Microeconomics Theory

Johnew Zhang

December 9, 2017

Contents

Background

The first section is about making rational decisions, notion of uncertainty and probability density functions. There will be assignments, a midterm and a final (Please find the weight of each component on the syllabus).

1 Choice Under Constraint

Consumers are always going to choose the most rational decision (maximize their benefit) based on cost constraints.

1.1 Consumer Choice

Define a choice set $\mathbf{X}$. For a corporation, it wants the most profitable choices. In general, we need a way to rank those choices so that we can figure out which one is the best choice. Thus we need to define a preference ordering.

- $x \gtrsim y$ means "$\mathbf{X}$ is preferred or indifferent to y".
- if $x \gtrsim y$ and $y \gtrsim x \implies x \sim y$.
- Ranking is complete if $\forall x, y \in \mathbf{X}$, either $x \gtrsim y$ or $y \gtrsim x$.
- Transitive $x \gtrsim y$ and $y \gtrsim z \implies x \gtrsim z$.
- Ranking (ordering) is rational if it is complete and transitive.
- Strict preference $x > y$ if $x \gtrsim y$ and not $y \gtrsim x$.

With the above, we can conclude a few properties.

Proposition 1. If $\gtrsim$ is rational, then

- $>$ strict preference is transitive.
- $\sim$ is transitive.
- If $x > y \gtrsim z \implies x > z$

Proof.

- To prove $x > y, y > z \implies x > z$. $x > y \implies x \gtrsim y$ and $y > z \implies y \gtrsim z \implies x \gtrsim z$. Assume $x \sim z$, Let's prove this by contradiction. $x \gtrsim z$ and $z \gtrsim x$. Hence $z \gtrsim y$ (contradiction).
- $x \sim y, y \sim z \implies x \sim z$. We know $x \gtrsim y$, $y \gtrsim x$, $y \gtrsim z$, $z \gtrsim y$. Therefore, $x \gtrsim z$ and $z \gtrsim x$. Therefore, $z \sim x$.
- Assume not true, so $z \gtrsim x$. Therefore, $y \gtrsim z \gtrsim x \implies y \gtrsim x$. Contradiction.

$\square$

Now, let's assume two consumers are buying and selling commodities, N different commodities. Choices are $x \in \mathbb{R}^N$ where $x = (x_1, x_2, x_3, \cdots, x_N)$. Prices are also a vector in $\mathbb{R}^N$. Cost of consumption vector X is just $P \cdot x = \sum_{i=1}^{N} P_i x_i$. Let's also define the budget set $B = \{x \in \mathbb{R}^N : P \cdot x \leq W\}$ where W is wealth.

Problem: choose best point in B according to $\gtrsim$.

Demand correspondence $x(p, W) = \{x \in \mathbb{R}^N : x \in B \text{ and } \forall y \in B, x \gtrsim y\}$. x depends on p, W. Maybe single point or a set of points. Here $P \cdot x = W$ is called the budget hyperplane. Price vector P is orthogonal to the budget hyperplane $P \cdot x = W$.

Proof. Suppose y satisfies $P \cdot y = W$. Let x be in the hyperplane, $x \in \{z : P \cdot z = W\}$. Then $P \cdot (x - y) = P \cdot x - P \cdot y = w - w = 0$. $\square$

Definition 1. *Demand correspondence $x(P,W)$ is homogeneous of degree zero if*

$$x(P,W) = x(\alpha P, \alpha W) \forall \alpha > 0$$

$\{x : P \cdot x \leq W\} = \{x : \alpha P \cdot x \leq \alpha W\}$.

Definition 2. *Walras's Law: demand correspondence $x(P,W)$ satisfies Walras's Law if $\forall P \geq 0$ and $W > 0$. we have $P \cdot x = W, \forall x \in x(P,W)$.*

Assume that $x(P,w)$ is a function, i.e. $x(P,W)$ is a single point, not a a set. Let $x(\bar{P},W)$, function of $x, P = \bar{P}$ vary W. Show how choice varies with respect to the income and wealth. This path is sometimes called "Engel Curve".

$$D_p(P,W) = (\frac{\partial x_i}{\partial P_j})_{i,j}$$

where each diagonal entry is called own price effect and the other entries are called cross price effects. Price elasticity of demand (PED) is

$$\epsilon_{n,k} = \frac{\partial x_n(P,W)}{\partial P_k}\frac{P_k}{x_n(P,W)} = \frac{\frac{\partial x_n}{x_n}}{\frac{\partial P_k}{P_k}}$$

Similarly, we can also at the income and call it income elasticity of demand

$$\epsilon_{n,W} = \frac{\partial x_n(P,W)}{\partial W}\frac{W}{x_n(P,W)} = \frac{\frac{\partial x_n}{x_n}}{\frac{\partial W}{W}}$$

We call negative income elasticity of demand good inferior good.

Proposition 2. Sum of all elasticities for good n is zero.

Proof. $x(\alpha P, \alpha W), \alpha > 0$.

Differentiate the demand by α, set $\alpha = 1$.

$$\frac{\partial x_n}{\partial \alpha} = \sum_k \frac{\partial x_n}{\partial \alpha}\frac{\partial \alpha P_k}{\partial \alpha} + \frac{\partial x_n}{\partial \alpha W}\frac{\partial \alpha W}{\partial \alpha}$$

Set $\alpha = 1$, then the above will be

$$= (\frac{\partial x_n}{\partial P_1}, \cdots, \frac{\partial x_n}{\partial P_N}) \cdot (P_1, \cdots, P_N) + \frac{\partial x_n}{\partial W}W \tag{1}$$

$$= \sum \frac{\partial x_n}{\partial P_k}\frac{P_k}{x_n} + \frac{\partial x_n W}{\partial W x_n} = 0 \tag{2}$$

□

Ordering $\succsim$ is monotone if $x \in X$ and $y \gg x \implies y > x$. Strongly monotone if $y \succsim x$ and $y \neq x \implies y > x$. $\succsim$ is locally non-satiated if for every x and every $\epsilon > 0$. $\exists y$ such that $\|x - y\| < \epsilon$ and $y > x$. This means points that are arbitrarily close to x that are preferred to it.

Indifference set for $x = \{y : y \sim x\}$. Upper contour set for $x = \{y : y \succsim x\}$. Reversely, we have lower contour set where $\{y : y \precsim x\}$.

S is convex if for any $x, y \in S$, $\lambda x + (1-\lambda)y \in S, \forall \lambda \in [0,1]$.

S is strictly convex if for any $x, y \in S$, $\lambda x + (1-\lambda)y$ is in the interior of S, $0 < \lambda < 1$. Here, $y \in S$ is an interior point if there exists neighbourhoods of x containing only points in S.

Proposition 3. If upper contour sets are strictly convex, then $x(P,W)$ is a single point.

Proof. Assume not true. Assume $x_1(P,W)$ and $x_2(P,W)$ both satisfy definition of $x(P,W)$ and $x_1(P,W) \neq x_2(P,W)$. Then $x_3 = \lambda x_1(P,W) + (1-\lambda)x_2(P,W), 0 < \lambda < 1$. Claim $x_3 > x_1$ or x_2 (Contradiction) □

$\succsim$ is homothetic if $x \sim y \implies \alpha x \sim \alpha y$, for any $\alpha > 0$. Once we know one indifference surface, we know all of them.

Definition 3. *Quasi-linearity with respect to good 1 if*

1. *all indifference surfaces are parallel displacements along axis of commodity 1, i.e. if* $x \sim y \implies x + \alpha e_1 \sim y + \alpha e_1, 0 < \alpha, e_1 = (1, 0, \cdots, 0)$.

$\succsim$ on X is continuous if its upper contour sets and its lower contour sets are closed.

Representation of $\succsim$ by a Normal-Valued Function

Let $U : \mathbb{R}^N \to \mathbb{R}, x \in \mathbb{R}^N$. $U(x) \to \mathbb{R}^1$, $u(x) \succsim u(y) \iff x \succsim y$.

Theorem 1. *If the order is rational and complete, then it is continuous.*

Suppose we are lexicographical ordering on $\mathbb{R}^2$. Define $\succsim$ as $x \succsim y$ if either $x_1 > y_1$ or $x_1 = y_1$ and $x_2 > y_2$. This is an argument that the upper contour sets are not closed.

Proposition 4. If $u(x)$ represents $\succsim$ then so does $\phi(u(x))$ for any increasing function $\phi : \mathbb{R}^N \to \mathbb{R}^1$. u is unique up to an order preserving transformation.

Proof. Need to show $Iu(x) = I\phi(u(x))$. $Iu(x) = \{y : u(y) = u(x)\}$
and $I\phi(u(x)) = \{y : \phi(u(y)) = \phi(u(x))\}$. Suppose

$$u(x) = u(y) \implies \phi(u(x)) = \phi(u(y)) \implies Iu(x) \subset I\phi(u(x))$$

$$\phi(u(x)) = \phi(u(y)) \implies u(x) = u(y)$$

$$I\phi(u(x)) \subset Iu(x)$$

□

Proposition 5. Suppose $u(x)$ represents $\succsim$ strictly convex and non-satiated. Then the demand function $x(P, W)$ satisfies

1. Homogeneous of degree zero
2. Walras' Law

1.2 Consumer Maximization Problem (UMP)

Choose x to $\max u(x)$ subject to $P \cdot x = W$.

We can solve the above using Lagrangean function such that $\mathcal{L} = u(x) + \lambda[w - p \cdot x]$ where λ is called the lagrange multiplier and sometimes it is called shadow price.

