Sums

January-05-11
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Assignments due on Fridays

Let f be any bounded function over a closed interval. i.e. $f: |a, b| \to \mathbb{R}$

f may be +ve, -ve, and possibly discontinuous.

Let $\mathcal{P}$ be a partition of [a, b]
Since f is bounded(bded) over each $[x_{j-1}, x_j]$ we get the numbers
$\sup\{f(x): x_{j-1} \le x \le x_j\} = \sup f[x_{j-1}, x_j]$
$\inf\{f(x): x_{j-1} \le x \le x_j\} = \inf f|x_{j-1}, x_j|$

Partition

A partition of [a, b] is a strictly increasing list of numbers starting at a and ending at b.
Denoted
$\mathcal{P}: a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

Uniform Partition

$\mathcal{P}$ is called uniform when the x_j are equally spaced.

A Distance Problem

You go from A to B in a car, odometer broken, speedometer is working, and you have a watch. The trip takes two hours. Estimate the distance traveled.

Take time samples between 0 and 2
$0 = t_0 < t_1 < t_2 < t_3 < \cdots < t_{n-1} < t_n = 2$
On each time interval $|t_{j-1}, t_j|$, record the maximum speed V_j attained on that interval.
Over the interval $|t_{j-1}, t_j|$ you travel at most a distance max $speed * time = V_j(t_j - t_{j-1})$
Over the full time interval [0, 2] you travelled at most a distance

$$D = \sum_{j=1}^{n} V_j(t_j - t_{j-1})$$

If v_j is the minimum speed recorded over time interval $[t_{j-1}, t_j]$ then total distance travelled is at least

$$d = \sum_{j=1}^{n} v_j(t_j - t_{j-1})$$

If each interval $[t_{j-1}, t_j]$ is small we expect $V_j - v_j$ to be small.
Then the difference

$$D - d = \sum_{j=1}^{n} (V_j - v_j)(t_j - t_{j-1})$$

should be small.

Roughly

$$D - d = \sum small \times small = \sum really\ small = fairly\ small$$

So actual distance covered is pinched between two estimates that are close to each other.

An Area Problem

Suppose a continuous (cts.) function (fun) f is defined over an interval [a, b] and f ≥ 0.
Estimate the are under f and over [a, b].
Well, chop up [a, b] into a pieces.
$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$
On each $|x_{j-1}, x_j|$ let $M_j = \max(\{f(x): x \in |x_{j-1}, x_j|\})$ and let $m_j = \min(\{f(x): x \in |x_{j-1}, x_j|\})$
If A_j is the actual area under f and over $[x_{j-1}, x_j]$ then,
$m_j(x_j - x_{j-1}) \le A_2 \le M_j(x_j - x_{j-1})$
Add up to get

$$\sum_{j=1}^{n} m_j(x_j - x_{j-1}) \le \sum_{j=1}^{n} A_j = total\ exact\ area\ under\ f\ and\ over\ |a, b| \le \sum_{j=1}^{n} M_j(x_j - x_{j-1})$$

If we make each $|x_{j-1}, x_j|$ small we expect $M_j - m_j$ to be small and thus

$$\sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}) = \sum small \times small = \sum very\ small = smallish$$

So we have a good estimate for the area, since the difference between the bounds is small.

Upper and Lower Sums

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Let f be any bounded function over a closed interval. i.e. $f: |a, b| \to \mathbb{R}$

Let $\mathcal{P}$ be a partition of [a, b]

Lower Sum

The lower sum for f using $\mathcal{P}$ is

$$L(f, \mathcal{P}) = \sum_{j=1}^{n} \inf f|x_{j-1}, x_j| \, (x_j - x_{j-1})$$

Upper Sum

$$U(f, \mathcal{P}) = \sum_{j=1}^{n} \sup f|x_{j-1}, x_j| \, (x_j - x_{j-1})$$

Note:
$L(f, \mathcal{P}) \le U(f, \mathcal{P})$
since $\inf f[x_{j-1}, x_j] \le \sup f|x_{j-1}, x_j|$ and add up inequalities

Refinement

A partition $\mathcal{Q}$ of [a, b] refines $\mathcal{P}$ when the points of $\mathcal{P}$ are also in $\mathcal{Q}$

Proposition 1

If $\mathcal{Q}$ refines $\mathcal{P}$ then
$L(f, \mathcal{P}) \le L(f, \mathcal{Q}) \le U(f, \mathcal{Q}) \le U(f, \mathcal{P})$

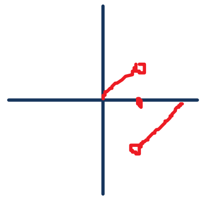
Proposition 2 (Corollary)

If $\mathcal{P}, \mathcal{Q}$ are any partitions of $|a, b|$, then
$L(f, \mathcal{P}) \le U(f, \mathcal{Q})$

Let $f: |a, b| \to \mathbb{R}$ be a bounded function and
$\mathcal{P}: a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ a partition of $|a, b|$

For each $[x_{j-1}, x_j]$ we have
$\sup\{f(x): x_{j-1} \le x \le x_j\} = \sup f[x_{j-1}, x_j]$ and
$\inf\{f(x): x_{j-1} \le x \le x_j\} = \inf f[x_{j-1}, x_j]$

Example



$$f(x) = \begin{cases} x \text{ on } [0, \frac{1}{2}) \\ x - 1 \text{ on } (\frac{1}{2}, 1] \\ 0 \text{ at } \frac{1}{2} \end{cases}$$

Use $\mathcal{P}: 0 < \frac{1}{2} < \frac{2}{3} < 1$

$\sup f \left|0, \frac{1}{3}\right| = \frac{1}{3}$, $\inf f \left|0, \frac{1}{3}\right| = 0$

$\sup f \left|\frac{1}{3}, \frac{2}{3}\right| = \frac{1}{2}$, $\inf f \left|\frac{1}{3}, \frac{2}{3}\right| = -\frac{1}{2}$

$\sup f \left|\frac{2}{3}, 1\right| = 0$, $\inf f \left|\frac{2}{3}, 1\right| = -\frac{1}{3}$

Example

$$f(x) = \begin{cases} 1 \text{ when } x \in \mathbb{Q} \\ 0 \text{ when } x \notin \mathbb{Q} \end{cases}$$

For every $\mathcal{P}: 0 = x_0 < x_1 < \cdots < x_n = 1$
we get

$$L(f, \mathcal{P}) = \sum \inf f|x_{j-1}, x_j| \, (x_j, x_{j-1}) = 0$$

$$U(f, \mathcal{P}) = \sum \sup f|x_{j-1}, x_j|(x_j - x_{j-1}) = \sum_{j-1}^{n} 1(x_j - x_{j-1}) = 1$$

Example

$f(x) = x^2$ on $|0, 1|$
Take the uniform partition

$$\mathcal{P}_n: 0 = \frac{0}{n} < \frac{1}{n} < \frac{2}{n} < \frac{n-1}{n} < \frac{n}{n} = 1$$

Now

$$U(f, \mathcal{P}_n) = \left(\frac{1}{n}\right)^2 \left(\frac{1}{n} - 0\right) + \left(\frac{2}{n}\right)^2 \left(\frac{2}{n} - \frac{1}{n}\right) + \left(\frac{3}{n}\right)^2 \left(\frac{3}{n} - \frac{2}{n}\right) + \cdots + \left(\frac{n}{n}\right)^2 \left(\frac{n}{n} - \frac{n-1}{n}\right)$$

$$= \frac{1^2}{n^3} + \frac{2^2}{n^3} + \frac{3^3}{n^3} + \cdots + \frac{n^n}{n^3} = \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

Similarly,

$$L(f, \mathcal{P}) = \frac{1}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)$$

Refinements

Example

$0 < \frac{1}{2} < 3 < 3.2 < 5$ is refined by $0 < \frac{1}{2} < 1.7 < 3 < 3.2 < 4 < 5$

Proof of Proposition 1

Show $U(f, \mathcal{Q}) \le U(f, \mathcal{P})$
It suffices to check this when $\mathcal{Q}$ has just one point more than $\mathcal{P}$ since we can induct over the number of points.
Say $\mathcal{P}: a = x_0 < x_1 < \cdots < x_{k-1} < x_k < \cdots < x_n = b$
$\mathcal{Q}: a = x_0 < x_1 < \cdots < x_{k-1} < y < x_k < \cdots < x_n = b$
Now
$U(f, \mathcal{P})$

$$= \sum_{j=1}^{k-1} \sup f[x_{j-1}, x_j] \, (x_j - x_{j-1}) + \sup f|x_{k-1}, x_k| \, (x_k - x_{k-1})$$

$$+ \sum_{j=k+1}^{n} \sup f|x_{j-1}, x_j| \, (x_j - x_{j-1})$$

$U(f,\mathcal{P})$
$= \sum_{j=1}^{k-1} \sup f[x_{j-1},x_j]\,(x_j - x_{j-1}) + \sup f|x_{k-1},y|\,(y - x_{k-1})$
$+ \sup f|y,x_k|\,(x_k - y) + \sum_{j=k+1}^{n} \sup f|x_{j-1},x_j|\,(x_j - x_{j-1})$
So we need to see that
$sup\, f|x_{k-1},x_k|\,(x_k - x_{k-1}) \geq sup\, f|x_{k-1},y|\,(y - x_{k-1}) + sup\, f|y,x_k|\,(x_k - y)$
We know that $\sup f|x_{k-1},x_k| \geq \sup f[x_{k-1},y]$ and $\sup f|x_{k-1},x_k| \geq \sup f[y,x_k]$
and thus
$\sup f|x_{k-1},y|\,(y - x_{k-1}) + \sup f|y,x_k|\,(x_k - y)$
$\leq \sup f|x_{k-1},x_k|\,(y - x_{k-1}) + \sup f|x_{k-1},x_k|\,(x_k - y)$
$= \sup f|x_{k-1},x_k|\,(x_k - x_{k-1})$
$\mathcal{QED}$

Proof of Proposition 2

Let $\mathcal{R}$ be the partition of [a, b] that includes all points of $\mathcal{P}$ and $\mathcal{Q}$
$\mathcal{R}$ is called the common refinement of $\mathcal{P}$ and $\mathcal{Q}$
By Proposition 1, we get
$L(f,\mathcal{P}) \leq L(f,\mathcal{R}) \leq U(f,\mathcal{R}) \leq U(f,\mathcal{Q})$ ■

Integrable Definition

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Integrable Function and Integral

A function f is said to be integrable over [a, b] iff
$\sup_P L(f,P) = \inf_Q U(f,Q)$

The common number is the integral of f over [a, b]
We write:
$\int_a^b f = \sup_P L(f,P) = \inf_Q U(f,Q)$

Since $U(f,\mathcal{Q})$ is an upper bound for all $L(f,\mathcal{P})$'s we get
$\sup\{L(f,\mathcal{P}): \mathcal{P} is\ any\ partition\ of\ [a,b]\} \le U(f,\mathcal{Q})$

Short notation:
$\sup_{\mathcal{P}} L(f,\mathcal{P}) \le U(f,\mathcal{Q})$

Since $\sup_P L(f,P)$ is a lower bound for all $U(f,Q)$ we get
$\sup_P L(f,P) \le \inf_Q U(f,Q)$

Example

$f(x) = \begin{cases} 1\ for\ x \in \mathbb{Q} \\ 0\ for\ x \notin \mathbb{Q} \end{cases} on\ [a,b]$
We saw all $L(f,P) = 0$ and all $U(f,Q) = 1$
So
$\sup_P L(f,P) = 0 < 1 = \inf_Q U(f,Q)$
So f is not integrable

Example

$f(x) = x^2$ on $[0,1]$
Using uniform partitions $\mathcal{P}_n$ we got
$L(f,\mathcal{P}_n) = \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$
$U(f,\mathcal{P}_n) = \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

Hence
$\inf_Q U(f,Q) \le \frac{1}{3}\ since \inf_Q U(f,Q) \le all\ U(f,P_n) and \lim_{n\to\infty} U(f,P_n) = \frac{1}{3}$
Similarly, $\frac{1}{3} \le \inf_P L(f,Q)$
$\frac{1}{3} \le \sup_P L(f,P) \le \inf_Q U(f,Q) \le \frac{1}{3}$
so
$\int_0^1 f = \frac{1}{3}$

Riemann's Integrability Criterion

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Proposition 3 - proof to know

Riemann's Integrability Criterion

$f: |a,b| \to \mathbb{R}$ is integrable if and only if for every ε > 0, there is a partition R of [a, b] such that $U(f,R) - L(f,R) < \varepsilon$

Proposition 4

Every increasing/decreasing $f: |a,b| \to \mathbb{R}$ is integrable

Riemann Sum

Instead of using upper and lower sums, pick some value $f(a_i)$ in each section of the partition $\mathcal{P}$

$$\sum_{i=1}^{n} f(a_i)(x_i - x_{i-1})$$

Approaches the integral as the partition gets finer.

