f(s_j)} - \sum_{\substack{i=1 \\ i\ne j}}^m \gamma_i F(s_j - s_i)}^{g(s_j)} \pm [f(s_j^+) - f(s_j^-)](G_s + \tfrac{1}{2}) + \sum_{\substack{i=1 \\ i\ne j}}^m \gamma_i F(s_j - s_i) - f(s_j^\pm) &= \pm[f(s_j^+) - f(s_j^-)]G_s \\ &= \pm \gamma_f(s_j) G_s. \end{aligned}$$
□

7.5 Remark. Recall that we had the function

$$F(t) = \frac{1}{2} - \frac{t}{2\pi}$$

which was designed specifically to put a gap of size 1 at 0. Suppose we take a look at the Fourier sums of F at this "travelling point" $\frac{\pi}{n}$:

$$\lim_{n\to\infty} \left[\underbrace{\sigma_n}_{\text{Césaro sums}} (F, \tfrac{\pi}{n}) - F(\tfrac{\pi}{n}) \right] \approx -0.11.$$

[DIAGRAM with G_s and G_σ].

7.6 Theorem. Let $f \in L(\mathbb{T})$ satisfy

- f is piecewise differentiable (i.e. f is differentiable except at finitely many points in $[-\pi, \pi]$)
- $\displaystyle \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'| = \|f'\|_1 < \infty.$
- f is bounded, i.e. $\|f\|_\infty < \infty$.

Then in fact,

$$\sup_{k\in\mathbb{Z}} |kc_k(f)| < \infty.$$

Proof. See website. □

7.7 Remark (application). Let $f(t) = \sqrt{|t|}$, $t \in [-\pi, \pi]$ continued 2π-periodically. Notice that f is piecewise differentiable, and

$$|f'(t)| = \frac{1}{\sqrt{|t|}}$$

almost everywhere, so that $\|f'\|_1 < \infty$. Thus

$$\sup_{k \in \mathbb{Z}} |kc_k(f)| < \infty$$

(i.e. $c_k(f) = O(\frac{1}{|k|})$). Then by Hardy's Tauberian Theorem,

$$\lim_{n \to \infty} \|s_n(f) - f\|_\infty = 0.$$

Notice that $f \notin D(\mathbb{T})$ from A6.

7.8 Remark (fact). It is true that $f \in A(\mathbb{T})$, which can be gleamed from a theorem in a book of Katznelson.

Important points for the final exam:

- continuous functions are dense in L_p (the case $p = 1$ too).
- development of the Fourier algebra (functions whose Fourier coefficients are summable).
- important function spaces and their related spaces of Fourier coefficients (ordered by inclusion)
  - $A(\mathbb{T})$ and $\ell_1(\mathbb{Z})$ by definition
  - $C(\mathbb{T})$ and ?
  - $L_2(\mathbb{T})$ and $\ell_2(\mathbb{Z})$ due to Riesz-Fischer and Plancherel
  - $L_1(\mathbb{T})$ and $A(\mathbb{Z})$
  - ?? and $c_0(\mathbb{Z})$

  It turns out that ? $= C^*(\mathbb{Z})$, and ?? $= C^*(\mathbb{T})$. They are not identifiable as a space of "functions". Also, Riemann-Lebesgue tells us that $A(\mathbb{Z}) \subsetneq c_0(\mathbb{Z})$.
- there may be questions about which function spaces a given function is a member of.

Review diagram is posted on `mlbaker.org`.