If $\exists \lambda^* \geq 0$ such that (x^*, λ^*) form a saddle point of $\mathcal{L}$. Then x^* solves the constrained max problem. A saddle point means the local derivative of the point is zero, that is

$$x^* \text{ maximize } \mathcal{L}(x, \lambda^*)$$

$$\lambda^* \text{ minimize } \mathcal{L}(x^*, \lambda)$$

Proof. $\mathcal{L}(x^*, \lambda^*) = u(x^*) + \lambda^*[w - px^*]$. Derivative with respect to $\lambda = 0$ here. That implies $w - px^* = 0$. This is the budget constrain.

$$\mathcal{L}(x^*, \lambda^*) \geq \mathcal{L}(x, \lambda^*) \forall x$$

. This implies $u(x^*) + \lambda^*(w - px^*) \geq u(x) + \lambda^*(w - px), \forall x$. Therefore, $u(x^*) \geq u(x) + \lambda^*[w - px], \forall x$. □

Choose x to $\max u(x)$, $p \cdot x = w$. $\mathcal{L} = u(x) + \lambda[w - p \cdot x]$. Then $\frac{\partial u}{\partial x_i} - \lambda p_i = 0$. Therefore, $\frac{\partial u(x)}{\partial x_i} = \lambda p_i$ and $\frac{\partial u(x)}{\partial x_j} = \lambda p_j$. Therefore, $\frac{\partial u / \partial x_i}{\partial u / \partial x_j} = \frac{p_i}{p_j}, \forall i, j, i \neq j$.

Side note: implicit derivative $\frac{\partial y}{\partial x} = -\frac{\partial F / \partial x}{\partial F / \partial y}$

1.2.1 Interpretation of λ

Suppose we make small changes in $x, \Delta x$. What's the change in utility?

$$\Delta u = \sum_i \frac{\partial u}{\partial x_i} \Delta x_i, \frac{\partial u}{\partial x_i} = \lambda p_i$$

$$\Delta u = \sum_i \lambda p_i \Delta x_i = \lambda \sum_i p_i \Delta x_i (\text{ change u expenditure } \Delta w)$$

$$\Delta u = \lambda \Delta w$$

$$\frac{\Delta u}{\Delta w} = \lambda$$

The above implies if λ is large, then the more change in wealth, the more utility.

1.3 Indirect Utility Function

$u(x)$ through the constraint maximization problem $u(x^*)$ depends on p, w.

We define $V(p, w) = u(x^*)$. Here $\frac{\partial V}{\partial w} = \lambda$; $\frac{V}{\partial p_n} = -\lambda x_n(p, w)$. Now let's prove the second one.

Proof.

$$\frac{\partial V}{\partial p_n} = \sum_j \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial p_n} = \sum_j \lambda p_j \frac{\partial x_j}{\partial p_n}$$

We know $\sum x_j p_j = w$. $\lambda x_n + \lambda \sum \frac{\partial x_j}{\partial p_n} p_j = 0$. Therefore, the first equation is equal to $-\lambda x_n$. □

Examples

Cobb-Douglas Utility Function $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$. This function is homogenous. Choose x_1, x_2 to maximize. We will find that $x_1 = w\frac{\alpha}{p_1}$ and $x_2 = w\frac{1-\alpha}{p_2}$. The price elasticity to the demand is constant (1). In other words, $\frac{p_1 x_1}{w} = \alpha$ and $\frac{p_2 x_2}{w} = 1 - \alpha$.

If we write this as an indirect utility function. Then $V(p, w) = u(x^*) = w(\frac{\alpha}{p_1})^\alpha (\frac{1-\alpha}{p_2})^{1-\alpha}$.

Linear Utility $u(x) = ax_1 + bx_2$. The optimal point is always at the either side of corners depending on the slope (corner solutions). In other words, the solution is dependent on the slopes of both utility function and budget line.

Fixed Coefficient $u(x_1, x_2) = \min\{ax_1, bx_2\}$. The solution will be $x_2 = x_1 \frac{a}{b}$.

Constant Elasticity Substitution $u(x_1, x_2) = \{x_1^p + x_2^p\}^{1/p}$. $p_1 x_1 + p_2 x_2 - w = 0$. First order condition is

$$\frac{1}{p}\{x_1^p + x_2^p\}^{1/p-1} p x_i^{p-1} = \lambda p_i, i = 1, 2$$

Divide $i = 1, 2$, then we get

$$\frac{x_1}{x_2} = \frac{p_1}{p_2}^{1/p-1}$$

Let $\rho. = \frac{p}{p-1}$. Using the budget constraint, the solution is $x_1 = w\frac{p_2^{\rho-1}}{p_1^\rho + \rho_2^r}$ and $x_2 = w\frac{p_1^{\rho-1}}{p_1^\rho + \rho_2^r}$

In addition, the indirect utility function is $V(p, w) = w(p_1^\rho + p_2^\rho)^{-\frac{1}{\rho}}$.

1.4 Expenditure Minimization Problem (EMP)

Choose x to minimize $p \cdot x$ such that $u(x) = \bar{u}$. The EMP is dual of the UMP.

Definition 4. *The expenditure function $e(p, U)$ is the solution to the EMP problem for prices p and required utility U.*

In other words, $e(p, U) = p \cdot x^*(p, U)$.

Example

Cobb-Douglas We can write $U(x) = \alpha \ln x_1 + (1-\alpha)\ln x_2$. Then we can use Lagaranean to solve the constraint problem. Hence the expenditure function $e^U(\frac{p_1}{\alpha})^\alpha(\frac{p_2}{1-\alpha})^{1-\alpha}$.

Constant Elasticity Substitution Expenditure is $U(p_1^r + p_2^r)^{\frac{1}{r}}$.

Definition 5. *The optimal commodity vector in the EMP, denoted $h(p,U)$: $\mathbb{R}^N \to \mathbb{R}^N$, is known as Hicksian or compensated demand function.*

From the definition, we can see that

- $x^* = h(p, \bar{U}) = x(p, e(p, U))$
- $U(h(p,U)) = U$
- $e(p,U) = p \cdot h(p,U)$
- $h(p,U) = x(p, e(p,U)$.
- $x(p,W) = h(p, V(p,w))$.

Hicksian compensated demand is the same as derivative of expenditure function with respect to prices. In other words

$$\frac{\partial e(p,U)}{\partial p_n} = \frac{\partial}{\partial p_n}\sum_i h_i(p,U)p_i = \sum_i p_i \frac{\partial h_i(p,U)}{\partial p_n} + h_n(p,u)$$

where $\sum_i p_i \frac{\partial h_i(p,U)}{\partial p_n} = 0$.

$$\lambda \sum_i \frac{\partial u_i}{\partial h_i}\frac{\partial h_i}{\partial p_n} = \frac{dU}{dp_n} = 0$$

since $\bar{U} = u(h(p,\bar{U}))$.

Here we call the change in price with the same utility the substitution effect.

Proposition 6. Slutsky Equation:

$$\frac{\partial x_n(p,w)}{\partial p_k} = \frac{\partial h_n}{\partial p_k} - \frac{\partial x_n}{\partial w}x_n(p,w)$$

The Slutsky is trying to separate the relative price effect and substitution effect out the demand function. In other words, the uncompensated change in demand is the residual effect of the compensated change in demand and change in income.

Proof. $h_n(p,U) = x_n(p, e(p,U))$. This implies

$$\frac{\partial h_n}{\partial p_k} = \frac{\partial x_n}{\partial p_k} + \frac{\partial x_n}{\partial w}\frac{\partial e}{\partial p_k} = \frac{\partial x_n}{\partial p_k} + \frac{\partial x_n}{\partial w}h_k(p,U) = \frac{\partial x_n}{\partial p_k} + \frac{\partial x_n}{\partial w}x_k$$

since $\frac{\partial e}{\partial p_k} = h_k = x_k$. □

To explain this more intuitively, we can first assume the price does not change, then the change in the demand would be the change in income. Then because of price change, the optimal demand we move away from the change in income. Therefore, the combined effect would result the demand change in uncompensated utility.

The above equation can be rewritten as the following

$$\frac{\partial x_n/x_n}{\partial p_n/p_n} = \frac{\partial h_n/h_n}{\partial p_n/p_n}. - \frac{\partial x_n/x_n}{\partial W/W}\frac{p_n x_n}{W}$$

that is, the price elasticity of uncompensated demand is the price elasticity of compensated demand minus income expenditure of demand times the expenditure share.

1.4.1 Labors Supply and Wage Rate

$U(Y, L)$ where L is leisure and Y is the income. Let $y = (K - L)w + A$ where K is the maximum number of hours of work and A is the non-labor income. $S = wK + A$ is the maximum possible income. We can then write

$$U((K - L)w + A, L)$$

Solving using Lagrane, we can get $x_L(w, S)$ regular demand for leisure. We can write this using the Slutsky equation,

$$\begin{aligned}
\frac{\partial x_L}{\partial w} &= \frac{\partial h_L}{\partial w} - \frac{\partial x_L}{\partial S} \cdot L \\
\frac{\partial x_L}{\partial w} - \frac{\partial x_L}{\partial S} K &= \frac{\partial h_L}{\partial w} - \frac{\partial x_L}{\partial S} \cdot L \\
\frac{\partial x_L}{\partial w} &= \frac{d h_L}{d w} - K \frac{\partial x_L}{\partial S} \\
\frac{d x_L}{d w} &= \frac{d h_L}{d w} + \frac{\partial x_L}{\partial S}(K - L)
\end{aligned}$$

If we assume Cobb-Douglas $U(y, L) = y^\alpha L^{1-\alpha}$, then the labor and wage will be independent.

1.5 Welfare Impact of A Price Change

Indirect utility function $V(p, W)$. If you change the price $p^0 \to p^1$, then $[V(p^0, W) - V(p^1, W)]$. We can use the expenditure function $e(p, U)$ is the minimum you have to spend to reach U at prices p; $[e(p, V(p^1, W) - e(p, V(p^0, W))]$.

Let $U^0 = V(p^0, W), U^1 = V(p^1, W)$. Then $e(p^0, U^0) = e(p^1, U^1) = W$.

Define

$$\begin{aligned}
EV(p^0, p^1, W) &= e(p^0, U^1) - e(p^0, U^0) = e(p^0, U^1) - W = e(p^0, U^1) - e(p^1, U^1) \\
&= -\int_{p_0}^{p_1} \frac{\partial e}{\partial p_1}(p, U^1) dp \\
&= -\int_{p_0}^{p_1} h_i(p, U^1) dp
\end{aligned}$$

where EV is called the equivalent variation; the difference in cost of reaching $U^1 + U^0$ at prices p^0.