We have seen that all
$L(f,P) \le all\ U(f,Q)$
Thus
$\sup_P L(f,P) \le \inf_Q U(f,Q)$
If = happens we say f is integrable on [a, b] and its integral is

$$\int_a^b f = \sup_P L(f,P) = \inf_Q U(f,Q)$$

Proof of proposition 3

Suppose f is integrable and take $\varepsilon > 0$. Then
$\sup_P L(f,P) = \inf_Q U(f,Q)$
Hence there exist partitions P_1, Q_1 such that
$\sup_P L(f,P) - \frac{\varepsilon}{2} < L(f,P_1)$
$U(f,Q_1) < \inf_Q U(f,Q) + \frac{\varepsilon}{2}$
Let R be a common refinement of P_1 and Q_1
Then
$\sup_P L(f,P) - \frac{\varepsilon}{2} < L(f,P_1) \le L(f,R) \le U(f,R) \le U(f,Q_1), < \inf_Q U(f,Q) + \frac{\varepsilon}{2}$
But
$$\int_a^b f = \sup_P L(f,P) = \inf_Q U(f,Q)$$
so
$$\int_a^b f - \frac{\varepsilon}{2} < L(f,R) \le U(f,R) < \int_a^b f + \frac{\varepsilon}{2}$$
And therefore
$U(f,R) - L(f,R) < \varepsilon$

Conversely, say for every $\varepsilon > 0$ there is a partition R such that $U(f,R) - L(f,R) < \varepsilon$
Then we have
$L(f,R) \le \sup_P L(f,P) \le \inf_Q U(f,Q) \le U(f,R)$
So for every $\varepsilon > 0$, we get
$0 \le \inf_Q U(f,Q) - \sup_P L(f,P) < \varepsilon$
But $\inf_Q U(f,Q) - \sup_P L(f,P)$ is constant, so
$\inf_Q U(f,Q) - \sup_P L(f,P) = 0$
■

Example

$a < c < b$, Put
$$f(x) = \begin{cases} 0, & a \le x < c \\ 1, & x = c \\ 0, & c < x \le b \end{cases}$$
Use Proposition 3. Take ε > 0
Pick x_1, x_2 such that $a < x_1 < c < x_2 < b$ and $x_2 - x_1 < \varepsilon$
Take $R: a < x_1 < x_2 < b$, a partition of $[a, b]$
$L(f,R) = 0 \times (a - x_1) + 0 \times (x_2 - x_1) + 0 \times (b - x_2) = 0$
$U(f,R) = 0 \times (a - x_1) + 1 \times (x_2 - x_1) + 0 \times (b - x_2) < \varepsilon$
So $U(f,R) - L(f,R) < \varepsilon - 0 = \varepsilon$

So f is integrable and $\int_a^b f = 0$

Proof of Proposition 4

Suppose $f: |a,b| \to \mathbb{R}$ is increasing (i.e. $a \le x_1 \le x_2 < b \Rightarrow f(x_1) \le f(x_2)$)

If $f(x) = c = const$ then a simple calculation gives all $U(f,P) = all\ L(f,P) = c(b-a)$
So
$$\int_a^b f = \sup_P L(f,P) = \inf_Q L(f,Q) = c(b-a)$$

Now, suppose $f(x) \ne constant$, so $f(b) > f(a)$
Take any ε > 0

Pick a partition P: $a = x_0 < x_1 < \cdots < x_n = b$ such that all $x_j - x_{j-1} < \frac{\varepsilon}{f(b)-f(a)}$

Then

$$U(f,P) - L(f,P) = \sum_{j=1}^{n} \left(\sup f|x_{j-1}, x_j| - \inf f|x_{j-1}, x_j|\right)(x_j - x_{j-1})$$

$$= \sum_{j=1}^{n} \left(f(x_j) - f(x_{j-1})\right)(x_j - x_{j-1}) < \sum_{j=1}^{n} \left(f(x_j) - f(x_{j-1})\right)\frac{\varepsilon}{f(b) - f(a)}$$

$$= \frac{\varepsilon}{f(b) - f(a)} \times \left(f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(x_n) - f(x_{n-1})\right)$$

$$= \frac{\varepsilon}{f(b) - f(a)}\left(f(b) - f(a)\right) = \varepsilon$$

■

Example

$$f(x) = \begin{Bmatrix} 0\ on\ \left|0, \frac{1}{2}\right) \\ \frac{1}{2}\ on\ \left|\frac{1}{2}, \frac{2}{3}\right) \\ \frac{2}{3} on\ \left|\frac{2}{3}, \frac{3}{4}\right) \\ \cdots \\ 1\ at\ 1 \end{Bmatrix}$$

Uniform Continuity

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Fact

$|\sin b - \sin a| \leq |b - a|$

Triangle Inequality

On a triangle, the distance between any two points is less than or equal to the sum of the distances between the other points, and greater than or equal to the difference in the distances of the other points.

$|a + b| \leq |a| + |b|$
$|a - b| \geq ||a| - |b||$

Uniform Continuity

On midterm

A function $f: I \to \mathbb{R}$ is uniformly continuous on the interval I when for every ε > 0 there is a δ > 0 such that $|f(x) - f(p)| < \varepsilon$ when $x, p \in I$ and $|x - p| < \delta$

Comparison of Continuities

Normal:

f cts. on I
$\forall \varepsilon > 0\ \forall p \in I\ \exists \delta > 0\ s.t.$
$\forall x \in I\ |x - p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$

Uniform:

f unif. cts. on I
$\forall \varepsilon > 0\ \exists \delta > 0\ s.t. \forall p \in I\ \forall x \in I$
$|x - p| < \delta \Rightarrow |f(x) - f(p)| < \varepsilon$

Example

$f(x) = x + \sin(x)\ on\ \mathbb{R}$
Take any p ∈ ℝ and show f is continuous at p

Take any ε > 0. Let's find δ > 0 such that $|x + \sin x - (p + \sin p)| < \varepsilon$ when $|x\ - p| < \delta$
$|x + \sin x - p - \sin p| \leq |x - p| + |\sin x - \sin p| \leq |x - p| + |x - p| = 2|x - p|$
Take $\delta = \frac{\varepsilon}{2}$
When $|x - p| < \delta$, we will get
$|x + \sin(x)\ - (p + \sin p)| \leq 2|x - p| < 2\delta = 2\left(\frac{\varepsilon}{2}\right) = \varepsilon$

Example

$f(x) = x^2\ on\ \mathbb{R}$
Take $p \in \mathbb{R}$. Check f is continuous at p. Take ε > 0
Need δ > 0 so that $|x - p| < \delta \Rightarrow |x^2 - p^2| < \varepsilon$
$|x^2 - p^2| = |x + p||x - p|$
If we keep $|x - p| < 1, then\ |x| - |p| < 1$, so $|x| < |p| + 1$
Then when $|x - p| < 1$:
$|x^2 - p^2| \leq (|x| + |p|)|x - p| \leq (|p| + 1 + |p|)|x - p| = (2|p| + 1)|x - p|$
Take $\delta = \min\left\{1, \frac{\varepsilon}{2|p|+1}\right\}$

Now when $|x - p| < \delta$ we get
$|x^2 - p^2| \leq (2|p| + 1)|x - p| < (2|p| + 1)\left(\frac{\varepsilon}{2|p| + 1}\right) = \varepsilon$

Note:

In the first case, δ did not depend on p, while in the second case δ did depend on p. There is not a single δ that works for all possible points.
$f(x) = x + \sin x$ is uniformly continuous on ℝ. Right now don't know that $f(x) = x^2$ is not uniformly continuous.

Proof that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$

Suppose f were unif. cts. on ℝ and look for contradiction.
So for ε = 1 we have a δ > 0 such that $x, p \in \mathbb{R}$ and $|x - p| < \delta \Rightarrow |x^2 - p^2| < 1$
Let n be an integer so big that $\frac{1}{n} < \delta$
Then take $p = n\ and\ x = n + \frac{1}{n}$. Clearly $|x - p| = \frac{1}{n} < \delta$
$|x^2 - p^2| = \left|\left(n + \frac{1}{n}\right)^2 - n^2\right| = \left|n^2 + 2 + \frac{1}{n^2} - n^2\right| = 2 + \frac{1}{n^2} > 1$

Sequences and Unif. Ctn.

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$f: I \to \mathbb{R}$ is uniformly continuous on the interval I means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ when $x, p \in I$ and $|x - p| < \delta$

Proposition 5

$f: I \to \mathbb{R}$ is not uniformly continuous on I $\Leftrightarrow$ there exist sequences $x_n, p_n \in I$, such that $x_n - p_n \to 0$ while $f(x_n) - f(p_n) \nrightarrow 0$

equivalently

$f: I \to \mathbb{R}$ is uniformly continuous on I $\Leftrightarrow$ $\forall$ sequences $x_n, p_n \in I, x_n - p_n \to 0 \Rightarrow f(x_n) - f(p_n) \to 0$

Proposition 6

$If\ f: |a, b| \to \mathbb{R}$ is continuous on a closed interval [a, b], then f is uniformly continuous.

Proof of Proposition 5

Say f is unif. cts. on I.
Take $x_n, p_n \in I$ and $x_n - p_n \to 0$
Want $f(x_n) - f(p_n) \to 0$
Take $\varepsilon > 0$, we need to show $|f(x_n) - f(p_n)| < \varepsilon$ eventually
By uniform continuity of f, we have $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ when x, p ∈ I and $|x - p| < \delta$
Eventually $|x_n - p_n| < \delta\ \forall$ n ≥ N and so $|f(x_n) - f(p_n)| < \varepsilon\ \forall n \geq N$
So $f(x_n) - f(p_n) \to 0$

Now suppose f is not unif. cts. on I
So there is a "bad" ε >0 that no δ > 0 can please
No $\delta = \frac{1}{n}$ can please this ε. For each such $\frac{1}{n}$ we pick up $x_n, p_n \in I$ such that
$|x_n - p_n| < \frac{1}{n}\ while\ |f(x_n) - f(p_n)| \geq \varepsilon$
By the squeeze theorem, $x_n - p_n \to 0$ and clearly $|f(x_n) - f(p_n)| \nrightarrow 0$
■

Example

Show $f(x) = \ln x$ is not uniformly continuous on $(0, 1)$
Well, $\frac{1}{e^n}\ and\ \frac{1}{e^{n+1}} \in (0, 1)$ and $\frac{1}{e^n} - \frac{1}{e^{n+1}} \to 0$
But $\ln\left(\frac{1}{e^n}\right) - \ln\left(\frac{1}{e^{n+1}}\right) = -n - (-(n+1)) = 1 \nrightarrow 0$

Proof of Proposition 6

Suppose f is not uniformly continuous.
Then there is a "bad" ε > 0 such that no δ > 0 can please.
For all $\delta = \frac{1}{n}$, pick $x_n, p_n \in I$ such that $|x_n - p_n| < \frac{1}{n}$ but $|f(x_n) - f(p_n)| \geq \varepsilon$

Using Bolzano-Weierstrass we pick up a subsequence p_{n_k} of p_n such that $p_{n_k} \to p \in [a, b]$ as k → ∞
Notice $x_{n_k} = p_{n_k} + (x_{n_k} - p_{n_k}) \to p + 0 = p$
So $f(x_{n_k}) \to f(p)$ as $k \to \infty$ and $f(p_{n_k}) \to f(p)$
Therefore $f(x_{n_k}) - f(p_{n_k}) \to p - p = 0$ so $\exists K \in \mathbb{N}$ such that $|f(x_{n_k}) - f(p_{n_k})| < \varepsilon$ ∀ k ≥ K
But $|f(x_n) - f(p_n)| \geq \varepsilon\ \forall n$, a contradiction.
So f is uniformly continuous. ■

Integrability of Continuous

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Theorem 7

Every continuous function on a closed interval is integrable on that interval.

If $f: |a, b| \to \mathbb{R}$ is continuous and $\varepsilon > 0$ is given, take $\delta > 0$ such that $|x - p| < \delta \Rightarrow |f(x) - f(p)| < \frac{\varepsilon}{b-a}$
If $\mathcal{P}: a = x_0 < x_1 < \cdots < x_n = b$ is a partition constructed such that all $x_j - x_{j-1} < \delta$ then
$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$
So f is integrable on [a, b].

Proof of Theorem 7

On each $|x_{j-1}, x_j|$ f gets a maximum and a minimum value by the extreme value theorem.
Pick u_j, v_j *such that* $f(u_j) = \sup f[x_{j-1}, x_j]$ *and* $f(v_j) = \inf f[x_{j-1}, x_j]$
$x_{j-1} \le v_j \le u_j \le x_j$ so $u_j - v_j \le x_j - x_{j-1} \Rightarrow \sup f|x_{j-1}, x_j| - \inf f|x_{j-1}, x_j| < \frac{\varepsilon}{b-a}$

$$U(f, \mathcal{P}) - L(f, P) = \sum_{i=1}^{n} (\sup f|x_{i-1}, x_i| - \inf f|x_{i-1}, x_i|)(x_i - x_{i-1}) < \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(x_i - x_{i-1})$$

$$= \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (x_i - x_{i-1}) = \frac{\varepsilon}{b-a}(b-a) = \varepsilon$$

Estimating Integrals

To make an estimate of the integral of a continuous bounded function on [a, b], for an estimate within ε of the true integral, partition the interval into $[x_{j-1}, x_j]$ with $x_j - x_{j-1} < \frac{\varepsilon}{b-a}$ and sum the area of those rectangles.

Fundamental Theorem of Calculus I

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Observation

If f is integrable on [a, b] and S is a number such that
$L(f,\mathcal{P}) \le S \le U(f,\mathcal{P})$ for all partitions $\mathcal{P}$ then $S = \int_a^b f$

Theorem 8

Fundamental Theorem of Calculus pt. 1

(Learn Proof)

If F, f are functions on [a, b] such that

- f is integrable
- F is continuous on [a, b]
- F' = f over (a, b)

Then

$$\int_a^b f = F(b) - F(a)$$

F(x) is known as the antiderivative of f or the indefinite integral

Question: Is there a function F such that F' is not integrable?

Notation

Non-mathematical reasoning

When f is continuous, we see $\int_a^b f \approx U(f,\mathcal{P})$ when $\mathcal{P}$ is very fine.

$$U(f,\mathcal{P}) = \sum_j \sup f|x_{j-1},x_j|\,(x_j - x_{j-1}) \approx \sum_j f(x_j)(x_j - x_{j-1})$$

Pretend your $\mathcal{P}$ is so fine that you make a cut at every x in $[a,b]$
Now you get "nano-thin" rectangles of "thickness" dx, height $f(x)$, and "area" $f(x)dx$.
"Add up" these "values" $f(x)dx$ using the "limiting sum" $\int_a^b \square$ and we can write

$$\int_a^b f = \int_a^b f(x)dx$$

Another Useful Notation

$F(x)\Big|_a^b$ *or* $|F(x)|_a^b$
means $F(b) - F(a)$

Proof of Fundamental Theorem

If $\mathcal{P}: a = x_0 < x_1 < \cdots < x_n = b$ is any partition of [a, b] we will show that
$L(f,\mathcal{P}) \le F(b) - F(a) \le U(f,\mathcal{P})$

$$F(b) - F(a) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1}))\text{, }\textit{rebuilt the telescope}$$

Apply the Mean Value Theorem to F over each $[x_{j-1},x_j]$, we pick up some $t_j \in (x_{j-1},x_j)$ such that

$F(x_j) - F(x_{j-1}) = F'(t_j)(x_j - x_{j-1}) = f(t_j)(x_j - x_{j-1})$

$\inf f|x_{j-1},x_j| \le f(t_j) \le \sup f|x_{j-1},x_j|$

$$\Rightarrow \sum_{i=1}^{n} \inf f|x_{i-1},x_i|\,(x_i - x_{i-1}) \le \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \le \sum_{i=1}^{n} \sup f|x_{i-1},x_i|\,(x_i - x_{i-1})$$

$\Rightarrow L(f,\mathcal{P}) \le F(b) - F(a) \le U(f,\mathcal{P})$

So

$$\int_a^b f = F(b) - F(a)$$

∎

Example

Let $f(x) = \sin x$ over $[0,\pi]$
We know $F(x) = -\cos x$
By FTC (part 1)

$$\int_0^\pi f = -\cos\pi + \cos 0 = -(-1) + 1 = 2$$

Example

$$\int_0^1 \frac{1}{1+x^2}dx = \arctan x\Big|_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

Example

$$\int_1^2 \frac{1}{x}dx = |\ln x|_a^b = \ln 2$$

Example

$$\int_{-1}^0 (x^3 + 2x^2)dx = \Big|\frac{1}{4}x^4 + \frac{2}{3}x^3\Big|_{-1}^0 = 0\ - \left(\frac{1}{4} - \frac{2}{3}\right) = \frac{5}{12}$$

Anti-Derivatives

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Integral
Riemann Integral
Conventional integral over an interval using upper and lower sums

Indefinite Integral
The anti-derivative of a function plus a constant.

Integrand
That which is to be integrated.

Terminology
In order to calculate $\int_a^b f(x)dx$ using FTC(I) we need a function F such that $F' = f$
Then we know
$$\int_a^b f(x)dx = F(b) - F(a)$$

Any function F such that $F' = f$ is called an anti-derivative of f and is denoted by
$$\int f(x)dx$$
with no endpoints. This is a function, while with endpoints is a number.

So FTC(I) said
$$\int_a^b f(x)dx = \int f(x)dx\Big|_a^b$$

If F, G are two anti-derivatives of f on some interval I then $F' = f = G' \Rightarrow (G - F)' = 0$
$\Rightarrow G - F = c = const$
$\Rightarrow G = F + c$

So one we have one anti-derivative F of f, we write
$$\int f(x)dx = F(x) + C$$

Because of FTC(I), we also call
$$\int f(x)dx$$
an indefinite integral of f.

Remember:
The left hand side (integral) is defined on its own. It is not defined through the anti-derivative.

So we need to find these indefinite integrals:

Anti-Derivative Rules
Know by heart

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C\,, a \in \mathbb{R}, a \neq -1$$
$$\int \frac{1}{x} = \ln|x| + C$$
$$\int \sin x\, dx = -\cos x + C$$
$$\int \cos x\, dx = \sin x + C$$
$$\int \frac{1}{\cos^2 x} dx = \tan x + C$$
$$\int \frac{1}{1+x^2} dx = \arctan x + C$$
$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$$
$$\int e^x dx = e^x + C$$

The Substitution Method
Suppose F, f, g, are functions. Here is the chain rule:

Derivative Style	Integration Style
If $F'(u) = f(u)$ Then $F(g(x))' = f(g(x))g'(x)$	If $\int f(u)du = F(u)$ then $\int f(g(x))g'(x)dx = F(g(x))$

So in order to find some
$$J = \int f(g(x))g'(x)dx$$
play the following substitution game.
Put $u = g(x)$
$$\frac{du}{dx} = g'(x)$$
$du = g'(x)dx$
Find

$\int f(u)du = F(u)$

$J = F\big(g(x)\big) + C$

Example

$J = \int \frac{2x}{1+x^2}dx$

Put $u = 1 + x^2 \Rightarrow \frac{du}{dx} = 2x\,dx$

$J = \int \frac{1}{u}du = \ln|u| = \ln(1+x^2) + C$

Example

$J = \int \frac{x}{1+x^4}dx = \frac{1}{2}\int \frac{2x}{1+(x^2)^2}dx$

$Put\ u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

$So\ J = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\arctan u = \frac{1}{2}\arctan(x^2) + C$

Example

$J = \int \frac{1}{x\ln x}dx$

$Put\ u = \ln x \Rightarrow du = \frac{1}{x}dx$

$J = \int \frac{1}{u}du = \ln|u| = \ln\ln x + C$

Example

$J = \int \sqrt{1-x^2}\,dx$

More obscure - trig substitution. Cleverly notice

$J = \int (1-x^2) \times \frac{1}{\sqrt{1-x^2}} = \int (1-\sin^2(\arcsin x))\left(\frac{1}{\sqrt{1-x^2}}\right)dx$

$Put\ u = \arcsin x \Rightarrow du = \frac{1}{\sqrt{1-x^2}}dx$

$J = (1-\sin^2 u)du = \int \cos^2 u\,du = \frac{1}{2}\int (\cos 2x + 1) = \frac{1}{2}\int \cos 2u\,du + \frac{1}{2}\int 1du = \frac{1}{4}\sin 2u + \frac{1}{2}u$

$= \frac{1}{2}\sin u \cos u + \frac{1}{2}u = \frac{1}{2}\sin(\arcsin x)\cos(\arcsin x) + \frac{1}{2}\arcsin x = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$

Integration Methods

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Integration by Parts

$J = \int u\,dv = uv - \int v\,du$

Memorise

Integrating Rationals

Key Theorem

Every rational function can be expressed as a linear combination of the following functions:

$1, x, x^2, \dots, x^n \dots$

$\frac{1}{x-a}, \frac{1}{(x-a)^2}, \frac{1}{(x-a)^3}, \dots, \frac{1}{(x-a)^n}, \dots$ for any $a \in \mathbb{R}$

$\frac{1}{x^2+bx+c}, \frac{1}{(x^2+bx+c)^2}, \dots, \frac{1}{(x^2+bx+c)^n}, \dots$ Where x^2+bx+c is irreducible

$\frac{x}{x^2+bx+c}, \frac{x}{(x^2+bx+c)^2}, \dots, \frac{x}{(x^2+bx+c)^n}, \dots$ Where x^2+bx+c is irreducible

In other words, these functions form a basis for the set of all rational functions.

Thus we need to be able to integration the functions on this list, and write a rational function as a linear combination of these.

Change of Variables for Definite Integrals

$If\ F(u) = \int f(u)du$

$\int_a^b f(g(x))g'(x)dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a))$

$= \int_{g(a)}^{g(b)} f(u)du$

Integration by Substitution

To integrate stuff like $J = \int f(g(x))g'(x)dx$
Put $u = g(x) \Rightarrow du = g'(x)dx$
Find $F(u) = \int f(u)du$
Write $J = F(g(x)) + C$

But it's sometimes not easy to see what $u = g(x)$ to try. Try something and hope

Inverse Substitution Method

Example

We had

$J = \int \sqrt{1-x^2}dx$

and discovered that $u = \arcsin x \Rightarrow du = \frac{1}{\sqrt{1-x^2}}dx$ let to $J = \int \cos^2 u\,du$ then we got to

$J = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$

But what if we did not know to try $u = \arcsin x$? Here is a way to $\int \cos^2 u\,du$
Put $x = \sin u \Rightarrow dx = \cos u\,du$
$u = \arcsin x$
$\sqrt{1-x^2} = \sqrt{1-\sin^2 u} = \cos u$
$J = \int \sqrt{1-x^2}dx = \int \cos^2 u\,du$
Then continue as before.

Example

$J = \int \sqrt{1+e^x}dx$

$Put\ u = \sqrt{1+e^x} \Rightarrow e^x = u^2 - 1$

$du = \frac{1}{2\sqrt{1+e^x}}e^x dx = \frac{u^2-1}{2u}dx$

$dx = \frac{2u}{u^2-1}du$

So

$J = \int u \times \frac{2u}{u^2-1}du = 2\int \frac{u^2}{u^2-1}du = 2\left(\int \frac{u^2-1}{u^2-1}du + \int \frac{1}{u^2-1}du\right)$

Call

$J_1 = \int \frac{1}{u^2-1}du$

Use Partial Fractions

$\frac{1}{u^2-1} = \frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$
$1 = A(u+1) + B(u-1) =$
$u = 1 \Rightarrow A = \frac{1}{2}$
$u = -1 \Rightarrow B = -\frac{1}{2}$

So

$J_1 = \frac{1}{2}\int \frac{du}{u-1} - \frac{1}{2}\int \frac{du}{u+1} = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1|$

$J = 2\left(u + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1|\right) = 2u + \ln|u-1| - \ln|u+1|$

$= 2\sqrt{1+e^x} + \ln(\sqrt{1+e^x} - 1) - \ln(\sqrt{1+e^x} + 1) + C$

Integration by Parts

Say f, g are differentiable on I
Here is the product rule

Differentiation Style	**Integration Style**
$(f(x)g(x))'$ $= f(x)g'(x) + f'(x)g(x)$	$f(x)g(x)$ $= \int (f(x)g'(x) + f'(x)g(x))dx$

So

$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)\,dx$

To exchange $\int f(x)g'(x)dx$
Put $u = f(x) \Rightarrow du = f(x)dx,\ \frac{dv}{dx} = g'(x) \Rightarrow dv = g'(x)dx$
$v = \int dv = \int g'(x)dx$
Here you need to integrate this "part"

Write

$J = \int u\,dv = uv - \int v\,du$

Example

$J = \int x\, e^x dx$

Put $u = x \Rightarrow du = dx$

$dv = e^x dx \Rightarrow v = \int e^x\, dx = e^x$

This

$J = xe^x - \int e^x dx = xe^x - e^x + C$

Example

$J = \int x^2 \cos x\, dx$

Put $u = x^2, dv = \cos x\, dx$

$du = 2x\, dx, v = \int \cos x\, dx = \sin x$

$J = x^2 \sin x - 2\int x \sin x\, dx$

$Put\ u = x, dv = \sin x\, dx$

$du = dx, v = \int \sin x\, dx = -\cos x$

$J = x^2 \sin x - 2\left(-x\cos x + \int \cos x\ \ dx\right) = x^2 \sin x + 2x\cos x - 2\sin x + C$

Example

$J = \int \ln x\ \ dx$

$Put\ u = \ln x\, , dv = dx$

$du = \frac{1}{x} dx\ , v = x$

$J = x\ln x - \int \frac{1}{x} x\ dx = x\ln x - x + C$

Example

$J = \int \arctan x\, dx$

$Put\ u = \arctan x\, , dv = dx$

$du = \frac{1}{1+x^2} dx, v = x$

$J = x\arctan x - \int \frac{x}{1+x^2} dx$

$J_1 = \frac{1}{2}\int \frac{2x}{1+x^2} dx = \frac{1}{2}\ln(1+x^2)$

$J = x\arctan x - \frac{1}{2}\ln(1+x^2) + C$

Example

$J = \int e^x \sin x\ \ dx$

Put $u = e^x, dv = \sin x\, dx$
$du = e^x\ dx\ , v = -\cos x$

$J = -e^x \cos x + \int e^x \cos x\, dx$

$J_1 = \int e^x \cos x\, dx$

$Put\ u = e^x, dv = \cos x\, dx$
$du = e^x, v = \sin x\, dx$

$J_1 = e^x \sin x - \int e^x \sin x\, dx$

$J_1 = e^x \sin x - J$
$J = -e^x \cos x + J_1 = -e^x \cos x + e^x \sin x - J$
$2J = e^x \sin x - e^x \cos x$
$J = \frac{e^x \sin x - e^x \cos x}{2} + C$

Example
Constant over irreducible quadratic - complete the square and use arctan

$J = \int \frac{dx}{x^2+x+1}$
Complete square

$x^2 + x + 1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2 + 1\right) = \frac{3}{4}\left(\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2 + 1\right)$

$Put\ u = \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right), du = \frac{2}{\sqrt{3}} dx \Rightarrow dx = \frac{\sqrt{3}}{2} du$

So

$$J = \int \frac{1}{\frac{3}{4}(u^2+1)}\left(\frac{\sqrt{3}}{2}\right)du = \frac{4}{3}\left(\frac{\sqrt{3}}{2}\right)\int \frac{1}{u^2+1}du = \frac{2}{\sqrt{3}}\arctan u = \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)+C$$

Example of Rational Theorem

$$J = \int \frac{x^3+x+1}{x^2-2x-3}dx$$

Here, deg $top \geq$ deg $bottom$

$$J = \int x+2+\frac{8x+7}{x^2-2x-3}dx$$

Easily, $\int x+2\,dx = \frac{1}{2}x^2+2x$

$$J_1 = \int \frac{8x+7}{x^2-2x-3}dx = \int \frac{8x+7}{(x-3)(x+1)}dx$$

We try to solve

$$\frac{8x+7}{x^2-2x-3} = \frac{A}{x-3}+\frac{B}{x+1}$$

Get, check myself

$$A = \frac{31}{4}, B = \frac{1}{4}$$

$$J_1 = \frac{31}{4}\int \frac{1}{x-3}+\frac{1}{4}\int \frac{1}{x+1} = \frac{31}{4}\ln(|x-3|)+\frac{1}{4}\ln(|x+1|)$$

$$J = \frac{1}{2}x^2+2x+\frac{31}{4}\ln(|x-3|)+\frac{1}{4}\ln(|x+1|)$$

Rational Expansion

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A basis for spaces of rational functions is all:

$x^n, n \in \mathbb{N}$

$\frac{1}{(x-a)^n}, a \in \mathbb{R}, n \in \mathbb{N}$

$\frac{1}{(x^2+bx+c)^n}, b, c \in \mathbb{R}, b^2-4c > 0, n \in \mathbb{N}$

$\frac{x}{(x^2+bx+c)^n}, b, c \in \mathbb{R}, b^2-4c > 0, n \in \mathbb{N}$

Example

$$J = \int \frac{3x^2+2}{(x+1)(x^2+x+1)}dx$$

For the partial fraction expansion (write rational function in terms of basis)

$$\frac{3x^2+2}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{B}{x^2+x+1} + \frac{Cx}{x^2+x+1}$$