Define

$$CV(p^0, p^1, W). = e(p^1, U^1) - e(p^1, U^0) = W - e(p^1, U^0) = e(p^0, U^0) - e(p^1, U^0) = -\int_{p_0}^{p_1} h_i(p, U^0) dp$$

where CV is called the compensated variation.

Hence

$$EV - CV = \int_{p_i^1}^{p_i^0} [h_i(p, U^1) - h_i(p, U^0)] dp_i$$

Sufficient condition for $EV = CV$ is $h(p, U^0) = h(p, U^1)$, i.e., h is independent of U.

Example

Suppose there are two goods 1 and 2, $U(x_1, x_2) = f(x_1) + x_2$.

$$\min(p_1 x_1 + p_2 x_2)$$

$$\text{s. t. } f(x_1) + x_2 = \hat{U}$$

This is equivalent to

$$\min p_1 x_1 + p_2(\hat{U} - f(x_1)$$

That is $p_1 = p_2 f'(x_1)$. This implies the solution is

$$x_1^* = (f')^{-1}(\frac{p_1}{p_2})$$

and

$$x_2^* = \hat{U} - f((f')^{-1}(\frac{p_1}{p_2}))$$

.

The expenditure function is

$$p_1(f')^{-1}(\frac{p_1}{p_2}) + p_2[\hat{U} - f(f')^{-1}(\frac{p_1}{p_2}))]$$

Here good 2 always has the fixed demand.

1.5.1 Deadweight Loss From Commodity (Sales) Taxation

Suppose we have two goods (p_1^0, p_2^0). Change $p_1^0 \to (p_1^0 + t)$ where t is the tax. The revenue raised $T = tx_1(p^1, W)$ where $p^1 = (p_1^0 + t, p_2^0)$. We can compare this with tax T on wealth reducing this from W to $W - T$.

Is $EV(p^1, p^2, W)$ bigger or less than $-T$?

$$\begin{aligned}
-T - EV &= e(p^1, U^1) - e(p^0, U^1) - th_1(p^1, U^1) \\
&= \int_{p_1^0}^{p_1^0 + t} h_1(p_1, p_2, U^1)dp_1 - th_1(p_1, U^1) \\
&= \int_{p_1^0}^{p_1^0 + t} [h_1(p_1, p_2, U^1) - h_1(p_1^0 + t, p_2, U^1)]dp_1 > 0
\end{aligned}$$

Hence this implies the welfare loss is more than the tax loss.

2 Preference & Demand Aggregation

Individual demand $x_i(p, W_i)$ for person i. The aggregate demand $D = \sum_i x_i(p, W_i)$. Can we write this as

$$D(p, W_1, \cdots, W_I) \text{ as a function of } D(p, \sum_i W_i)$$

Consider changes ΔW_i in income levels such that $\sum_i \Delta W_i = 0$.

$$D(p, \sum_i W_i) = D(p, \sum_i (W_i + \Delta W_i))$$

$$D(p, W) = \sum_i x_i(p, W_i) \implies \sum_i [x_i(p, W_i + \Delta W_i) - x_i(p, W_i)] = 0, \forall \sum_i \Delta W_i = 0$$

$$\implies \sum_i \frac{\partial x_i}{\partial W_i} \Delta W_i = 0 \text{ if } \sum_i \Delta W_i = 0 \implies \frac{\partial x_i}{\partial W_i} = K, \forall i \text{ or } \frac{\partial x_{il}}{\partial W_i} = \lambda_l, \forall i, l$$

$$\implies \sum_i K \Delta W_i = k \sum_i \Delta W_i = 0$$

This is the sufficient condition but not if and only if. Preferences are homothetic and identical. For example, the quasi-linear shows the same but it does not need to follow homothetic and identical.

Quasi-Linear Case

$$u_i(x_{i1}, x_{i2}) = x_{i1} + f(x_{i2})$$

considering budget constraints

$$p_1x_1 + p_2x_2 = W_i \implies x_{i1} = -x_{i2}\frac{p_2}{p_1} + \frac{W_i}{p_1}$$

Therefore

$$u() = -x_{i2}\frac{p_2}{p_1} + \frac{W_i}{p_1} + f(x_{i2})$$

$$f'(x_{i2}) = \frac{p_2}{p_1}, x_{i2} = (f')^{-1}(\frac{p_2}{p_1})$$

$$x_{i1} = -\frac{p_2}{p_1}[(f')^{-1}(\frac{p_2}{p_1})] + \frac{w_i}{p_1}$$

Population of N people,

$$x_2 = N(f')^{-1}(\frac{p_2}{p_1})$$

$$x_1 = -\frac{Np_2}{p_1}(f')^{-1}(\frac{p_2}{p_1}) + \sum_i \frac{W_i}{p_1}$$

2.1 Social Choice Theory

How to move from individual preferences to some "collective" or "social" preference? First, define a finite set of alternatives. Each person has a ranking of these $\succeq_i$. Let $\Pi =$ set of all possible rankings of alternatives. Social choice rule is defined as a mapping from $\phi(\succeq_1, \cdots, \succeq_n) \to \Pi$. For example, we can take three people A, B, C with three alternatives α, β, γ.

A	B	C
α	γ	β
β	α	γ
γ	β	α

Vote between α and β. Vote between β and γ. Then $\alpha > \beta$, $\beta > \gamma$ and $\gamma > \alpha$ since γ beats α most of the time. Hence it is not transitive. This example is from Condocet.

2.1.1 Conditions that a Social Choice Rules Should Satisfy

Should produce a transitive social ordering.

1. Unrestricted Domain: works whatever views and preferences.

2. Respect of Unanimity: if everyone prefers α to β, then society ranks $\alpha \succ \beta$.

3. Independence of Irrelevant Alternatives: Social preferences between α and β depends only on individuals preferences over (α, β) but not on their preferences about their alternatives.

   For any α, β and any 2 sets of preferences $\succeq_i, \succeq_i'$, if $\succeq_i$, $\succeq_i'$ all agree on α, β, the social choice is the same for both sets of preferences.

4. None dictatorship: nobody decides the preference.

There is no social choice rule satisfying 1, 2, 3 and 4.

Theorem 2. *Assuming the above four condition holds, for any alternative b, if everyone ranks b either best or worst, then the society ranks b either best or worst.*

Proof. Assume not true, then $\exists a, c$ such that $a \succeq b \succeq c$ by society. Let's change the preference as follows

- For anyone who ranks b at the top, move c to the second place.
- For anyone who ranks b at the bottom, move c to the top.

This does not change anyone's ranking of b v.s. a. Hence social ranking of b, a is unchanged. Haven't changed anyone's ranking of b v.s. c so social ranking of b v.s. c has not changed. Thus $a \succ b$, $b \succ c \implies a \succ c$. But everyone ranks $c \succ a$ so by respect of unanimity, it is impossible. □

Theorem 3. *Assume $\exists$ alternative b such that everyone ranks b best or worst.*

Proof. Assume all rank b worst. Then society ranks b worst by respect of unanimity.

Now, we can start moving b to the top. Let's suppose B is the first person to move b from the worst to the best. Let's construct four sets of preferences

1. Everyone from 1 to $B-1$, they rank b top and the rest rank b bottom. Then society ranks b worst. In other words, $\forall a, a \succeq b$.
2. Everyone from 1 to B, they rank b top and the rest rank b bottom. Then society ranks b best. In other words, $\forall c, b \succeq c$.
3. From (4), we move a above b. Everyone from 1 to $B-1$ ranks b top and everyone from $B+1$ to n ranks b bottom.
4. Suppose everyone ranks arbitrarily except that $\exists a, c$, person B ranks a over c.

Note, in (3), the society's ranking of a and c remains the same. In other words, (1), (3), (4), everyone's ranking of a, c is unchanged. Society's ranking of a, c remains the same in (3), (4). From (1) to (3), ranking of a and b is the same. Therefore, $a \succ b$. Similarly, from (2) to (3), $b \succ c$. Therefore, in (1), (2), (3), we will have the order $a \succ b \succ c$ by the society. Hence B has to be a dictator. □

2.2 Restoring Domain Of Preferences

An ordering on the line $\succeq$ is single-peaked if $\exists m$, such that for $y, z > m$, $y \succeq z$ if and only if $z \succ y$ and for $m \succ y, z$, $y \succeq z$ if and only if $y \succ z$.

Median agent k is median for preferences $\succeq_i$, $i = 1, \cdots, I$, if $N(i : m_i \geq m_k) \geq \frac{I}{2}$ and $N(i : m_i \leq m_k) \geq \frac{I}{2}$ where $N(X)$ is the number of things in set X.

Theorem 4. *Median Voting Theorem: Suppose k is a median agent. Then in voting the peak of agent k, m_k, cannot be defeated by any other alternative.*

Proof. Pick any alternative y, assume $m_k > y$. Consider set of agents with peaks $\geq m_k$. $\{i : m_i \geq m_k\}, m_i \geq m_k \geq y$ so i prefers m_k to y. □

Let's define firms as $y \in \mathbb{R}^N$ is a production plan. Sign convention: input is negative and output is positive. The price vector is $p \in \mathbb{R}^N$ and the profit is $\pi = p \cdot y = \sum_i p_i y_i$. Production possibility set $Y \in \mathbb{R}^N$ is the set of plans y open to firm. Technology and resources owned by firms may be affected by legal constraints.

Firm's aim to maximize profits. Pick $y^* \in Y$ such that $p \cdot y^* \geq p \cdot y, \forall y \in Y$. Pick x such that $\max p \cdot (x, y)$ such that $y = f(x)$ where f is the production function, $y \in \mathbb{R}, x \in \mathbb{R}^{N-1}$.

Cobb-Douglas $y = x_1^\alpha x_2^\beta$ where y is output and x_1, x_2 are inputs.