And solve for A, B, C. We get:

$3x^2+2 = A(x^2+x+1) + B(x+1) + Cx(x+1)$

$Put\ x = -1$, get $A = 5$

$Put\ x = 0$, get $2 = 5 + B \Rightarrow B = -3$

$Put\ x = 1$, get $5 = 15 - 6 + C \times 2 \Rightarrow C = -2$

So

$$\frac{3x^2+2}{(x+1)(x^2+x+1)} = \frac{5}{x+1} - \frac{3}{x^2+x+1} - \frac{2x}{x^2+x+1}$$

Need

$$J_1 = \int \frac{1}{x+1}dx = \ln(|x+1|)$$

$$J_2 = \int \frac{1}{x^2+x+1}dx = \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right), see\ last\ lesson$$

$$J_3 = \int \frac{x}{x^2+x+1}dx$$

Force $(x^2+x+1)'$ on top and fix the damage

$$J_3 = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int \frac{dx}{x^2+x+1} = \frac{1}{2}\ln(x^2+x+1) - \frac{1}{2}J_2$$

Put everything together again

$$J = 5\ln|x+1| - \frac{4}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) - \ln(x^2+x+1) + C$$

Example

$$J = \int \frac{x}{(x^2-4x+5)^2}dx$$

First, force derivative of x^2-4x+5 on top and fix

$$J = \frac{1}{2}\int \frac{2x-4}{(x^2-4x+5)^2}dx + 2\int \frac{1}{(x^2-4x+5)^2}dx$$

$$J_1 = \int \frac{2x-4}{(x^2-4x+5)^2}dx$$

$Put\ u = x^2-4x+5$

$$J_1 = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{x^2-4x+5}$$

$$J_2 = \int \frac{dx}{(x^2-4x+5)^2}$$

Complete square of bottom

$x^2-4x+5 = (x-2)^2+1$

Put $u = (x-2)$

$$J_2 = \int \frac{du}{(u^2+1)^2}$$

Next do a trick.

$$J_2 = \int \frac{(u^2+1)}{(u^2+1)^2}du - \int \frac{u^2}{(u^2+1)^2}du$$

$$J_2 = \int \frac{du}{u^2+1} - J_3 = \arctan(x-2) - J_3$$

Now do

$$J_3 = \int \frac{u^2}{(u^2+1)^2}du = \int u \times \frac{u}{(u^2+1)^2}du$$

$$Put\ v = u, dw = \frac{u}{(u^2+1)^2}du$$

$$dv = du, w = \int \frac{u}{(u^2+1)^2}du = \frac{1}{2}\int \frac{2u}{(u^2+1)^2}du = -\frac{1}{2} \times \frac{1}{u^2+1}$$

$$J_3 = -\frac{u}{2(u^2+1)} + \int \frac{1}{2(u^2+1)}du = -\frac{u}{2(u^2+1)} + \frac{1}{2}\arctan(u)$$

$$J_2 = \arctan(x-2) + \frac{u}{2(u^2+1)} - \frac{1}{2}\arctan(u) = \frac{1}{2}\arctan(x-2) + \frac{x-2}{2x^2-8x+10}$$

$$J = -\frac{1}{2(x^2-4x+5)} + \frac{x-2}{x^2-4x+5} + \arctan(x-2)$$

Example

$$J = \int \frac{dx}{(x^2+1)^3}$$

Trick like before

$$J = \int \frac{x^2+1}{(x^2+1)^3}dx - \int \frac{x^2}{(x^2+1)^3}dx$$

$$J_1 = \int \frac{dx}{(x^2+1)^2}, done\ in\ previous\ problem$$

$$J_2 = \int x \times \frac{x}{(x^2+1)^3}dx$$

Use parts.

$$Put\ u = x, dv = \frac{x}{(x^2+1)^3}dx, du = dx$$

$$v = \frac{1}{2}\int \frac{d(x^2+1)}{(x^2+1)^3} = \frac{1}{2}\left(-\frac{1}{2}\times\frac{1}{(x^2+1)^2}\right) = -\frac{1}{4(x^2+1)^2}$$

And now keep going with easier problems.

Properties of Integrals

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Proposition 1
A bounded $f: [a,b] \to \mathbb{R}$ is integrable iff there is a sequence of partitions $\mathcal{P}_n$ of [a, b] and a number S such that $L(f, \mathcal{P}_n) \to S$ and $U(f, \mathcal{P}_n) \to S$ as $n \to \infty$ then $S = \int_a^b f$

Proposition 2
If f, g are integrable or [a, b], then so is $f + g$ and
$$\int_a^b f + g = \int_a^b f + \int_a^b g$$

Linearity (follows from Prop 3, next lesson)
$$\int_a^b (c_1 f_1 + c_2 f_2 + \cdots + c_n f_n)$$
$$= c_1 \int_a^b f_1 + c_2 \int_a^b f_2 + \cdots + c_n \int_a^b f_n$$

Proof of Proposition 1
Suppose such P_n and S exist.
Clearly $U(f, P_n) - L(f, P_n) \to S - S = 0$
So for $\varepsilon > 0$, $U(f, P_n) - L(f, P_n) < \varepsilon$ eventually.
Thus f is integrable.

Also,
$$L(f, P_n) \le \sup_P L(f, P_n) = \int_a^b f = \inf_P U(f, P) \le U(f, P_n)$$
Hence $S \le \int_a^b f \le S$ so $S = \int_a^b f$

Conversely suppose f is integrable over $[a, b]$
For each $\frac{1}{n}$ we get at P_n such that
$$U(f, P_n) - L(f, P_n) < \frac{1}{n}$$
Also $L(f, P_n) \le \int_a^b f \le U(f, P_n)$
Thus
$$0 \le U(f, P_n) - \int_a^b f < \frac{1}{n}$$
$$0 \le \int_a^b f - L(f, P_n) \le \frac{1}{n}$$
So $U(f, P_n) \to \int_a^b f$ and $L(f, P_n) \to \int_a^b f$

Proof of Proposition 2
By proposition 1, we have partitions $P_n \text{ and } Q_n$ such that
$$L(f, P_n) \to \int_a^b f \leftarrow U(f, P_n)$$
$$L(g, Q_n) \to \int_a^b g \leftarrow U(g, Q_n)$$
Let R_n be the common refinement of P_n and Q_n
Then $L(f, P_n) \le L(f, R_n) \le U(f, R_n) \le U(f, P_n)$
Squeeze and get $L(f, R_n) \to \int_a^b f \leftarrow U(f, R_n)$
Likewise, $L(g, R_n) \to \int_a^b g \leftarrow U(f, R_n)$

So $L(f, R_n) + L(g, R_n) \to \int_a^b f + \int_a^b g \leftarrow U(f, R_n) + U(g, R_n)$
What we really wanted was
$$L(f + g, R_n) \to \int_a^b f + \int_a^b g \leftarrow U(f + g, R_n)$$

We need to observe that for any
$$\mathcal{P}: a = x_0 < x_1 < \cdots < x_n = b$$
$L(f, \mathcal{P}) + L(g, \mathcal{P}) \le L(f + g, \mathcal{P}) \le U(f + g, \mathcal{P}) \le U(f, \mathcal{P}) + U(g, \mathcal{P})$

For each $x \in [x_{j-1}, x_j]$ we have $f(x) + g(x) \le \sup f[x_{j-1}, x_j] + \sup g[x_{j-1}, x_j]$
$\Rightarrow \sup(f + g)[x_{j-1}, x_j] \le \sup f[x_{j-1}, x_j] + \sup g[x_{j-1}, x_j]$
Now add up to get
$$\sum_i \sup(f + g)[x_{j-1}, x_j](x_j - x_{j-1})$$
$$\le \sum_i \sup f[x_{j-1}, x_j](x_j - x_{j-1}) + \sum_i \sup g[x_{j-1}, x_j](x_j - x_{j-1})$$
Hence $U(f + g, \mathcal{P}) \le U(f, \mathcal{P}) + U(g, \mathcal{P})$
And similarly, $L(f, \mathcal{P}) + L(g, \mathcal{P}) \le L(f + g, \mathcal{P})$

Back to R_n we get
$L(f, R_n) + L(g, R_n) \le L(f + g, R_n) \le U(f + g, R_n) \le U(f, R_n) + U(g, R_n)$
By squeeze
$$L(f + g, R_n) \to \int_a^b f + \int_a^b g \leftarrow U(f + g, R_n)$$
So by Proposition 1, $f + g$ is integrable and
$$\int_a^b f + g = \int_a^b f + \int_a^b g$$

■

Mult. and Splicing

February-04-11
9:33 AM

Proposition 3

If f is integrable on $[a, b]$ then so is -f and
$\int_a^b -f = -\int_a^b f$

Proposition 4

If f is integrable on [a, b] and c ≥ 0 then cf is integrable and $\int_a^b cf = c\int_b^a f$

Proposition 5

If $c \in \mathbb{R}$ and f is integrable on [a, b] then cf is integrable and $\int_a^b cf = c\int_a^b f$

Proposition 6: Splicing Property

Let $a < c < b$
A function f is integrable on $[a, b] \Leftrightarrow$
f is integrable on [a, c] and on [c, b] then

$$\int_a^b f = \int_a^c f + \int_c^b f$$

We saw that f is integrable on $[a, b] \Leftrightarrow$ there is a sequence of partitions P_n and a number S such that
$L(f, P_n) \to S \leftarrow U(f, P_n)$ and that $S = \int_a^b f$

Proof of Proposition 3

For any bounded set A, let $-A = \{-a: a \in A\}$
We have
$\sup(-A) = -\inf(-A)$
$\inf(-A) = -\sup(-A)$
So for any partition $P: a = x_0 < x_1 < \cdots < x_n = b$ we have

$$L(-f, P) = \sum_j \inf(-f)|x_{j-1}, x_j|(x_j - x_{j-1}) = \sum_i -\sup f|x_{j-1}, x_j|(x_j - x_{j-1}) = -U(f, P)$$

Likewise, $U(-f, P) = -L(f, P)$

Sine f is integrable, have partitions P_n such that

$$L(f, P_n) \to \int_a^b f \leftarrow U(f, P_n)$$

Hence

$$L(-f, P_n) = -U(f, P_n) \to -\int_a^b f \leftarrow -L(f, P_n) = U(-f, P_n)$$

So by proposition 1 applied to $-f$ we get -f is integrable and $\int_a^b -f = -\int_a^b f$

Proposition 4

Have P_n such that $L(f, P_n) \to \int_a^b f \leftarrow U(f, P_n)$
You can check $U(cf, P_n) = cU(f, P_n), L(cf, P_n) = cL(f, P_n)$
Hence $L(cf, P_n) = cL(f, P_n) \to c\int_a^b f \leftarrow cU(f, P_n) = U(cf, P_n)$
So cf is integrable and $\int_a^b cf = c\int_a^b f$

Proof of Proposition 5

If c <0, write $c = -(-c) where\ -c > 0$ and use Prop 3 & 4
Thus $cf = -(-cf)$ is integrable and

$$\int_a^b cf = \int_a^b -(-cf) = -\int_a^b -cf = -(-c)\int_a^b f = c\int_a^b f$$

Proof of Proposition 6

If $P: a = x_0 < x_1 < \cdots < x_n = c, Q: c = y_0 < y_1 < \cdots < y_m = b$
we can splice these to get
$P \vee Q: a = x_0 < x_1 < \cdots < x_n = c = y_0 < y_1 < \cdots y_m = b$
Easy(you do it)
$L(f, P \vee Q) = L(f, P) + L(f, Q)$
$U(f, P \vee Q) = U(f, P) + U(f, Q)$
Say f is integrable on $[a, c]$ and on $[c, b]$ thus have P_n of $[a, c]$ and Q_n of $[c, b]$ such that

$$L(f, P_n) \to \int_a^c f \leftarrow U(f, P_n)$$

$$L(f, Q_n) \to \int_c^b f \leftarrow U(f, Q_n)$$

Thus

$$L(f, P_n \vee Q_n) = L(f, P_n) + L(f, Q_n) \to \int_a^c f + \int_c^b f \leftarrow U(f, P_n) + U(f, Q_n) = U(f, P_n \vee Q_n)$$

By proposition 1, f is integrable on $[a, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$

Conversely, suppose f is integrable on [a, b]. Check f is integrable on [a, c] and on [c, b]
If R is a partition of [a, b] $R: a = x_0 < x_1 < \cdots < x_n = b$ we refine R by inserting c. Get $R \cup \{c\}$
With $P: a = x_0 < x_1 < \cdots < x_{j-1} < c, Q: c < x_{j+1} < \cdots < x_n = b$
Have $R \cup \{c\} = P \vee Q$
For $\varepsilon > 0$ have R such that $U(f, R) - L(f, R) < \varepsilon$
Taking P as shown, we get
$U(f, P) - L(f, P) \le U(f, P) - L(f, P) + U(f, Q) - L(f, Q) = U(f, R \cup \{c\}) - L(f, R \cup \{c\})$
$\le U(f, R) - L(f, R) < \varepsilon$
So f is integrable on $|a, c|$ and on $[c, b]$ and by above, $\int_a^b f = \int_a^c f + \int_c^b f$

Fundamental Theorem of Calculus II

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Fundamental Theorem of Calculus Pt. 2

Let f be continuous on an interval I
Then there is a function g defined on I such that
$g'(x) = f(x)$ for all x in I

More specifically, pick any $a \in I$, x and define the integral for

$$g(x) = \int_a^x f(t)dt$$

For each $x \in I$
Then $g'(x) = f(x)$

Summary

$$f \; cts \Rightarrow \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Integral Function

$$g(x) = \int_a^x f(t)dt$$

Is the integral function of f.

A Useful Convention

$$Declare \int_a^a f = 0$$

- Consistent with splicing
  $$\int_a^a f + \int_a^a f = \int_a^a f$$
- Consistent with FTC 1
  $$\int_a^a f = F(a) - F(a)$$

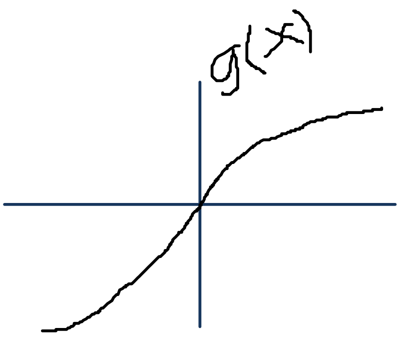
If $b < a$ declare

$$\int_a^b f = -\int_b^a f$$

- Consistent with splicing
  $$\int_a^b f + \int_b^a f = \int_a^a f = 0$$
- Consistent with FTC 1
  $$\int_a^b f = -\int_a^b f = -(F(a) - F(b)) = F(b) - F(a)$$

So we get general splicing

$$\int_a^b f + \int_b^c f + \int_c^d f + \int_d^e f = \int_a^e f$$



Proof of FTC(II)

Know for Midterm

Say $a < x$, we need to show

$$\frac{g(x+h) - g(x)}{h} \to f(x) \; as \; h \to 0$$

Do $h \to 0^+$ first

Examine

$$|g(x+h) - g(x) - f(x)h| = \left|\int_a^{x+h} f(t)dt - \int_a^x f(t)dt - f(x)h\right|$$

$$= \left|\int_x^{x+h} f(t)dt - f(x)h\right| = \left|\int_x^{x+h} (f(t) - f(x))dt\right| \le \int_x^{x+h} |f(t) - f(x)|dt$$

$$\le \int_x^{x+h} \left(\max_{t\in[x,x+h]} |f(t) - f(x)|\right) dt$$

By monotonicity of integrals

$$= \left(\max_{t\in[x,x+h]} |f(t) - f(x)|\right) \int_x^{x+h} 1dt = \max_{t\in[x,x+h]} |f(t) - f(x)|\, h$$

Divide by h and get

$$\left|\frac{g(x+h) - g(x)}{h} - f(x)\right| \le \max_{t\in[x,x+h]} |f(t) - f(x)| = |f(s) - f(x)|$$

For some $s \in [x, x+h]$ by EVT for $|f(t) - f(x)|$ on $[x, x+h]$

As $h \to 0^+, get \; s \to x \; and \; since \; f \; is \; continuous, |f(s) - f(x)| \to 0$

So squeeze and

$$\left|\frac{g(x+h) - g(x)}{h} - f(x)\right| \to 0$$

■

Variations in order of x, x+h, a can be handled with the conventions of sign on integrals.

Examples

$$\frac{d}{dx}\int_0^x \sin(t^2)\, dt = \sin(x^2)$$

$$\frac{d}{dx}\int_0^{\sqrt{x}} e^{t^2} dt = \frac{e^{\sqrt{x}^2}}{2\sqrt{x}} = \frac{e^x}{2\sqrt{x}}$$

Here we had $h(x) = \sqrt{x}, g(u) = \int_0^u e^{t^2} dt$

$$\int_0^{\sqrt{x}} e^{t^2} dt = g(h(x))$$

$$\left(g(h(x))\right)' = g'(h(x))h'(x) = e^{(h(x))^2} h'(x) = \frac{e^x}{2\sqrt{x}}$$

$$\frac{d}{dx}\int_{x^3}^{-5} \frac{\sin t}{t} dt = \frac{d}{dx} - \int_{-5}^{x^3} \frac{\sin t}{t} dt = -\frac{\sin(x^3)}{x^3} 3x^2 = -\frac{3\sin(x^3)}{x}$$

Example

$$\frac{d}{dx}\int_0^{x^3} \frac{1}{1+t^4} dt = \frac{1}{1+(x^3)^4} 3x^2 = \frac{3x^2}{1+x^{12}}$$

Example

Sketch

$$g(x) = \int_0^x e^{-t^2} dt$$

First check g is odd.
Verify $g(x) + g(-x) = 0$
$\left(g(x) + g(-x)\right)' = g'(x) - g'(-x) = e^{-x^2} - e^{-(-x)^2} = 0$
So $g(x) + g(-x) = c = const.$
Plug in $x = 0$ and get $g(0) - g(-0) = 0 - 0 = 0$
Thus $g(x) + g(-x) = 0$, so g is odd.

Now worry about $x \ge 0$.
Have $g'(x) = e^{-x^2} > 0 \Rightarrow g \; inc \; on \; [0, \infty)$
$g''(x) = -2xe^{-x^2} < 0$ g conc. down

One more issue: does $g(x) \to \infty$ as $x \to \infty$ or does g(x) tend to some finite B as $x \to \infty$?
Use a comparison trick:
Know $e^{-t^2} \le 2te^{-t^2} \; when \; t \ge 1$

$$g(x) = \int_0^x e^{-t^2} dt = \int_0^1 e^{-t^2} dt + \int_1^x e^{-t^2} dt \le \int_0^1 e^{-t^2} dt + \int_1^x 2te^{-t^2} dt$$

Now get

$J = \int 2te^{-t^2}dt\, , Put\ u = -t^2 \Rightarrow -du = 2tdt$

$J = -\int e^u du = -e^{-t^2}$

So

$g(x) \leq \int_0^1 e^{-t^2}dt + |-e^{-t^2}|_1^x = \int_0^1 e^{-t^2}dt + \frac{1}{e} - e^{-x^2} \leq \int_0^1 e^{-t^2}dt + \frac{1}{e} = fixed\ B$

Thus $g(x)$ has a horizontal asymptote as $x \Rightarrow\ \pm\infty$

Volume

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The Disk Method

Say $f \geq 0$ on $[a, b]$, f continuous and the region below f is rotated about the x-axis to make a solid. Find the volume of the solid.

$$V = \int_a^b \pi f^2(x)dx$$

The Shells Method

Say $0 \leq a < b$ and $f \geq 0$ and cts on $[a, b]$
Rotate region R about y-axis . Find resulting volume.

$$V = \int_a^b 2\pi x f(x)dx$$

Integral of Odd Functions

If f is continuous and odd, then

$$\int_{-a}^{a} f(t)dt = 0$$

Disk Method

Take partition $\mathcal{P}$ of $[a, b]$ with sample points t_i in each $[x_{j-1}, x_j]$
The stick over $[x_{j-1}, x_j]$ of height $f(t_j)$ rotates about an axis to make a disk of volume
$\pi f(t_j)^2(x_j - x_{j-1})$

The Riemann sum:

$$R(\pi f^2, \mathcal{P}, t_1 \dots t_n) = \sum_j \pi f^2(t_j)(x_j - x_{j-1})$$

This makes the volume when P is very fine, i.e. when all $x_j - x_{j-1} \to 0$
in the limit we get

$$V = \int_a^b \pi f^2(x)dx = \pi \int_a^b f^2(x)dx$$

Example

Rotate the region under $y = \sin x$, over $[0, \pi]$ about the x axis, and find volume of the football.

$$V = \pi \int_0^\pi \sin^2 x\, dx = \frac{1}{2}\pi \int_0^\pi 1 - \cos 2x\, dx = \frac{1}{2}\pi \left(|x|_0^\pi - \left|\frac{1}{2}\sin(2x)\right|_0^\pi \right) = \frac{1}{2}\pi(\pi - 0 - 0 + 0) = \frac{\pi^2}{2}$$

Shells Method

Take sample partition P of [a, b]
The stick of height $f(t_j)$ sitting on $[x_{j-1}, x_j]$ spins about y-axis to generate a shell. $t_j \in [x_{j-1}, x_j]$
Shell has radius t_j and height $f(t_j)$, and thickness $(x_j - x_{j-1})$
$V = 2\pi t_j f(t_j)(x_j - x_{j-1})$
The Riemann sum:

$$\sum_{j=1}^{n} 2\pi t_j f(t_j)(x_j - x_{j-1}) = R(2\pi x f(x), P, t_1 \dots t_n)$$

approximates our volume for small $x_j - x_{j-1}$
As $(x_j - x_{j-1}) \to 0, we\ get$

$$V = \int_a^b 2\pi x f(x)dx$$

Example

Rotate region under $y = \sin x$ over $[0, \pi]$ about y-axis to make a cake.
Find volume:

$$V = 2\pi \int_0^\pi x \sin x\ dx = 2\pi^2$$

Example

The disk of centre (2, 0) and radius 1 rotates about y-axis to make a donut.
Find volume of torus (donut)
$(x-2)^2 + y = 1 \Rightarrow y = \pm\sqrt{1-(x-2)^2}$
$1 \leq x \leq 3$ and $height = 2\sqrt{1-(x-2)^2}$
The stick at x of height $2\sqrt{1-(x-2)^2}$ and thickness dx revolves about y-axis to make shell of volume
$dV = 2\pi x \left(2\sqrt{1-(x-2)^2}\right)dx$

$$V = \int_1^3 4\pi x\sqrt{1-(x-2)^2}dx$$

$u = x - 2 \Rightarrow du = dx$

$$V = 4\pi \int_{-1}^{1} (u+2)\sqrt{1-u^2}\ du = 4\pi \int_{-1}^{1} u\sqrt{1-u^2}du + 8\pi \int_{-1}^{1} \sqrt{1-u^2}du$$

By looking at a circle $y = \pm\sqrt{1-u^2}$ we get

$$\int_{-1}^{1} \sqrt{1-u^2}du = \frac{\pi}{2}$$

And since $u\sqrt{1-u^2}$ is odd,

$$\int_{-1}^{1} u\sqrt{1-u^2}\ du = 0$$

So

$$V = 8\frac{\pi\pi}{2} = 4\pi^2$$

Proof of Integrals of Odd Function

Let's first check that the integral for

$$g(x) = \int_0^x f(t)dt$$

is even. Want $g(-x) = g(x)$
Calculate derivatives
$g'(x) = f(x)$
$(g(-x)) = g'(-x)(-1) = -f(-x) = f(x)$
So $g(-x) = g(x) + c$
Put $g(0) = g(0) + c \Rightarrow c = 0$

So $g(x) = g(-x)$

$$\int_{-a}^{a} f(t)dt = -\int_{0}^{-a} f(t)dt + \int_{0}^{a} f(t)dt = -g(-a) + g(a) = g(a) - g(a) = 0$$

Series

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Sequence

A sequence is a list of numbers $x_1, x_2, \dots, x_n, \dots$

Know $x_n \to p\ as\ n \to \infty$ means $\forall \varepsilon > 0, |x_n - p| < \varepsilon$ eventually.

Fact: If x_n is monotone and bounded, then
$x_n \to some\ p$
i.e.: $x_1 \le x_2 \le x_3 \le \dots \le some\ B$
or $x_1 \ge x_2 \ge x_3 \ge \dots \ge some\ B$
then $x_n \to some\ p$

Series

A series is made up of 2 sequences.
Sequence of terms:

$x_1, x_2, x_3, \dots, x_n, \dots$

Sequence of sums (called partial sums)

$s_1 = x_1$
$s_n = s_{n-1} + x_n$

When the $s_n \to some\ s$ we say that our series converges to s.

Notation

$x_1 + x_2 + x_3 + \cdots + x_n + \cdots$

$$\sum_{k=1}^{\infty} x_k \ or \ \sum x_k$$

A series that converges is sometimes called summable.

Proposition 1

If $\sum_{k=1}^{\infty} x_k$ converges to s, then $x_n \to 0$

Caution:

If $x_n \to 0$, series $\sum x_k$ could still diverge

Geometric Series

Pick any $x \in \mathbb{R}$
and consider the geometric series:

$$\sum_{k=0}^{\infty} x_k = 1 + x + x^2 + x^3 + \cdots$$

This series converges $\Leftrightarrow |x| < 1$
in that case it converges to $\frac{1}{1-x}$

Proof of Proposition 1

Let $s_n = x_1 + x_2 + \cdots + x_n$
Note $x_{n+1} = s_{n+1} - s_n \to s - s = 0$

$e.g.$

$$\sum_{k=1}^{\infty} (-1)^{-k} = -1 + 1 - 1 + 1 - 1 + \cdots$$

Here $(-1)^k \nrightarrow 0$ and series diverges

$e.g.$

$$\sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{k}}}$$

Check $\left(\frac{1}{n^{\frac{1}{n}}}\right) \to 0?$

Have

$$\ln\left(\frac{1}{n^{\frac{1}{n}}}\right) = \ln 1 - \frac{\ln n}{n} \to 0$$

$$\left(\frac{1}{n^{\frac{1}{n}}}\right) \to e^0 = 1$$

Example

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$$

$$\left(\frac{1}{3} + \frac{1}{4}\right) \ge \frac{1}{2}$$

$$\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \ge \frac{1}{2}$$

etc.
We see that with n big enough, se can make $s_n \ge$ any multiple of $1/2$. Thus s is not bounded.

Proof of Geometric Convergence

If $|x| \ge 1$, we see that $|x^n| \ge 1$
so $x^n \nrightarrow 0$ so $\sum_{k=0}^{\infty} x^k$ diverges

If $|x| < 1$ we know

$$1 + x + \cdots + x^n = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

$\left|\frac{x^{n+1}}{1-x}\right| \to 0$ when $|x| < 1$
So $1 + x + \cdots + x^n \to \frac{1}{1-x}$ as $n \to \infty$

Properties of Series

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Basic Facts

Addition

$$\sum_{k=1}^{\infty} x_k \to s, \sum_{k=1}^{\infty} y_k \to u$$
$\Rightarrow$
$$\sum_{k=1}^{\infty} (x_k + y_k) \to x + y$$

Multiplication

$$\sum_{k=1}^{\infty} x_k \to s, \qquad c \in \mathbb{R}$$
$\Rightarrow$
$$\sum_{k=1}^{\infty} cx_k \to cs$$

Modifications

Any changes or deletions of a finite number of terms in $\sum x_k$ has no effect on convergence (although it may change the value converged to)

Monotonicity

If $x_n \geq 0$, the partial sums s_n are increasing.
and s_n converges iff s_n is bounded.