2.2.1 Assumptions about Y

Constant return to scale: $y \in Y \implies \alpha y \in Y, \forall \alpha > 0$. Geometrically, Y is a cone. Decreasing returns on scale then Y is strictly convex. Increasing returns on scale, then $y \in Y \implies \alpha y \in Y, \alpha \geq 1$ and $\exists y \in Y : \beta y \notin Y, \beta \in (0, 1)$.

2.3 Profit Maximization and Cost Minimization

Profit Max

$$\max p \cdot y$$

$$\text{s.t.} F(y) = 0$$

Profit function $\pi(p) = \max_y (p \cdot y), y \in Y$. The supply function is $y(p) = \{y : p \cdot y = \pi(p)\}$.

Example

Input x and output y such that $y = f(x)$, Prices $px, py - \pi = p_y y - p_x x$

$$\max p_y y - p_x x$$

$$\text{s.t.} y = f(x)$$

$$\pi = p_y f(x) - p_x x$$

Therefore

$$p_y \left| \frac{\partial f}{\partial x} \right| = p_x$$

In other words, the price of the marginal product is the same as the price of the output.

2.3.1 Cost Minimization

x input vector, y is a single output. Look at

$$\min_x p_x x, f(x) \geq y$$

Therefore,

$$\frac{p_{x_i}}{p_{x_j}} = \frac{\partial f}{\partial x_i} / \frac{\partial f}{\partial x_j}$$

Cost function $C(p_x, y)$

Proposition 7. $Y_i, i = 1, \cdots, I$ be production sets. $Y = \sum_i Y_i = \sum p \in \mathbb{R}^N$ a price. Let $y^* \max p \cdot y, y \in Y$ and $y_i^* \max p \cdot y_i, y_i \in Y_i$, then $\sum_i y_i^* = y^*$.

Proof. $\sum_i y_i^* \in \sum_i Y_i = Y$. $y \in Y$ can be written as $y = \sum_i y_i, y_i \in Y_i$. We know $p \cdot y_i^* \geq p \cdot y_i \forall y_i \in Y_i$ for all i. $\sum_i p \cdot y_i^* = p \cdot \sum_i y_i^* \geq p \cdot \sum_i y_i, \forall y_i \in Y_i, \forall i$. $p \cdot \sum_i y_i^* \geq p \cdot y, \forall y \in Y$. □

3 Choice Theory

Lottery, $\mathcal{L}$, is a list of outcomes $1, \cdots, N$ with probabilities $p_i, \cdots, p_n, \sum_n p_n = 1, p_n \geq 0, \forall n$. (Von Neumann&Morgenstein and Savage view). This is a simple lottery.

K simple lotteries $L_k = (p_1^k, \cdots, p_n^k)$. Choose lottery K with probability a_k, $0 \leq a_k \leq 1, \sum_k a_k = 1$. This is a compound lottery. Reducing compound to simple lottery.

Probability of outcome j is $\sum_{i=1}^K a_i p_j^i$

Preference relation $\succeq$ on space of simple lotteries is continuous if for any l, l' and $l'' \in L$ the sets $\{\alpha \in \{0,1\} : al + (1-a)l' \succeq l''\}$ and $a \in [0,1] : l'' \succeq al + (1-a)l'\}$ are closed.

Independence: For all $l, l', l'' \in L$, $l \succeq l' \iff \alpha f + (1-\alpha)l'' \succeq \alpha l' + (1-\alpha)l'', \forall l'', \forall \alpha \in [0,1]$. If $l \sim l' \iff \alpha l + (1-\alpha)l'' \sim \alpha l' + (1-\alpha)l''$.

Before we start our discussion, we assume that our lotteries follow the conditions below

- Continuity: consider l, l', l'', $l \succeq l' \succeq l''$ Then continuity if $\exists \alpha$ such that $\alpha l + (1-\alpha)l'' \sim l', 0 \leq \alpha \leq 1$. Consider the following example, $l \succ l' \iff$ either $l_1 \succ l_1'$ or $l_1 = l_1' \& l_2 \succ l_2'$. It is similar to the lexical-graphical order.

- Independence (transitive; complete): the independence condition implies that indifferent curves are parallel straight lines.

  Proof. Indifference curve is a straight line if $l_1 \sim l_2 \implies \alpha l_1 + (1-\alpha) l_2 \sim l_1 \sim l_2, \alpha \in [0,1]$.

  Assume it is not true, $0.5l_1 + 0.5l_2 \succ l_2$. Then $0.5l_1 + 0.5l_2 \succ 0.5l_2 + 0.5l_2$. We know that $l_1 \sim l_2$. From the independence, we have

$$0.5l_1 + 0.5l'' \sim 0.5l_2 + 0.5l''$$

  Let's $l'' = l_2$. Then $0.5l_1 + 0.5l_2 \sim 0.5l_2 + 0.5l_2$. This is a contradiction. □

Definition 6. *Utility has expected utility form if* $\exists$ *a set of numbers* $u_1, \cdots, u_N$ *assigned to the* N *outcomes such that*

$$U(l) = \sum_n u_n p_n$$

l^n degenerate lottery if it yields outcome n probability 1.

Proposition 8. Utility $U : L \to \mathbb{R}$ has expected utility form if and only if

$$U\left(\sum_{i=1}^{k} \alpha_k l_k\right) = \sum_{i=1}^{k} \alpha_k U(l^k)$$

In other words, the utility of a compound lottery is the expectation of a set of simple lotteries.

Proof. Suppose U has linearity property

$$l = (p_1, \cdots, p_n) \text{ as } l = \sum_n p_n l^n$$

$$U(l) = U(\sum_n p_n l^n) = \sum_n p_n U(l^n) = \sum_n p_n U_n$$

Now suppose U is an expected utility. Compound lottery $(l_1, \cdots, l_k : a_1, \cdots, a_k)$. Probability of outcome

$$n = a_1 p_1^1 + a_2 p_n^2 + \cdots = \sum_k a_k p_n^k = p_n$$

$$U(\sum_k a_k l_k) = \sum_n U_n \left[\sum_n a_k p_n^k\right] = \sum_k a_k U(l_k)$$

□

Proposition 9. Suppose $U : L \to \mathbb{R}$ is an expected utility function for preference relation $\succeq$ on L. Then $\hat{U} : L \to \mathbb{R}$ is an alternative such that if and if $\exists$ scalar $\beta > 0, \gamma$ such that

$$\hat{U}(l) = \beta U(l) + \gamma$$

Proof. Choose 2 lotteries $\bar{l}, \underline{l}$ such that $\bar{l} \succeq l \succeq \underline{l}, \forall l \in L$. Suppose U is an expected utility function and $\hat{U} = \beta U + \gamma$.

$$\hat{U}\left(\sum_k a_k l_k\right) = \beta U\left(\sum_k a_k l_k\right) + \gamma = \beta\left[\sum_k a_k U(l_k)\right] + \gamma = \sum_k a_k \left[\beta U(l_k) + \gamma\right] = \sum_k a_k \hat{U}(l_k)$$

Show that $\hat{U}, U$ are the expected utilities, then $\hat{U}$ is a affine transformation of U.

$$U(l) = \lambda_l U(\bar{l}) + (1-\lambda_l) U(\underline{l}) = U(\lambda_l \bar{l} + (1-\lambda_l)\underline{l})$$

Therefore, we get

$$\lambda_l = \frac{U(l) - U(\underline{l})}{U(\bar{l}) - U(\underline{l})}$$

Since

$$\lambda_l U(\bar{l}) + (1-\lambda_l)U(\underline{l}) = U(\lambda_l \bar{l} + (1-\lambda_l)\underline{l})$$
$$\implies l \sim \lambda_l \bar{l} + (1-\lambda_l)\underline{l}$$

Hence

$$\hat{U}(l) = \hat{U}(\lambda_l \bar{l} + (1-\lambda_l)\underline{l}) = \lambda_l \hat{U}(\bar{l}) + (1-\lambda_l)\hat{U}(\underline{l}) = \lambda_l[\hat{U}(\bar{l}) - \hat{U}(\underline{l}) + \hat{U}(l)]$$
$$\hat{U}(l) = \beta U(l) + \gamma$$

where $\beta = \frac{\hat{U}(\bar{l}) - \hat{U}(\underline{l})}{U(\bar{l}) - U(\underline{l})}$ and $\gamma = \hat{U}(\underline{l}) - \beta U(\underline{l})$ □

Let's define $U(l) = \sum_n p_n U_n$. Then we know

$$U(l_1) - U(l_2) > U(l_3) - U(l_4)$$

$\hat{U}$ is an alternative utility represent the same preferences

$$\exists B > 0, \gamma, \text{s.t.} \hat{U}(l). = BU(l') + \gamma$$

Therefore,

$$\hat{U}(l_1) - \hat{U}(l_2) = BU(l_1) + \gamma - BU(l_2) - \gamma = B(U(l_1) - U(l_2))$$
$$\hat{U}(l_3) - \hat{U}(l_4) = B(U(l_3) - U(l_4))$$

Proposition 10. Suppose $\succeq$ on L is rational continuous and satisfies independence. Then it admits representation in the expected utility form.

However, in some experiments, we observe the inconsistency between people's choices and mainly, the discrepancy is derived from the ambiguity of the choices presented.

3.1 Risk Aversion

Lotteries over money, i.e.over $\mathbb{R}^1$. The probability of any outcome is defined by density function $f(t)$, $t \in [0, \infty]$ without loss of generality. There is a cumulative density function $F(x) = \int_0^x f(t)dt$. We say $\int_x^{x+\Delta x} f(t)dt$ is the probability an outcome happens between x, $x + \Delta x$.

Let's define $u(x)$ be utility of outcome X increasing as x increases. Now the expected utility is now defined as the following

$$U(F) = \int u(x)dF(x) = \int u(x)f(x)dx$$

where $f(x)$ is linear in probability and $u(x)$ is not necessarily linear. In addition, it is used to be called VN-M utility.