Integral Test

Let $f: |1, \infty) \to \mathbb{R}$ be such that:

- f is continuous
- f decreases
- $f \geq 0$

Put $x_k = f(k)$ where $k = 1, 2, 3, \dots$
Then
$$\sum_{k=1}^{\infty} x_k \ \ cges$$
$$\Leftrightarrow the\ sequence\ of\ integrals\ \int_1^n f(t)dt\ \ cges.$$
$$\Leftrightarrow \int_1^{\infty} f(t)dt\ \ exists$$

Example

If $x_1 + x_2 + \cdots + x_n + \cdots + \cdots \to s$
and if we replace x_1 by 7 and drop x_2 then
$7 + x_3 + x_4 + \cdots + x_n + \cdots \to x + 7 - x_1 - x_2$

Example

Look at
$$\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \cdots$$
Let's verify $s_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$
Make a comparison of terms
$$\frac{1}{2!} \leq \frac{1}{2}$$
$$\frac{1}{3!} \leq \frac{1}{2^2}$$
$$\frac{1}{4!} \leq \frac{1}{2^3}$$
$$\frac{1}{n!} \leq \frac{1}{2^{n-1}}$$

$$s_n \leq 1 + \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \leq 1 + \frac{1}{1 - \frac{1}{2}} = 3$$

So s_n converges to some e ≤ 3
Also notice $s_n > 2$ for $n \geq 3$ so $2 < e \leq 3$

Proof of Integral Test

Both $s_n = x_1 + x_2 + \cdots + x_n$ and $\int_1^n f$ are increasing sequences.
So check s_n and $\int_1^n f$ are bounded, or not, together.

Since f decreases,
$x_2 \leq f(x) \leq x_1\ on\ |1, 2|$
$x_{k+1} < f(x) \leq x_k$ on $|k, k+1|$
$x_n \leq f(x) \leq x_{n-1}$ on $|n-1, n|$

Integrate over $|k, k+1|$
$$x_{k+1} = \int_k^{k+1} x_{k+1}dt \leq \int_k^{k+1} f(t)dt \leq \int_k^{k+1} x_k dt = x_k$$
So
$$x_2 + x_3 + \cdots + x_n \leq \sum_{k=1}^{n-1} \int_k^{k+1} f(t)dt \leq x_1 + x_2 + \cdots + x_{n-1}$$
Splice
$$s_n - x_1 \leq \int_1^n f(t)dt \leq s_{n-1}$$

Say
$$\sum x_k \ \ cges$$
Then all $s_n \leq some\ bound\ B$.
Then $\int_1^n f \leq B$ for all n
Since $\int_1^n f$ increases with n, we get $\int_1^n f \to some\ limit\ L$

Say $\int_1^n f$ converges. Then all $\int_1^n f \leq some\ B$
Then $s_n - x_1 \leq this\ B$ so $So\ s_n \leq x_1 + B$ and since s_n increases, we get s_n converges.

Example. P-Series

For $p > 0$, e.g. $p = \frac{1}{2}, 1, 1.1, 2, \pi, \dots$
The function
$$f(x) = \frac{1}{x^p}$$
is continuous, decreasing, and ≥ 0
We know
$$\int_1^n \frac{1}{x^p} dx$$
converges iff p>1
Then

$\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges iff p > 1

Example

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

This is ≥ 0, cts, and decreasing

Look at

$$f(x) = \frac{1}{x \ln x}$$

$$\int_2^n \frac{dx}{x \ln x} = |\ln \ln x|_2^n = \ln(\ln n) - \ln(\ln 2) \to \infty$$

So

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Exercise

$$\sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)^2}$$

Show this converges

Estimation of Sum

February-18-11
9:29 AM

Integral Estimation

If

$$\sum_{k=1}^{\infty} x_k = s$$

$x_k > 0$, decreasing

$$\int_{n+1}^{\infty} f \le s - s_n \le \int_{n}^{\infty} f$$

Where $f(k) = x_k$

Estimation of Sum

Likely to be on final

Say we know

$$\sum_{k=1}^{\infty} x_k$$

converges, but the sum s is a mystery.

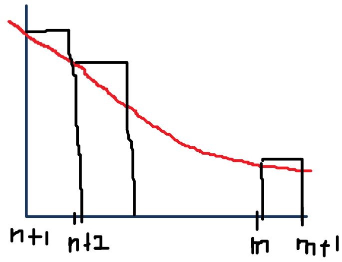
Know $s_n = x_1 + \cdots + x_n \approx s$ for large n
How close?

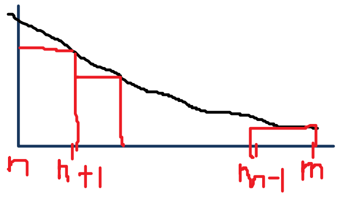
Given an $\varepsilon > 0$
Find n such that $s_n \approx s$ with error $< \varepsilon$
$|s - s_n| < \varepsilon$

If

$$s = \int_{k=1}^{\infty} x_k$$

was obtained by the integral test, here's how to answer our problem.





For $m > n \ge 1$ we have

$$\int_{n+1}^{m+1} f \le x_{n+1} + x_{n+2} + \cdots + x_m \le \int_{n}^{m} f$$

So

$$\lim_{m\to\infty} \int_{n+1}^{m+1} f \le \lim_{m\to\infty} x_{n+1} + \cdots + x_m \le \lim_{m\to\infty} \int_{n}^{m} f$$

$$\int_{n+1}^{\infty} f \le s - s_n \le \int_{n}^{\infty} f$$

Example

Let $s = \sum_{k=1}^{\infty} \frac{1}{k^3}$

If $s - s_n < \frac{1}{100}$, *then*

$$\int_{n+1}^{\infty} \frac{1}{t_3} dt < \frac{1}{100}$$

We see that

$$\int_{n+1}^{m} \frac{dt}{t^3} = \left| -\frac{1}{2t^2} \right|_{n+1}^{m} = -\frac{1}{2m^2} + \frac{1}{2(n+1)^2} \to \frac{1}{2(n+1)^2} \, as\ n \to \infty$$

So

$$\frac{1}{2(n+1)^2} < \frac{1}{100} \Rightarrow (n+1) > \sqrt{50} \Rightarrow n > \sqrt{50} - 1 \approx 6.07$$

So $n \ge 7$

Convergence Tests

March-02-11
12:12 AM

Proposition

If $0 \leq x_k \leq y_k$ and

$$\sum_{k=1}^{\infty} y_k$$

converges then

$$\sum_{k=1}^{\infty} x_k$$

converges.

Note:
When using comparison test, only care about end behaviour, not initial values.

Limit Comparison

If $0 \leq x_k \;\&\; 0 < y_k \;\&\, \frac{x_k}{y_k} \to some\ L\ where\ L \in (0, \infty)$ then $\sum x_k$ & $\sum y_k$ converge or diverge together.

Condensation Test

Let $x_1 \geq x_2 \geq \ \ldots \geq 0$
Then $x_1 + x_2 + \cdots + x_n + \cdots$ converges iff $x_1 + 2x_2 + 4x_4 + \cdots + 2^k x_{2^k}$ converges.

Proof of Proposition

$Let\ s_n = x_1 + x_2 + \cdots + x_n$ and $t_n = y_1 + \cdots + y_n$
Clearly s_n is increasing. Just check s_n bounded.
Know t_n ≤ some bound B. $s_n \leq t_n$ is obvious so $s_n \leq B$ so s_n converges.

Example

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+5}{n^2-3} \text{ converge?}$$

$$\frac{\sqrt{n}+5}{n^2-3} \leq \frac{2\sqrt{n}}{n^2-3} = \frac{2\sqrt{n}}{\frac{1}{2}n^2 + \frac{1}{2}n^2 - 3} \leq \frac{2\sqrt{n}}{\frac{1}{2}n^2} = \frac{4}{n^{\frac{3}{2}}}, \text{eventually, when } \frac{1}{2}n^2 - 3 > 0$$

Since

$$\sum^{\infty} \frac{4}{n^{3\backslash 4}}$$

converges, (p-series with $p = \frac{3}{2} > 1$), the original converges.

Example

$$\sum_{i=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} \text{ converge?}$$

Notice $n^{\frac{1}{n}} \to 1$

$$\frac{1}{n} ln(n) \to 0 \Rightarrow e^{\frac{1}{n} ln(n)} = n^{\frac{1}{n}} \to 1$$

So $n^{\frac{1}{n}} \leq \frac{3}{2}$ eventually, thus $n^{1+\frac{3}{2}n} \leq \frac{3}{2}n$ eventually

$$\frac{1}{n^{1+\frac{1}{n}}} \geq \frac{2}{3n} \text{ eventually}$$

But $\frac{2}{3n}$ diverges so

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} \text{ diverges}$$

Proof of Limit Comparison

Say $\sum y_k$ converges
Since $\frac{x_k}{y_k} \to L$ we get $\frac{x_k}{y_k} \leq L + 1$ eventually
Thus $0 < x_k < (L+1)y_k$ eventually
But

$$\sum_{k=1}^{\infty} (L+1)y_k$$

converges. By comparison, x_k converges too.

Conversely, say $\sum x_k$ converges. In this case use fact that $\frac{y_k}{x_k} \to \frac{1}{L}$ and $L \in (0, \infty)$ so y_n converges.

Example

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+n+1}}{n^2-5n+8}$$

We see that $\sqrt{n^3+n+1}$ is "like" $n^{\frac{3}{2}}$ and $n^2 - 5n + 8$ is "like" n^2, thus

$$\frac{\sqrt{n^3+n+1}}{n^2-5n+8} \text{ is "like"} \frac{n^{\frac{3}{2}}}{n^2} = \frac{1}{\sqrt{n}}$$

Try limit comparison with $\sum \frac{1}{\sqrt{n}}$

$$\frac{\frac{\sqrt{n^3+n+1}}{n^2-5n+8}}{\frac{1}{\sqrt{n}}} = \frac{\sqrt{n^4+n^2+n}}{n^2-5n+8} = \frac{\sqrt{1+\frac{1}{n^2}+\frac{1}{n^3}}}{1-\frac{5}{n}+\frac{8}{n^2}} \to 1$$

Since $\sum \frac{1}{\sqrt{n}}$ diverges, so does

$$\sum \frac{\sqrt{n^3+n+1}}{n^2-5n+8}$$

Example

Take

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

where p > 0
It's condensation is

$$\sum_{k=0}^{\infty} \frac{2^k}{(2^k)^p} = \sum_{k=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^k$$

The geometric series

$$\sum_{k=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^k$$

converges $\Leftrightarrow \frac{1}{p-1} < 1 \Leftrightarrow p - 1 > 0 \Leftrightarrow p > 1$

Proof of Condensation Test

Let $s_n = x_1 + \cdots + x_n$

$t_n = x_1 + 2x_2 + \cdots + 2^k x_{2^k}$

Since the x_n and t_k increase, it's enough to prove that s_n is bounded $\Leftrightarrow$ t_k is bounded

Say all $t_n \leq$ some bound B. For any n, take k so big that $n \leq 2^k$

Then $s_n \leq x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + \cdots + (x_{2^k} + \cdots + x_{2^{k+1}-1})$

$\leq x_1 + 2x_2 + 4x_4 + \cdots + 2^k x_{2^k} = t_k \leq B$

So t_k bounded $\Rightarrow$ s_n bounded

Next say all $s_n \leq some\ B$. For any k we get

$t_k = x_1 + 2x_2 + 4x_4 + 8x_8 + \cdots + 2^k x_{2^k} = 2\left(\frac{1}{2}x_1 + x_2 + 2x_4 + 4x_8 + \cdots + 2^{k-1}x_k\right)$

$\leq 2(x_1 + x_2 + (x_3 + x_4) + (x_5 + x_6 + x_7 + x_8) + \cdots + (x_{2^{k-1}+1} + \cdots + x_{2^k}) = 2s_{2^k} \leq 2B$

So s_n bounded $\Rightarrow$ t_k bounded.

∎

Example

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}, \text{where p} > 0 \text{ and fixed}$$

Condensation is:

$$\sum_{k=1}^{\infty} 2^k \times \frac{1}{2^k(\ln 2^k)^p} = \sum_{k=1}^{\infty} \left(\frac{1}{\ln 2}\right)^p \left(\frac{1}{k^p}\right)$$

Since

$$\sum_{k=1}^{\infty} \left(\frac{1}{\ln 2}\right)^p \left(\frac{1}{k^p}\right) \ converges \Leftrightarrow p > 1$$

$$\sum \frac{1}{n(\ln n)^p} \ converges \Leftrightarrow p > 1$$

Convergence of Primes

March-02-11
9:55 AM

Convergence of Primes

Let $2, 3, 5, 7, 11, p_n$ be sequences of primes in increasing order.
Does
$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{p_n}+\cdots$ converge? No

Say $\sum\frac{1}{k}$ converges to s. So there is an index n such that
$s-s_n=\frac{1}{p_{n+1}}+\frac{1}{p_{n+2}}+\cdots+\frac{1}{p_k}+\cdots\leq\frac{1}{2}$
For any positive integer a, let
$J(n,a)=\#\ of\ integers\ from\ 1\ to\ a\ that\ can\ be\ factored\ using\ only\ p_1,\ldots,p_n$

E.g.
$L(3,23)=\#\ integers\ from\ 1\ to\ 23\ that\ can\ be\ factored\ using\ 2,3,5$
$L(3,23)=\#\{1,2,3,4,5,6,8,9,10,12,15,16,18,20\}=14$

If m is an integer from 1 to a that factors using only $p_1,\ldots,p_n$, write
$m=(p_1^{c_1}p_2^{c_2}\ldots p_n^{c_n})(p_1^{d_1}\ldots p_n^{d_n})^2, where\ c_j\in\{0,1\}\&\ d_j\geq 0$
$p_1^{c_1}p_2^{c_2}\ldots p_n^{c_n}\ has\ at\ most\ 2^n\ options$
$p_1^{d_1}\ldots p_n^{d_n}\ has\ at\ most\ \sqrt{a}$ options
So $J(n,a)\leq 2^n\sqrt{a}$
Now get an upper bound for $a-J(n,a)$. If $p_k>p_n$, the number of integers from 1 to a that have p_k as a factor is $\leq\frac{a}{p_k}$

Thus
$a-J(n,a)\leq\frac{a}{p_{n+1}}+\frac{a}{p_{n+1}}+\cdots+\frac{a}{p_k}+\cdots=a\sum_{i=n+1}^{\infty}\frac{1}{p_i}\leq\frac{a}{2}$
So $a-J(n,a)\leq\frac{a}{2}$
$\frac{a}{2}\leq J(n,a)\leq 2^n\sqrt{a}\Rightarrow\sqrt{a}\leq 2^{n+1}\Rightarrow a\leq 4^{n+1}\ \forall a\in\mathbb{N}$
Clearly this is a contradiction.