3.1.1 St. Petersburg Paradox

Argument about why $U(x)$ is bounded above. Assume $U(x)$ unbounded. x_m be such that $U(x_m) > 2^m$. Consider the following lottery: toss coin repeatedly till it comes up heads. When it comes up heads on the mth toss you win x_m. Then the expected winning is

$$\sum_m u(x_m)\frac{1}{2}^m > \sum_m 2^m \frac{1}{2}^m > \infty$$

This paradox can be found in Friedman and Savage.

3.1.2 Risk Aversion Factor

Risk averses if any lottery $F()$ the degenerate lottery yielding expected outcome $\int x dF(x)$ with certainty is at least as good as $F()$. In addition, certainty equivalent of lottery $F()$ written $C(U, F)$ is amount of money with certainty that person regards as equivalent to $F()$.

Certainty equivalent C of F is defined as

$$U(C) = \int u(x)dF(x)$$

The followings are equivalent

- Decision-maker is risk averse
- $U()$ is concave
- $C(U, F) \leq \int x dF(x), \forall F$.

Define Arrow-Prutt coefficient of absolute risk aversion

$$\text{(IAA) } r_A(x) = \frac{-u''(x)}{u'(x)} > 0$$

3.1.3 Insurance

Suppose the initial wealth is W and a person can lose amount \$$D$ with probability π. Units of insurance costs \$$q$ and pays \$1 if loss occurs. Assume a units of insurance purchase. There are two possible outcomes

1. No loss: wealth is $(W - qa)(1 - \pi)$.
2. If there is a loss: wealth is $(W - D - qa + a)\pi$.

$$E[U] = u(W - qa)(1 - \pi) + u(W - D - qa + a)\pi$$

Then by FOC, we get

$$\frac{\partial E[U]}{\partial a} = -(1 - \pi)qu'(W - qa) + \pi(1 - q)u'(W - D - qa + a) \leq 0$$

Suppose $a \geq 0$. Assume $q = \pi$, then this insurance is actuarially fair. Therefore,

$$-u'(W - qa) + u'(W - D - qa + a) \leq 0$$

Suppose $a = 0$, then $-u'(W) + u'(W - D) \leq 0$. Therefore, we know that $u'(W - D) - u'(W) \leq 0$. However, this is impossible.

Therefore, $a > 0$, then we get $u'(W - qa) = u'(W - D - qa + a)$. Therefore, $a = D$. This is what we call full insurance.

3.1.4 Investing in Risky Asset

Two assets safe returns 1/\$ invested. Risky asset return Z/\$ invested where Z is a random variable distributed as $F(Z)$ such that $\int z dF(z) > 1$. How much of risky assets to buy? Suppose a is the risky asset purchased and b is the safe asset. We know $a + b = W$ (wealth). The overall problem is

$$\max_{a,b} \int u(az + b)dF(z)$$

$$\text{s.t. } a + b = W$$

Hence we can solve this by replacing b with $W - a$

$$\max_a \int u(a(z-1)+w)dF(z)$$

That is

$$\frac{\partial \int u(a(z-1)+w)dF(z)}{\partial a} = \int u'(w+a(z-1))(z-1)dF(z) \leq 0$$

Here if $a < W$, then ≤ 0; else ≥ 0 if $a > 0$.

Suppose $a = 0$, then the above FOC becomes $u'(W)\int(z-1)dF(z)$ but $\int(z-1)dF(z) > 0$. Therefore, it is impossible.

3.1.5 Absolute Risk Aversion

Proposition 11. The followings are equivalent:

(1) $\Gamma_A(x, u_2) \geq \Gamma_A(x, u_1), \forall x.$

(2) $\exists$ increasing concave function $\psi(\cdot)$, such that $u_2(x) = \psi(u_1(x))$

(3) $c(F, u_2) \leq c(F, u_1), \forall F.$

Proof. (1) $\iff$ (2) $\exists\psi, \psi(u,(x)) = u_2(x)$. Differentiate both side

$$\psi' u_1'(x) = u_2'(x)$$

Differentiate again

$$\frac{\psi''(u_1')^2}{\psi' u_1'} + \frac{\psi' u_1''}{\psi' u_1'} = \frac{u_2''}{u_2'} \implies -\frac{u_2''}{u_2'} = -\frac{u_1''}{u_1'} - \frac{\psi'' u_1'}{\psi}$$

Hence

$$\Gamma_A(2) = \Gamma_A(1) - \frac{\psi'' u_1'}{\psi'} \geq \Gamma_A(1), \forall\psi$$

□

Consider two assets, one risky and one safe. For the safe asset, it costs b and returns 1. The risky asset costs a and returns z with expectation $\int z dF(z) > 1$.

For individual i, she is solving the following

$$\max_{a_i, b_i} \int u_i(a_i z + b_i)dF(z)$$

Then the first order condition, we have

$$\phi_i(a_i) = \int u_i'(w_i + a_i(z-1))(z-1)dF(z) = 0$$

We want to prove: assume 2 is more risk averse than 1, i.e. $a_2 < a_1$ or $\phi_2(a_1) < 0$. Note that, $u_i'(w_i + a_i(z-1))$ is decreasing in a.

Since 2 is more risk averse than 1, then there exists a increasing concave function θ such that $\theta(u_1(x)) = u_2(x)$.

$$\begin{aligned}
\phi_2(a_2) &= \int u_2'(w + a_2(z-1))(z-1)dF(z) = 0 \\
\phi_1(a_1) &= \int u_1'(w + a_1(z-1))(z-1)dF(z) = 0 \\
\phi_2(a_1) &= \int \theta'(u_1(w + a(z-1)))u_1'(w + a_1(z-1))(z-1)dF(z) < \max(\theta')\phi_1(a_1) = 0
\end{aligned}$$

3.1.6 Relative Risk Aversion

Relative risk aversion is $-\frac{xu''(x)}{u'(x)}$. Constant relative risk aversion is $u(x) = \log(x)$. Another one is called iso-elastic utility,

$$u(x) = \left[\frac{x^{1-\eta}}{1-\eta}\right]$$

3.2 Mean-Variance Analysis of Risks

Assume y random variable with mean y^* and distributed as $F(y)$. The expected utility is $\int u(y)dF(y) = EU$.

Define x (i.e. cost of risk bearing) as follows

(i) $u(y^* - x) = EU = \int u(y)dF(y)$

(ii) $u(y^*-x)-u(y^*) = \int[u(y)-u(y^*)]dF(y) = \int \left[u'(y^*)(y-y^*) + \frac{u''(y^*)}{2}(y-y^*)^2\right] dF(y) = \frac{u''(y^*)}{2}Var(y)$.

(iii) (ii) is equivalent to $-u'(y^*)x = \frac{u''(y^*)}{2}\sigma^2$. Hence $x = \left[-\frac{u''(y^*)}{u'(y^*)}\right]\frac{\sigma^2}{2}$

(iv) $EU = \int u(y)dF(y) = u(y^* - x) = u(y^*) + \frac{1}{2}u''(y^*)\sigma^2$. That is $EU = G(y^*, \sigma^2)$. Use the implicit function theorem to get slopes of the indifference curve in $y^* \to \sigma^2$ plane.

$$\frac{\partial y^*}{\partial \sigma} = -\frac{u''\sigma}{u' + 0.5\sigma^2 y^* u'''} \approx -\frac{u''\sigma}{u'} \qquad \text{Assume } u''' \approx 0$$

3.3 Stochastic Dominance

3.3.1 First Order Stochastic Dominance

If every person who maximize $E[\mu]$ with $\mu'(x) > 0$ prefers F to G, i.e.

$$\int \mu(x)dF(x) \geq \int \mu(x)dG(x)$$

Proposition 12. Distribution F is first order stochastic dominant over G if and only if

$$F(x) \leq G(x)$$

Proof. $F(x) \leq G(x) \implies F$ FOSD G. Then

$$A = \int_a^b \mu(x)dF(x) = \int_a^b \mu(x)f(x) = [\mu(x)F(x)|_a^b - \int_a^b \mu'(x)F(x)d(x) = \mu(b) - \int_a^b \mu'(x)F(x)d(x)$$

$$B = \int_a^b \mu(x)dG(x) = \mu(b) - \int_a^b \mu'(x)G(x)dx$$

$$A - B = \int_a^b \mu'(x)(G(x) - F(x)dx \geq 0$$

because $\mu(x)' \geq 0$ and $F(x) \leq G(x)$. Thus F is FOSD over G.

On the other hand, suppose $\exists x^* : F(x^*) > G(x^*)$. Then $A - B < 0$ for an interval around x^* and $\mu'(x) \geq 0$ such that μ' takes very large values in that interval and small elsewhere. This implies G FOSD F so contradiction. $\square$

3.3.2 Second Order Stochastic Dominance

Now we compare F, G with the same mean. Comparison with concave μ. F is second order stochastic dominant over G (F less risky than than G if $\int \mu(x)dF(x) \geq \int \mu(x)dG(x)$ for all μ with $\mu' > 0$ and $\mu'' < 0$ where $\mu : \mathbb{R}^N_+ \to \mathbb{R}^N_+$

Alternative approach is mean preserving spread. G is a MPS of distribution F if G is the reduction of a compound lottery made up of the distribution F with an additive lottery so that when F selects the final outcome is $x + z$, where $E[z] = 0$.

Proposition 13. Consider F, G with same mean. The following are equivalent:

1. F SOSD G
2. G is a MPS of F
3. $\int_0^x G(t)dt \geq \int_0^x F(t)dt$

3.4 Geometric Approach to Insurance

- Initial endowment: z_1
- $P[state1] = P_1$
- $P[state2] = P_2$
- Expected unity at z_1 is $\mu(z_{11})P_1 + \mu(z_{12})P_2$
- Indifference curve is $\mu(z_{11})P_1 + \mu(z_{12})P_2 = k$
- Slope is $-\frac{\mu'(z_{11})P_1}{\mu'(z_{12})P_1}$
- On the 45 degree line: slope is $-P_1/P_2$.
- Consider the move $z_1 \to z_2$. You are selling $z_{11} - z_{01}$ and buying $z_{22} - z_{12}$
- Expected value of trans is $-P_1(z_{11} - z_{01}) + P_2(z_{22} - z_{12})$
- Expected value is 0 when zero cost transaction.

$$\frac{P_1}{P_2} = \frac{z_{22} - z_{12}}{z_{11} - z_{01}}$$

Hence you afford to move to point z_2 and as a risk aversion investor you prefer z_2 than z_1. You always wants more.

4 Subjective Probabilities

de Finetti defined subjective probabilities. He used a more economist point of view. Here in this course, we will use Savage's view.

4.1 Savage Framework

Set of states S. $s \in S$ is specific state. States are uncertain once know state, no remaining uncertainty. Set of outcomes X, $x \in X$ on outcome. Outcome is what affects your well-being. Acts (policies, state, governments) F, $f \in F$ is an act (policy). A choice of policy is $f : S \to X$. f tells us what outcome x is associated with my states. $x = f(s)$. f, g are two policies $A \subset S$ is called an event. Then

$$f_A^g(s) = \begin{cases} g(s) & s \in A \\ f(s) & s \notin A \text{ i.e. } S \in A^c \end{cases}$$

4.2 Savage's Axiom

(1) Preferences are a complete transitive relation on F.

(2) **Savage Sure Thing Principle**: A is an event, A^c its complement, $f(s) = g(s), \forall s \in A^c$. $f(s) \neq g(s)$ some states in A. Introduce $f'(s)$, $g'(s)$ such that $f'(s) = g'(s), \forall s \in A^c$. $f'(s) = f(s) + g'(s) = g(s), \forall s \in A$. $f'(s) \neq f(s), s \in A^c$ and $g'(s) \neq g(s), s \in A^c$. Therefore, $f \geq g \iff f' \geq g'$.

(3) Let f_A^x be policy that produces outcome x for any $s \in A$, $f_A^x(s) = x, \forall s \in A$. For every $f \in F$, every $A \subset S$, $x, y \in X$ such that

$$x \geq y \iff f_A^x \geq f_A^y$$

(4) For every $A, B \subset S$, every x, y, z, w with $x \geq y$, $z \geq w$.

$$y_A^x \geq y_B^x \iff w_A^2 \geq w_B^2$$

(5) $\exists, f, g$ such that $f > g$.

(6) Continuity: for any acts $f(S) > g(S)$ and outcome x, $\exists$ a finite set of events $\{A_i\}_i$ whose union is S such that

$$f > \begin{cases} x & \text{if } S \in A_i \\ g(s) & \text{if } S \notin A_i \end{cases}$$

and

$$\begin{cases} x & \text{if } S \in A_j \\ f(s) & \text{if } S \notin A_j \end{cases} > g$$

Theorem 5. *Savage Theorem: Assume X is finite. Preference ordering satisfies (1) to (6) if and only if there exists probability p on states S and a non-constant utility $U : X \to \mathbb{R}$ such that*

$$f \succeq g \iff \int_S U(f(s))dp(s) \geq \int_S U(g(s))dp(s)$$

4.2.1 Sure Thing Principle (2)

Horse race A as horse

Table 1: default

	A wings	A not wins
1	Paris	Rome
2	London	Rome
3	Paris	LA
4	London	LA

$1 \succeq 2 \iff 3 \succeq 4$

4.3 Ambiguity

It is situation that people don't have subjective probability. Here we will talk about risk vs known probabilities and uncertainty vs unknown probabilities.

4.3.1 Ellsberg Paradox

2 Urns and 100 balls in each and they are black and red.

Option 1 In Urn 1, you have 50 black and 50 red. In Urn 2, you have 100 balls black and red. You get paid \$10 for black ball.

Option 2 In Urn 1, you have 50 black and 50 red. In Urn 2, you have 100 balls black and red. You get paid \$10 for red ball.

4.3.2 MaxMin Approaches (Wald)

f is a policy, maps states S to outcomes x saying that

$$f \geq g \iff \min_{s \in S} f(s) \geq \min_{s \in S} g(s)$$

Here we assume there is no probabilistic information at all.

4.3.3 MinMax Regret (Savage)

Regret associated with states

$$r(s, g) = \max_{f \in F}[f(x)] - g(s)$$

The Max regret is

$$\max_{s \in S} r(g) = \max_{s \in S}\{\max_{f \in F}[f(s)] - g(s)\}$$

You should choose a g to minimize the max regret.

$$\min_{g \in F} \max_{s \in S}\{\max_{f \in F}[f(s) - g(s)]\}$$

4.3.4 MaxMin Expected Utility (Gilboa and Schmildler)

1. Complete transitive ordering
2. Continuity: For every $f, g, h \in F$, if $f > g > h$ then

$$\exists \alpha, \beta \in (0, 1) \text{ s.t. } \alpha f + (1-\alpha)h > g > \beta f + (1-\beta)h$$

3. For every $f, g,\ f(s) \geq g(s), \forall s \in S \implies f \geq g$.
4. Independence :For every f, g, every constant $h \in F$, $\forall \alpha \in (0, 1), f \geq g \iff \alpha f + (1-\alpha)h \geq \alpha g + (1-\alpha)h$.
5. Uncertainty Aversion: $\forall f, g \in F, \forall \alpha \in (0, 1),\ f \sim g \implies \alpha f + (1-\alpha)g \geq f \sim g$.

Preference satisfies assumptions if and only there exists closed convex sex of probabilities C and a non-constant function $u : X \to \mathbb{R}$ such that

$$\min_{p \in C} \int u(f(S))dp(s) \geq \min_{p \in C} \int_S u(g(s))dp(s)$$

4.3.5 Smooth Ambiguity Aversion

Multiple probability distributions given arise naturally from different models of economy, stock market, etc. P_i is distribution over states, $i = 1, \cdots, I$. Each P_i comes from model M_i,

1. $\exists u : X \to \mathbb{R}$ such that

$$f \geq g \iff \int_S u(f(s))dP_i \geq \int_S u(g(s))dP_i$$

2. $\exists$ weights $\pi(p) \geq 0, \int_P \pi(P) = 1$, a function $\phi_o : \mathbb{R} \to \mathbb{R}$ such that

$$f \geq g \iff \int_P \pi(P)\phi(E_P(f))d\pi \geq \int_P \pi(P)\phi(E_p g)d\pi$$

4.3.6 Ellsberg Paradox

2 Urns 100 balls each. 1 is 50 red and 50 yellow. 2 is 100 red and yellow. \$10 if you select red. Urn 1: $0.5 \times 10 + 0.5 \times 0 = \$5$. Urn 2: $P_n =$ probability of choosing red of n = # of red in urn 2 = $\frac{n}{100}$.

$$\pi_n = \text{second order probability that \# of red} = n$$

$$u(x) = \text{linear}, u(x) = x$$

$$\phi(x)\text{linear}, \phi(x) = x$$

Value of bet on 2 is

$$\sum_{n=0}^{100} \pi_n p_n 10 = \sum_{n=0}^{100} \pi_n \frac{n}{10} = \sum_{n=0}^{100} \frac{1}{101}\frac{n}{10} = 5$$

π_n is uniform, i.e. $\pi_n = \frac{1}{100}, \forall n$.
If the Ambiguity Aversion $\phi(x) = \sqrt{x}$, then the bet over urn 2 is

$$\sum_{n=0}^{100} \pi_n \phi(\frac{n}{10}) = \sum_{n=0}^{100} \frac{1}{101}\phi(\frac{n}{10}) = \frac{1}{101}\sum_{n=0}^{100}\sqrt{\frac{n}{10}} = 2.1$$

The certainty equivalent is 4.4.

4.4 Models

A model m_i is a map from policies $f \in F$ to probability distributions over outcomes $P_i(x|m_i)$. Expected utility of policy f contingent on model m_i is $E[u(f|m_i)] = \int u(x)dP_i(x|m_i)$. π_i is the estimate of likelihood that m_i is the right model. Choose f to maximize

$$\sum_i \pi_i \phi(E[u(f|m_i)])$$

Assume that the policy f is in $\mathbb{R}$. Maximize with respect to f.

$$\sum_i \pi_i \phi'(E[u(f|m_i)])E[u'(f|m_i)] = 0$$

$$\pi_i' = \frac{\pi_i \phi'(E[u(f|m_i)])}{\sum_j \pi_j \phi'(E[u(f|m_j)])}, \pi_i' \in [0,1]$$

If ϕ is linear, then $\pi_i' = \pi_i$.
Dividing FOC by denominators of π_i'

$$\sum_i \pi_i' E[u'(f|m_i)] = 0$$

Expectation of marginal expected utility must be zero.

4.4.1 Problem

University endowment: 2 advisors X and Y. They give different forecasts of movements of bonds B and equity E. The table is in the notes. Allocate fraction e to equities and $(1-e)$ to bonds. Expected utility according to X,

$$x_{11}u(1.1ew+1.1(1-e)w)+x_{12}u(0.9ew+1.1(1-e)w)+x_{21}u(1.1ew+0.9(1-e)w)+x_{22}u(0.9ew+0.9(1-e)w)$$

To simplify, we get

$$EU = k + x_{12}u(w(1.1-0.2e)) + x_{21}u(w(0.9+0.2e))$$

We need to maximize with respect choice of e.

$$\frac{\partial EU}{\partial e} = 0.2\{x_{21}u'(c_{21}) - x_{12}u'(c_{12})\}$$

in general we need solve the following

$$\max_e \{\pi_x \phi(E[U_x(e)) + \pi_y \phi(EU_y(e))\}$$

$$\pi_x \phi'(EU_x)E[u'_x(e)] + \pi_y \phi'(EU_y)E[u'_y(e)] = 0$$

Define the following

$$\pi'_x = \frac{\pi_x \phi'(E[u_x])}{\pi_x \phi'(E[u_x]) + \pi_y \phi'(E[u_y])}$$

Then the FOC is just

$$\pi'_x E[u'_x(e)] + \pi'_y E[u'_y(e)] = 0$$

Expected marginal utility is zero.