Alternating Series

March-04-11
9:32 AM

Proposition

If $x_1 \geq x_2 \geq x_3 \geq \ \dots \geq x_n \geq \ \dots all \geq 0\ and\ x_n \to 0$,
then the alternating series $x_1 - x_2 + x_3 - x_4 + \cdots +$
$(-x)^{n+1} x_n + \cdots$
converges.

Estimation of Limit

May be on exam

The error that s_n makes in estimating s is less than or equal to the next missing term.

$|s - s_n| \leq x_{n+1}$

Absolute Summability (Absolute Convergence)

A series

$$\sum_{k=1}^{\infty} x_k$$

converges absolutely, or is absolutely summable when

$$\sum_{k=1}^{\infty} |x_k|$$

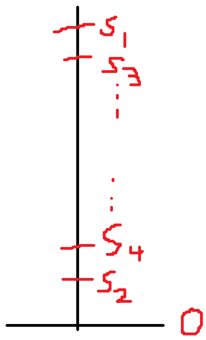
Proposition

If $\sum_{k=1}^{\infty} |x_k|$ converges, then $\sum_{k=1}^{\infty} x_k$ converges too

However, the converse fails.

Proof of Proposition

The decreasing assumption guarantees that the partial sums line up as shown:



$s_{2n} = s_{2n-1} - x_{2n}$
$s_{2n+1} = s_{2n} + x_{2n+1}$
Since s_{2n} are bounded by s_1 (all s_{2n+1}) and increasing then $s_{2n} \to some\ s\ as\ n\ \to \infty$
But $s_{2n+1} = s_{2n} + x_{2n+1} \to s + 0 = s$
Hence $s_n \to s$ ■

Also notice s is between all s_n and s_{n+1} because s_{2n} increase to s and s_{2n+1} decrease to s
Hence
$|s - s_n| \leq |s_{n+1} - s_n| = x_{n+1}$

Example

Does $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$ converge?
Clearly alternating.
Does $\frac{\ln n}{n} \to 0$? Yes
Does $\frac{\ln(n+1)}{n+1} \leq \frac{\ln n}{n}$?
Check:
Look at $\left(\frac{\ln x}{x}\right)'$ for all real x ≥ 1

$$\left(\frac{\ln x}{x}\right)' = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0\ for\ x > e$$

So eventually, $\frac{\ln x}{x}$ decreases. Hence $\frac{\ln n}{n}$ decreases eventually

So AST applies to

$$\sum (-1)^n \frac{\ln n}{n} \to s$$

Also

$$|s - s_{10}| \leq \frac{\ln 11}{11} \approx 0.22$$

Caution

For AST be sure x_n decreases.

$$1 - \frac{1}{2^1} + \frac{1}{2} - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{2^3} + \frac{1}{4} - \frac{1}{2^4} + \frac{1}{5} - \frac{1}{2^5} + \cdots$$

Clearly $1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{8}, \frac{1}{4}, \frac{1}{16}, \frac{1}{5}, \frac{1}{32}, \dots \to 0$, but is not decreasing.
Now

$$s_{2n} = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right) - \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\right)$$

Then

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) = s_{2n} + \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}\right)$$

If $s_{2n} \to s$ as $n \to \infty$, then right side would converge to $s + 1$, but left side diverges, so s_{2n} does not converge.

Absolute Summability Example

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots$$

converges by AST to s.
Rearrange the order of summation to get

$$s = \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{16} + \left(\frac{1}{9} - \frac{1}{18}\right) - \frac{1}{20} + \cdots$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) = \frac{1}{2}s \Rightarrow s = 0$$

By error estimate in AST we know
$|s - s_1| = |s - 1| \leq \frac{1}{2}$ So $s \geq \frac{1}{2}$
Contradiction.

Rearranging infinite terms in a series may lead to a different sum, or changing the existence of a limit.

Proof of Proposition

Let $s_n = x_1 + \cdots + x_n,\ t_n = |x_1| + \cdots + |x_n|$
Check that s_n is Cauchy.
Well, for $m > n \geq 1$
$|s_m - s_n| = |x_m + x_{m+1} + \cdots + x_{n+1}| \leq |x_m| + |x_{m+1}| + \cdots + |x_{n+1}| = |t_m - t_n| \to 0\ as\ n, m\ \to \infty$
So s_n converges

Ratio Test

March-07-11
9:32 AM

Ratio Test for Absolute Convergence

$Let\ x_n \neq 0\ and\ \left|\frac{x_{n+1}}{x_n}\right| \to L\ as\ n\ \to \infty$

If $L < 1$, then $\sum |x_n|$ converges
$L > 1$, then $x_n \nrightarrow 0\ and\ \sum x_n$ diverge.

Proof of Ratio Test

Say $L < 1$
Pick an r such that $L < r < 1$
We know $\left|\frac{x_{n+1}}{x_n}\right| < r$ when $n \geq some\ N$
Thus we get
$|x_N| \leq 1|x_N|$
$|x_{N+1}| \leq r|x_N|$
$|x_{N+2}| \leq r^2|x_N|$
...
$|x_{N+k}| \leq r^k|x_N|$
The geometric series

$$\sum_{k=0}^{\infty} |x_N| r^k \text{ converges since } |r| < 1$$

By comparison,

$$\sum_{k=1}^{\infty} |x_{N+k}| \text{ converges}$$

throw back in $|x_1|, |x_2|, \ldots, |x_{N-1}|$ and get

$$\sum_{n=0}^{\infty} |x_n| \text{ converges}$$

Say $L > 1$
Thus eventually

$$\left|\frac{x_{n+1}}{x_n}\right| > 1$$

So eventually we get
$|x_N| < |x_{N+1}| < |x_{N+2}| < \cdots$
So $x_n \nrightarrow 0$■

Example

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges, } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}$$

$$\left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \left|\frac{n}{n+1}\right| \to 1\ as\ n \to \infty$$

$$\left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| = \left|\frac{n^2}{(n+1)^2}\right| \to 1\ as\ n\ \to \infty$$

So L = 1 is useless

Example

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}(-1)^n \text{ converge absolutely?}$$

See if ratio test helps

$$\left|\frac{\left(\frac{(n+1)!\,(-1)^{n+1}}{(n+1)^{n+1}}\right)}{\frac{n!\,(-1)^n}{n^n}}\right| = \frac{(n+1)n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} < 1\ as\ n \to \infty$$

Yes

Limsup & Root Test

March-07-11
9:59 AM

Limit Superior

Let x_n be a bounded sequence.
Say $c \leq x_n \leq b$ for all x.

Put $t_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}$
Clearly
$b \geq t_1 \geq t_2 \geq \cdots \geq t_n \geq t_{n+1} \geq \cdots \geq c$

Thus $t_n \to some\ limit\ p$ and $t_n \geq p$
Write $p = limsup\ x_n =$ limit superior of our sequence x_n

Convention

If x_n is not bounded above, put $\limsup x_n = \infty$

Proposition

If x is bounded and $p = limsup\ x_n$ then for any $\varepsilon > 0$ we

- $x_n < p + \varepsilon$ eventually
- $p - \varepsilon < x_n$ infinitely often.

and $p = limsup\ x_n$ is the only number that does this trick.

Ordinary limits satisfy these properties, so if a sequence has a limit, then the limit is the limit superior.

Proposition

If $p = \limsup x_n$ then there is a subsequence x_{n_k} that converges to p. Also, if x_{n_k} is any subsequence with a limit q, then $q \leq p$

Root Test

Have a series

$$\sum_{k=1}^{\infty} x_k$$

and let $p = \limsup \sqrt[n]{|x_n|} \geq 0$

If $p < 1$, then $\sum |x_k|$ converges
$p > 1$, then $x_n \nrightarrow 0$ and $\sum x_k$ diverges

Example

$\frac{2}{1}, 0, \frac{3}{2}, 0, \frac{4}{3}, 0, \frac{5}{4}, 0, \frac{6}{5}, 0, \frac{7}{6}, \dots$

$p_n = 2, \frac{3}{2}, \frac{3}{2}, \frac{4}{3}, \frac{4}{3}, \frac{5}{4}, \frac{5}{4}, \dots$

$p = 1$

Proof of Proposition

Say $p = \limsup x_n\ and\ take\ \varepsilon > 0$
Know $t_n = \sup\{x_n, x_{n+1}, \dots\} \to$ p decreasing
So $t_N < p + \varepsilon$ for some N.
Clearly for all $n \geq N$ we also get $x_n < p + \varepsilon$

Also all $t_n \geq p$
so $\sup\{x_1, x_2, \dots\} \geq p \Rightarrow some\ x_{n_1} > p - \varepsilon$
$\sup\{x_{n_1+1}, x_{n_1+2}, \dots\} \geq p \Rightarrow some\ x_{n_2} > p - \varepsilon, where\ n_2 > n_1$
$\sup\{x_{n_2+1}, x_{n_2+2}, \dots, \} \geq p \Rightarrow some\ x_{n_3} > p - \varepsilon$
In this way, we come up with infinitely many $x_{n_k} > p - \varepsilon$

Next, suppose q also has the above traits. Want $q = p$
Say $p < q$ and get a contradiction.
Pick r such that $p < r < q$
Then we get $x_n < r$ eventually and $r < x_n$ infinitely often.
Impossible, so $q = p$

Proposition

Know $p - 1 < x_n < p + 1$ infinitely often, so pick one such x_{n_1}
Next, $p - \frac{1}{2} < x_n < p + \frac{1}{2}$ so pick one such $x_{n_2} > x_{n_1}$
Etc. Thus we pick up a subsequence x_{n_k} such that $p - \frac{1}{k} < x_{n_k} < p + \frac{1}{k}$
Let $k \to \infty$ and squeeze to get $x_{n_k} \to p$

Next say $x_{n_k} \to some\ q$. Want $q \leq p$
What if $p < q$? Pick r such that $p < r < q$
Thus $r < x_{n_k}$ eventually with k, since $x_{n_k} \to q$. But $x_n < r$ eventually by first property of p. This is a contradiction so $q \leq p$.

Proof of Root Test

$If\ p < 1$
Pick $p < r < 1$, thus $\sqrt[n]{|x_n|} < r$ eventually. So $|x_n| < r^n$ eventually.
But $\sum r^n$ converges, so by comparison, $\sum |x_n|$ converges

If $1 < p, then\ \sqrt[n]{|x_n|} > 1$ eventually.
Then $|x_n| > 1$ infinitely often, so $x_n \nrightarrow 0$

Example

$$x_n = \begin{cases} \frac{1}{2^n}\ n\ odd \\ \frac{1}{3^n}\ n\ even \end{cases}$$

Does $\sum x_n$ converge?

Try ratio test:

$$\left|\frac{x_{n+1}}{x_n}\right| = \begin{cases} \frac{\left(\frac{1}{3^{n+1}}\right)}{\frac{1}{2^n}} = \left(\frac{2}{3}\right)^n \left(\frac{1}{3}\right)\ n\ odd \\ \frac{\frac{1}{2^{n+1}}}{\frac{1}{3^n}} = \left(\frac{3}{2}\right)^n \left(\frac{1}{2}\right)\ n\ even \end{cases}$$

$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|$ not there
How about root test?

$$\sqrt[n]{|x_n|} = \begin{cases} \frac{1}{2}\ n\ odd \\ \frac{1}{3}\ n\ even \end{cases}$$

$limsup\ x_n = \frac{1}{2} < 1$

$\sum |x_n|$ converges

Permutations

March-09-11
10:05 AM

Proposition

Permutation on Absolutely Summable

If

$$\sum_{k=1}^{\infty} |x_k| \text{ and } s = \sum_{k=1}^{\infty} x_k$$

and σ is any permutation of $\{1, 2, 3, 4, \ldots\}$
then

$$\sum_{k=1}^{\infty} x_{\sigma(k)} = s$$

Power Series

Pick any $a_0, a_1, a_2, \ldots a_n, \ldots$ coefficients and $x \in \mathbb{R}$

$$The\ series\ a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Is a power series in x.

"Then I do the upside down
flippy thingy... the algebra."

Example Permutations

1 2 3 4 5 6 7 8 9 10 11 ...
2 1 4 3 6 5 8 7 10 9 ...

1	2	3	4	5	6	7	8	9	10	11	12	13	...
1	2	4	3	6	8	5	10	12	7	14	16	9	...