5 General Equilibrium

Firm i has a production possibility set $Y_i \subset \mathbb{R}^N$, $y_i \in Y_i$ production plan. (inputs are negative and outputs are positive). Price vector $p \in \mathbb{R}^N$ and profit is $p \cdot y_i$. In addition, let's only look at the relative price so the price vector is normalized (i.e. $\sum_{i=1}^n p_i = 1$). Firm's objective is

$$\max_{y_i \in Y_i} p \cdot y_i$$

Consumers: consumption vector $x_j \in X_j$. $x_j \subset \mathbb{R}_+^N$ consumption set. The utility function is

$$u_f : R^N \to \mathbb{R}$$

Consumers have endowments $w_j \in \mathbb{R}^N$. They can also own shares in firms θ_{ji} the fraction of the firm i owned by consumer j. Budget constraint is

$$p \cdot x_j \leq p \cdot w_j + \sum_i \theta_{ji}\pi_i$$

$$\max u_j(x_j) \text{ s.t. } px_j \leq pw_j + \sum_i \theta_{ij}\pi_i$$

Allocation: $x_j^*, j = 1, \cdots, I$ is an allocation, feasible if

$$\sum_j x_j^* \leq \sum_j w_j + \sum_i y_i^*, y_i, x_j \in \mathbb{R}^N$$

$$\sum_j x_{ji}^* \leq \sum_j w_j + \sum_i y_i^*$$

Pareto Efficient: An allocation $x_j^*, j = 1, \cdots, J$, $y_i^*, i = 1, \cdots, I$ is Pareto efficient if it is feasible. if there is no other feasible allocation $\hat{x}_j$, such that $u_j(\hat{x}_i) \geq u_j(x_j^*), \forall j, \exists k$, such that $u_k(\hat{x}_k) > u_k(x_k^*)$. Everyone is at least as well off as at x_j^* sand someone is better off.

Pareto Superior: (x_j^*, y_i^*) is Pareto superior to $(\hat{x}_j, \hat{y}_i)$ if $u_j(x_j^*) \geq u_j(\hat{x}_j), \forall j$ and there exists k such that $u_k(x_k^*) > u_k(\hat{x}_k)$

Competitive Equilibrium: price vector *, set of production plans y_i^* and consumption vector x_j^* such that

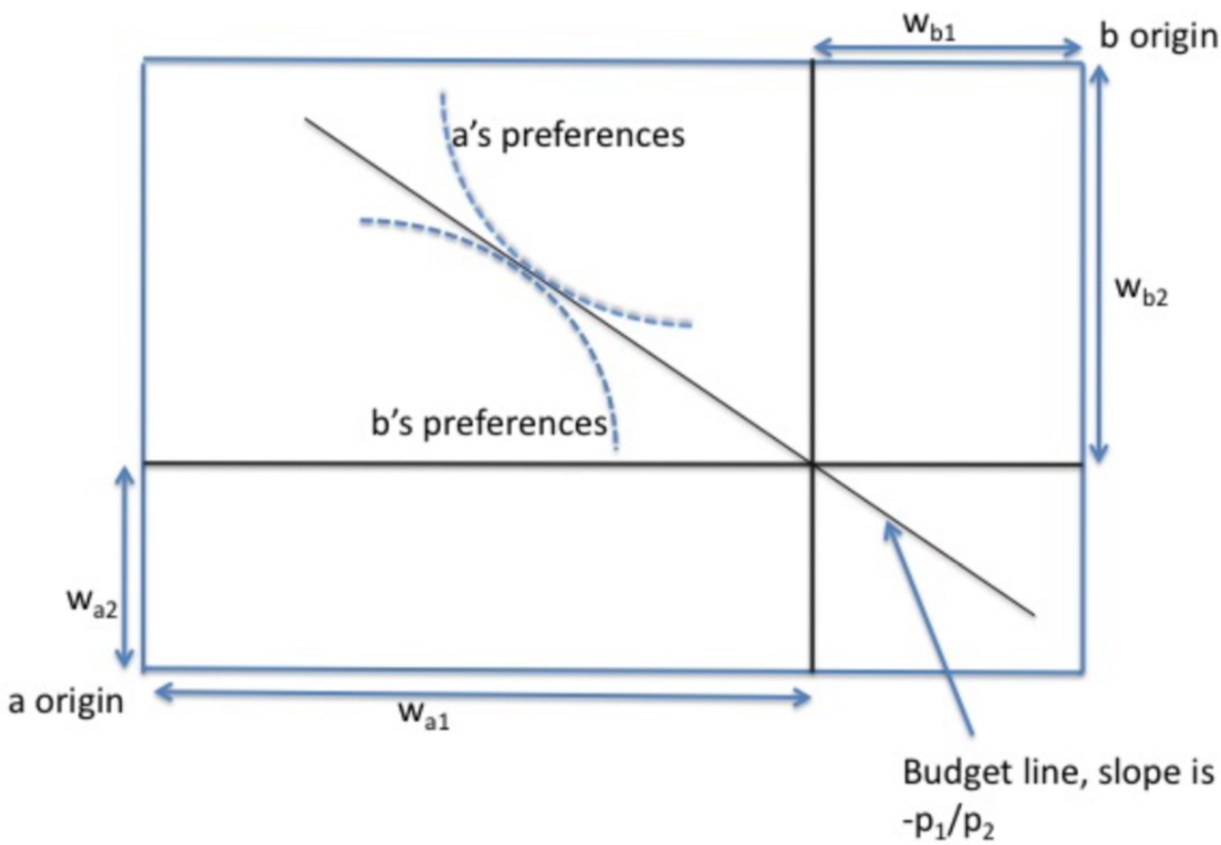
- $\forall i$, y_i^* maximizes $p^* \cdot y_i, y_i \in Y_i$
- $\forall j$, x_j^* maximizes $u_j(x_j)$ such that $p^* \cdot x_j \leq p^* \cdot w_j + \sum_i \theta_{ij}\pi_i$
- $\sum_i x_j^* \leq \sum_j w_j + \sum_i y_i^*$

5.1 Edgewood Box

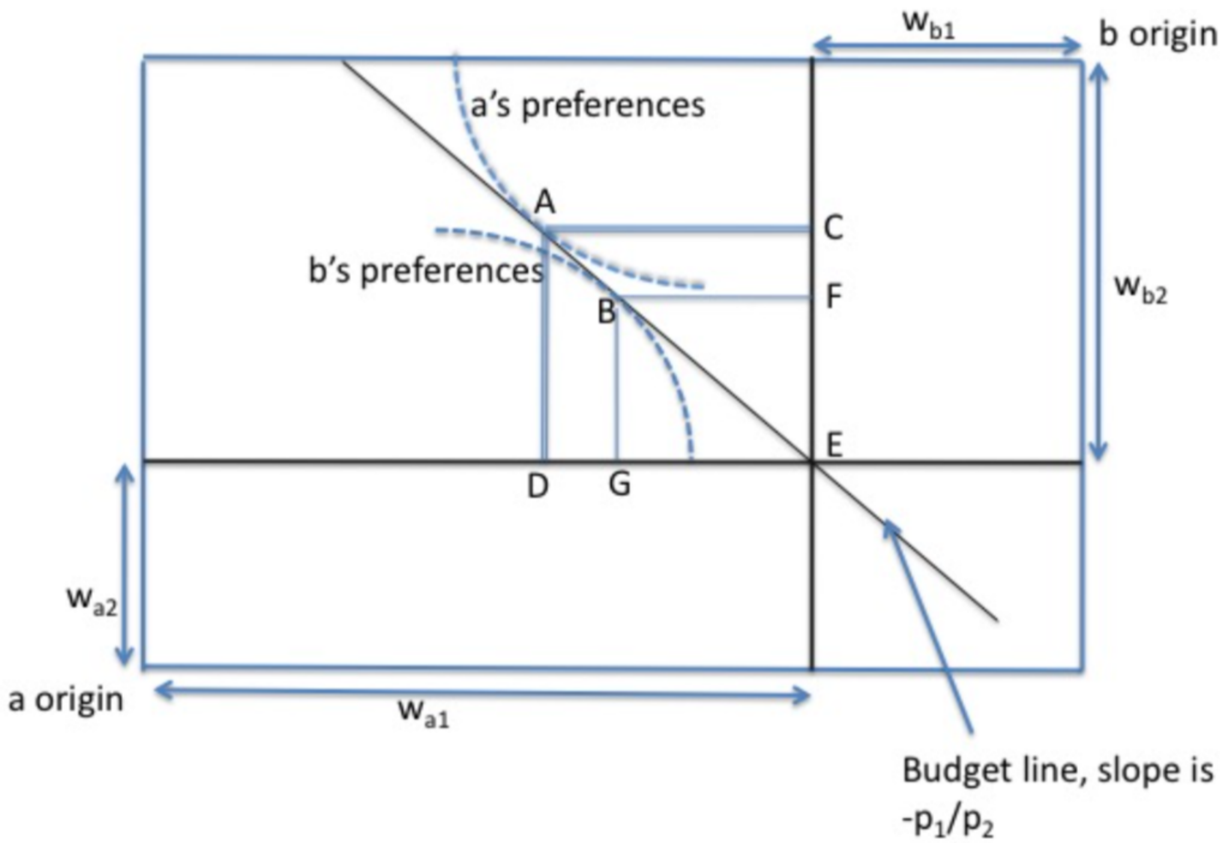
2 consumers a, b. Two goods, $1, 2$. No firms. Under exchange economy, total endowment good i, w_i,

$$w_i = w_{a_i} + w_{b_i}, i = 1, 2$$

Individual a has w_{a_1}, w_{a_2} and b has w_{b_1}, w_{b_2}



Any point in the box represents a division of w_1 and w_2 between a, b.



The graph above, it shows that A sells DE of 1 and buys AD of 2; B buys EG of good 1 and sells EF of 2. Here we can notice that demand does not match the supply.

We can easily see that all competitive equilibrium are Pareto efficient.

Proposition 14. If preferences are monotone, then any competitive equilibrium (p^*, x_j^*, y_i^*) is Pareto Efficient.

Proof. 1. $u_j(x_j) > u_j(x_j^*) \implies p^* \cdot x_j > p^* \cdot x_j^*$ (utility maximization)

2. $u_j(x_j) \geq u_j(x_j^*) \implies p^* \cdot x_j \geq p^* \cdot x_j^*$ (non-satiation)

3. (x_j', y_i') be Pareto superior to (x_j^*, y_i^*)
   From 1) and 2), $p^* \cdot x_j' \geq p^* \cdot x_j^*, \forall j$ and $p^* \cdot x_j' > p^* x_j^*$, for some j. Add up

$$\sum_j p^* \cdot x_j' > \sum_j x_j^* = \sum_j p^* w_j + \sum_i p^* y_i^*$$

4. Firms maximizes profits at y_i^*

$$p^* \cdot y_i' \leq p^* \cdot y_i^*, \forall i$$

$$p^* \sum_i y_i' \leq p^* \sum_i y_i^*$$

$$\sum_j p^* \cdot x_k' > \sum_j p^* \cdot w_j + \sum_i p^* \cdot y_i^* \geq \sum_j p^* w_j + \sum_i p^* \cdot y_i'$$

$$p^* \sum_j x_j' > p^* \sum w_j + p^* \sum_i y_i'$$

$$\sum_j x'_j \leq \sum_j w_j + \sum_i y'_i \text{ feasible}$$

This contradicts.

□

5.2 Existence of Competitive Equilibrium

$$Z(p) = \sum_j x_j(p) - \sum_j w_j - \sum_i y_i(p)$$

where $x_j(p)$ is the utility maximization choice at price p and $y_i(p)$ is profit maximization at prices p. $Z(p)$ is the excess demand function. We have an equilibrium if $Z(p) = 0$. Is there a p^* such that $Z(p^*) \leq 0$? Look at $Z(p)$ map from prices to $\mathbb{R}^N$ where $\sum_n p_n = 1$, $p_n \geq 0, \forall n$. And $p \in$ Simplex S^N and $Z(p) : S^N \to \mathbb{R}^N$

Define $Z^+(p)$ where $Z^+_\rho(p) = \max\{Z_\rho(p), 0\}$. Note:

$$Z^+(p) \cdot Z(p) = \sum_\rho \max\{Z_\rho, 0\} Z_\rho = 0 \implies Z(p) \leq 0$$

Let the following

$$a(p) = \sum_\rho [p_\rho + Z^+_\rho(p)] \in \mathbb{R}$$

Raising the price of goods for which demand is bigger the supply ($Z_\rho(p) > 0$). Let's define

$$f(p) = \frac{1}{a(p)}[p + Z^+(p)]$$

$$f(p) : S^N \to S^N$$

There exists p^* such that $f(p^*) = p^*$. Then

$$\begin{aligned}
0 = p^* \cdot Z(p^*) &= f(p^*) \cdot Z(p^*) \\
&= \frac{1}{a(p^*)}[p^* + Z^+(p^*)] \cdot Z(p^*) \\
&= \frac{1}{a(p^*)}[p^* \cdot Z(p^*) + Z^+(p^*) \cdot Z(p^*)] \\
&= \frac{1}{a(p^*)} Z^+(p)^* Z(p^*)
\end{aligned}$$

Here $p \cdot Z(p)$ is the value of demand minus the value of endowments minus the profits. It equals to the sum of all individuals' budget constraints. With non-satiation

$$Z(p^*) = 0$$

otherwise it is less than equal to 0.

5.3 Public Goods (Private Goods)

Non-excludable and non-rival goods. People who benefit from a public good but don't pay for it are "free rider". Non-rival means one person's consumption does not influence others'. Public good has both non-excludable and non-rival properties.

EXCLUDABLE↓/RIVAL→	**yes**	**no**
yes	private	Impure public (uncongested toll road)
no	Impure public (crowded road, fishery)	public (air quality)

5.3.1 Efficient Provision of Public Goods

Public good either provided or not $[0,1]$ choice. Two people $i, i = 1, 2$. Private good, x_i is i's consumption of private good.

$$x_i + g_i = w_i$$

where $x_i =$ spending on private on, $w_i =$ wealth and g_i is contribution to the cost of the public good.

The rule for provision $g_1 + g_2 >$ cost. $= c$, it is provided and $g_1 + g_2 < c$, not provided.

$$\text{Utility} = \begin{cases} u_i(1, w_i - g_i) & g_1 + g_2 > c \\ u_i(0, w_i) & g_1 + g_2 < c \end{cases}$$

When is the first Pareto superior to the second?

True if there are g_1, g_2 such that $g_1 + g_2 > c$ and $u_i(1, w_i - g_i) > u_i(0, w_i), i = 1, 2$.

Willingness-to-pay for public good (WTP) is defined as r_i such that (maximum it would make sense to pay)

$$u_i(1, w_i - r_i) = u_i(0, w_i)$$

$$u_1(1, w_1 - g_1) > u_1(0, w_1) = u_1(1, w_1 - r_1)$$
$$u_2(1, w_2 - g_2) > u_2(0, w_2) = u_2(1, w_2 - r_2)$$

Then $w_1 - g_1. > w_1 - r_1$ and $w_2 - g_2 > w_2 - r_2$. Also we know that $g_1 < r_1, g_2 < r_2$ so $r_1 + r_2 > g_1 + g_2 > c$. It is efficient to provide public good if and only if $r_1 + r_2 > c$, i.e., if sum of WTP's exceeds costs.

Proof. Suppose $r_1 + r_2 > c$ and choose g_i such that $g_i < r_i$ and $g_1 + g_2 > c$ and

$$u_1(1, w_1 - g_1) > u_1(0, w_1), u_1(1, w_1 - r_1) = u_1(0, w_1)$$

□

5.4 Game Theory

5.4.1 Prisoner's Dilemma

A↓/B→	Silent	Confess
Silent	1/1	3/0
Confess	0/3	2/2

The Nash equilibrium is the confess/confess case but the silent/silent can be the best outcome. This means that individually rational outcome is not Pareto efficient.

5.4.2 Nash Equilibrium

Players $i, i = 1, \cdots, I$. Chooses a strategy $s_i \in S_i$ such that S_i is the set of possible strategies.

$$S_{-i} = (S_1, S_2, \cdots, S_{i-1}, S_{i+1}, S_{i+2}, \cdots, S_I)$$

$$(S_i^*), i = 1, \cdots, I$$

forms a Nash Equilibrium if $\forall i$, $S_i^* \max_{s_i \in S_i} u_i(S_i, S_{-i}^*)$

Reaction function $f_i(s_{-i})$ is a person i's best response to the list of moves S_{-i} by others.

$$(S_1^*, \cdots, S_I^*)$$

such that $f_i^*(S_{-i}^*) = S_i^*$

Suppose there are two people, $i = 1, 2$. Public good, cost is c. Each person offers o_1 or o_2 towards cost. Good is provided if $o_1 + o_2 > c$. $r_1 + r_2 > c$. People will offer $o_i < r_i$. $(0 \le o_i \le r_i)$. Suppose person 1 offers $o_1 < .c$. Choose $o_2 = c - o_1$, then good is provided. The option is $u_2(1, w_2 - (c - o_1)) > u_2(0, r_2) = u_2(0, w_2)$.

5.4.3 The General Case

Proposition 15. Suppose that the consumption vectors x_j^* minimize the weighted utility sum $\sum_j \alpha_j U_j(x_j)$, $\sum_j x_j \in (\sum_i Y_i + \sum_j w_j)$ where $\alpha_j \geq 0$. Then x_j^* are Pareto efficient.

Proof. Suppose not. There exists x_j' feasible Pareto Superior, i.e. $u_j(x_j') \geq u_j(x_j^*), \forall j$ and for some j: $u_j(x_j') > u_j(x_j^*)$. Therefore

$$\sum_j \alpha_j u_j(x_j') > \sum_j \alpha_j u_j(x_j^*)$$

contradicts assumption $\alpha_j = 1, \forall j$. □

c_j is j's consumption of private good. w_j is j's endowment. g_f is what j pays to provision public good. $c_j + g_j = w_j$ and $g =$ amount of public good and $g = f(\sum_j g_j)$.

How much will j offer for the public good? Pick c_j, g_j to maximize $u(c_j, g)$ such that $c_j + g_j = w_j$ and $g = f(\sum_j g_j)$. First order condition:

$$\frac{\partial u_j / \partial g}{\partial u_j / \partial c_j} = \frac{1}{f'}$$

It shows the ratio of the marginal utility of the public good and marginal utility of private good. Above is the necessary and sufficient condition for Pareto efficient.

Efficiency: $\max \sum_j \alpha_j u_j(c_j, g)$ such that $g = f(\sum_j g_j), \sum_f c_f = \sum_j w_j - \sum_j g_j$. If maximize the weighted sum utility, then Pareto efficient.

$$\alpha = \sum u_j(c_j, g), \lambda(\sum w_j - \sum g_j - \sum c_j)$$

From first order condition we get

$$\sum \frac{\partial u / \partial g}{\partial u_j / \partial c_j} = \frac{1}{f'}$$

. Therefore, Nash equilibrium level of provision of public good is inefficient as $\partial u_j / \partial c_j < \sum_j \partial u_j / \partial c_j$. Therefore, $g_{NE} < g_{PE}$.

For firm, revenue is $g \sum_j p_j$. Then $\pi = f(z) \sum_j p_j - z$ where z is the input to public good. $g = f(z)$, $f' \sum_j p_j = 1$ and $\sum_j p_j = \frac{1}{f'}$. Therefore, $\sum_j \frac{\partial u_j / \partial g}{\partial u_j / \partial c_j} = \frac{1}{f'}$.

Market for public good, but with potential different price for all buyers (price discrimination). Each person pays p_j for each unit of public good.

$$\max u_j(c_j, g)$$

such that $w_j = c_j + g p_j$. The FOC is just $\frac{\partial u_j / \partial g}{\partial u_j / \partial c_j} = p_j$