Proof of Proposition

Take any $\varepsilon > 0$.
Want M such that

$$\left|\sum_{k=1}^{m} x_{\sigma(k)} - s\right| < \varepsilon, \text{when } m \geq M$$

First pick N such that

$$\sum_{k=N+1}^{\infty} |x_k| < \varepsilon$$

Next take M such that $x_{\sigma(1)}, \ldots, x_{\sigma(M)}$ includes all $x_1, \ldots, x_N$
Now when $m \geq M$ we get

$$\left|\sum_{k=1}^{m} x_{\sigma(k)} - \sum_{k=1}^{N} x_k\right| = |\text{ a sum of finitely many } x_j \text{ that excludes } x_1, \ldots x_N|$$

$$\leq \left|sum\ of\ finitely\ many\ |x_j|\ that\ excludes\ |x_1|, \ldots |x_N|\right| \text{ (by Triangle Inequality)}$$

$$\leq \sum_{k=N+1}^{\infty} |x_k| < \varepsilon$$

$$\left|\sum_{k=1}^{m} x_{\sigma(k)} - s\right| \leq \left|\sum_{k=1}^{m} x_{\sigma(k)} - \sum_{k=1}^{N} x_k\right| + \left|\sum_{k=1}^{N} x_k - s\right| < \varepsilon + \left|\sum_{k=N+1}^{\infty} x_k\right| \leq \varepsilon + \sum_{k=N+1}^{\infty} |x_k| \leq \varepsilon + \varepsilon = 2\varepsilon$$

Power Series

For which x does $\sum a_k x^k$ converge?
Always for $x = 0$.

$$\sum_{k=0}^{\infty} x^k\ converges \Leftrightarrow |x| < 1$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k\ converges\ for\ all\ x$$

Proof: Ratio test gives

$$\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \frac{1}{n+1}|x| \to 0 < 1$$

$$\sum_{k=0}^{\infty} k!\, x^k\ converges\ only\ if\ x = 0$$

$Ratio$:

$$\left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = (n+1)|x| \to \infty > 1$$

Power Series

March-14-11
9:32 AM

Power Series

$$\sum_{n=0}^{\infty} a_n x^n$$

Proposition

Every power series does one of three things:

- Converge for just $x = 0$
- Converge absolutely for all $x \in \mathbb{R}$
- For some $0 < R < \infty$, converges absolutely when $|x| < R$ and when $|x| > R$, $a_n x^n \nrightarrow 0$ and
$$\sum_{k=0}^{\infty} a_k x^k \text{ diverges}$$

Radius of Convergence

R is known as the radius of convergence for the power series. If converges for no x, $R = 0$ If converges for all x, $R = \infty$

Case 1: $R = 0$
Case 2: $R = \infty$
Case 3: $R = \dfrac{1}{limsup \sqrt[n]{|a_n|}}$

Interval of Convergence

$(-R, R), |-R, R|, (-R, R|, [-R, R)$
0
$\mathbb{R}$

Power Series Functions

Since $\sum a_k x^k$ depends on x, we can make a function on the interval of convergence defined by
$f(x) = \sum a_k x^k$

Proof of Proposition

Look at the sequence $\sqrt[n]{|a_n|}$
If $\sqrt[n]{|a_n|}$ is not bounded then for $|x| \neq 0$, $\sqrt[n]{|a_n x^n|} = \sqrt[n]{|a_n|}|x|$ is not bounded either. By the root test, $a_n x^n \nrightarrow 0$ and $\sum a_k x^k$ diverges. Case 1.

If $\sqrt[n]{|a_n|}$ is bounded then for any x we get $limsup \sqrt[n]{|a_n x^n|} = |x| limsup \sqrt[n]{|a_n|}$
If $limsup \sqrt[n]{|a_n|} = 0$, then so is $limsup \sqrt[n]{|a_n x^n|} = 0 < 1$ so the root test says $\sum a_n x^n$ converges absolutely for all $x \in \mathbb{R}$

If $limsup \sqrt[n]{|a_n|} > 0$, the root test tells us that $\sum a_k x^k$ converges absolutely when $|x| < \frac{1}{limsup \sqrt[n]{|a_n|}}$ and diverges when $|x| > \frac{1}{limsup \sqrt[n]{|a_n|}}$. Case 3

$$R = \frac{1}{limsup \sqrt[n]{|a_n|}}$$

Example

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n} x^n \text{ radius?}$$

$$R = \frac{1}{limsup\left(\sqrt[n]{\frac{n^2}{2^n}}\right)} = \frac{2}{limsup\,(\sqrt[n]{n})^2} = 2$$

So the Radius is 2.

Illustrations of what can happen at $\pm R$

E.g.

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n$$

Use ratio test, $\left|\frac{n x^{n+1}}{(n+1)x^n}\right| \to |x|\ as\ n \to \infty$ so R = 1
Know

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n \text{ converges absolutely when } |x| < 1 \text{ \& not when } |x| > 1$$

$$\text{Now for } x = 1, get \sum_{n=1}^{\infty} \frac{1}{n}(1)^n\ diverges$$

$$x = -1\ get \sum_{n=1}^{\infty} \frac{1}{n}(-1)^n \text{ converges but not absolutely}$$

E.g.

$$\sum_{n-1}^{\infty} \frac{1}{n^2} x^n, R = 1$$

For $x = \pm 1$, get $\sum \frac{1}{n^2}(-1)^n$ converges absolutely.

E.g.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^{2n} - -x^2 + \frac{1}{2}x^4 - \frac{1}{3}x^6 + \cdots$$

By ratio test

$$\left|\frac{\frac{(-1)^{n+1}x^{2(n+1)}}{n+1}}{\frac{(-1)^n x^{2n}}{n}}\right| = \frac{n}{n+1}|x|^2 \to |x|^2\ as\ n \to \infty$$

But ratio test says when $|x|^2 < 1\&\ not\ when\ |x|^2 > 1$
R=1

Derived Series

March-16-11
9:34 AM

Power Series Recap

Every power series

$$\sum_{n=0}^{\infty} a_n x^n$$

comes with a radius.
This is a quantity R where $0 \le R \le \infty$.
If $|x| < R, \sum |a_n x^n|$ converges and if $|x| > R, a_n x^n \nrightarrow 0$ and $\sum a_n x^n$ diverges.
Thus, when $R > 0$, power series create functions f on $(-R, R)$ by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

Derived Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$
The derived series is defined to be

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots$$

In other words, differentiate each term.

We will show that the radius of the derived series does not change (i.e. = R) and $f'(x)$ exists on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Here is why this is not obvious.
Here is

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = \lim_{t \to x} \frac{\sum_{n=0}^{\infty} a_n t^n - \sum_{n=0}^{\infty} a_n x^n}{t - x} = \lim_{t \to x} \sum_{n=0}^{\infty} a_n \frac{t^n - x^n}{t - x}$$

$$= \lim_{t \to x} \lim_{k \to \infty} \sum_{n=1}^{k} a_n \frac{t^n - x^n}{t - x}$$

Next,

$$\sum_{(n=1)}^{\infty} n a_n x^{n+1} = \lim_{k \to \infty} \sum_{n=1}^{k} n a_n x^{n-1} = \lim_{k \to \infty} \sum_{n=1}^{k} (a_n x^n)' = \lim_{k \to \infty} \sum_{n=1}^{k} a_n \lim_{t \to \infty} \frac{t^n - x^n}{t - x}$$

$$= \lim_{k \to \infty} \lim_{t \to x} \sum_{n=1}^{k} a_n \frac{t^n - x^n}{t - x}$$

Does

$$\lim_{k \to \infty} \lim_{t \to x} \square = \lim_{t \to x} \lim_{k \to \infty} \square ?$$

Note. Can't always switch limits

E.g.

$$x_{mn} = \begin{cases} 1 \text{ if } m \ge n \\ 0 \text{ if } m < n \end{cases}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$$\lim_{n \to \infty} \lim_{m \to \infty} x_{mn} = \lim_{n \to \infty} 1 = 1$$

$$\lim_{m \to \infty} \lim_{n \to \infty} x_{mn} = \lim_{m \to \infty} 0 = 0$$

Uniform Convergence

March-16-11
9:59 AM

Norm (Sup-Norm, Uniform Norm)

Let f be a bounded function on an interval I.
The sup-norm of f on I is
$\|f\|_I = \sup\{|f(x)| : x \in I\}$

Properties of sup-norm

$\|f\|_I = 0 \Leftrightarrow f = 0 = 0 \text{ function on } I$
$\|cf\|_I = |c|\|f\|_I$
$\|f + g\|_I \leq \|f\|_I + \|g\|_I$

Uniform Distance

For two functions f, g, on I their uniform distance is
$\|f - g\|_I = \sup_{x \in I} |f(x) - g(x)|$

Uniform Convergence of Sequences of Functions

Given f_n on I we say that $f_n \to f$ (tends to f) uniformly on I when $\|f_n - f\|_I \to 0 \text{ as } n \to \infty$

Notice

$If\ f_n \to f$ uniformly on I then
$|f_n(x) - f(x)| \leq \|f_n - f\|_I \to 0$
So $f_n(x) \to f(x) \forall x \in I.$

Pointwise Convergence

When $f_n(x) \to f(x) \forall x \in I$ we say that $f_n \to f$ pointwise on I.

Observation

Thus $f_n \to f \text{ unif on } I \Rightarrow f_n \to f \text{ ptw on } I$
However, $f_n \to f \text{ ptw on } I \not\Rightarrow f_n \to f \text{ unif on } I$

Continuity of Uniform Convergence

If $f_n \to f$ uniformly on I and the f_n are continuous on I, then f is continuous on I.

Integration of Uniform Convergence

If $f_n \to f$ uniformly on I and say f_n, f are integrable on I.
Then for every $|a, b| \subseteq I$ we get $\int_a^b f_n \to \int_a^b f$

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b \lim_{n \to \infty} f_n$$

Note

$f_n \to f$ pointwise on $[a, b] \not\Rightarrow \int_a^b f_n \to \int_a^b f$

Sup-Norm Examples

$\|\sin x\|_{\mathbb{R}} = 1$
$\|\sin x\|_{|0, \frac{\pi}{4}|} = \frac{1}{\sqrt{2}}$
$\|\arctan x\|_{\mathbb{R}} = \frac{\pi}{2}$

Find $\|x^3(1 - x)\|_{|0,1|}$
Use derivatives
$f(x) = x^3(1 - x) \Rightarrow f'(x) = x^2(3 - 4x)$
Max at $\frac{3}{4}$

$$\|x^3(1 - x)\| = f\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^3 \left(1 - \frac{3}{4}\right) = \frac{27}{256}$$

Proofs

$\|cf\|_I = \sup_{x \in I} |cf(x)| = \sup_{x \in I} |c||f(x)| = |c| \sup_{x \in I} |f(x)| = |c|\|f\|_I$

For every $x \in I$ we know
$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_I + \|g\|_I\ \forall x \in I$
So $\|f\|_I + \|g\|_I$ is an upper bound for $|f(x) + g(x)|$ so
$\|f + g\|_I \leq \|f\|_I + \|g\|_I$

Sequences of Functions Examples

$on\ |0, 1|, f_n(x) = x^n$

Take any power series $\sum_{n=0}^{\infty} a_n x^n$ with radius $R > 0$

Let $s_n(x) = \sum_{k=0}^{n} a_k x^k \text{ on } (-R, R)$

Let f be such that $f^{(n)}(p)$ all exist where $p \in I$
Get Taylor Polynomials:

$T_0(x) = f(p)$
$T_1(x) = f(p) + f'(p)(x - p)$
$T_2(x) = f(p) + f'(p)(x - p) + \frac{f''(p)}{2!}(x - p)$
...
$T_n(x) = f(p) + f'(p)(x - p) + \cdots + \frac{f^{(n)}(p)}{n!}(x - p)^n$

Counterexample to $f_n \to f \text{ ptw} \Rightarrow f_n \to f \text{ unif}$?

Example:

$f_n(x) = x^n \text{ on } |0, 1|$
See that:

$$f_n(x) \to \begin{cases} 0 \text{ when } 0 \leq x < 1 \\ 1 \text{ when } x = 1 \end{cases}$$

So $f_n \to f$ *pointwise on* $|0, 1|$ *where*

$$f(x) = \begin{cases} 0 \text{ when } 0 \leq x < 1 \\ 1 \text{ when } x = 1 \end{cases}$$

However,
$\|f_n - f\|_{|0,1|} = \sup_{x \in |0,1|} |x^n - f(x)| = 1 \nrightarrow 0$

Proof of Continuity of Uniform Convergence

May be on Exam

Take $p \in I$ and $\varepsilon > 0$
Need $\delta > 0$ so that $|f(x) - f(p)| < \varepsilon$ when $|x - p| < \delta$
Since $\|f_n - f\|_I \to 0$, we have an N such that $\|f_N - f\|_I < \frac{\varepsilon}{3}$
Now, f_N is continuous at p so take $\delta > 0$ such that $|f_N(x) - f_N(p)| < \frac{\varepsilon}{3}$ when $|x - p| < \delta$.
Now for $|x - p| < \delta$ *we get*
$|f(x) - f(p)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(p)| + |f_N(p) - f(p)|$
$\leq 2\|f_N - f\|_I + |f_N(x) - f_N(p)| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$
■

Example

$f_n(x) = x^n(1 - x) \text{ on } |0, 1|$
Clearly for all $x \in [0, 1]$, $f_n(x) \to 0$ i.e. $f_n \to 0$ pointwise on $|0, 1|$
Does $f_n \to 0$ uniformly on $[0, 1]$?
We need $\|f_n - 0\|_{|0,1|}$

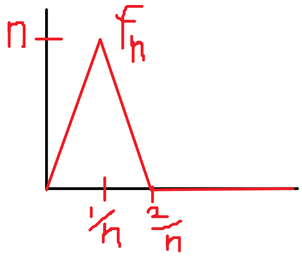
Proof of Integration of Power Series

Know for Exam

$$0 \le \left| \int_a^b f_n(t)dt - \int_a^b f(t)dt \right| \le \int_a^b |f_n(t) - f(t)| dt \le \int_a^b \|f_n - f\|_I dt = \|f_n - f\|_I (b-a) \to 0$$

So squeeze. ■

Example of failure for pointwise



Here, $\int_0^1 f_n = 1$

but $f_n \to 0$ pointwise on $[0, 1]$

$$\int_0^1 0 = 0$$

Series of Functions

March-21-11
9:39 AM

Series of Functions

Given a sequence of functions, $f_1(x), f_2(x), \dots, f_n(x), \dots$ on I form the partial sum functions:

$s_1(x) = f_1(x)$
$s_2(x) = f_1(x) + f_2(x)$
$\vdots$
$s_n(x) = f_1(x) + \cdots + f_n(x)$
$\vdots$

We say $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on I when $s_n \to some\ function\ s$ uniformly on I

The Weierstrass M-Test

Let f_n functions defined on I and $\|f_n\|_I \leq some\ const\ M_n$

If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on I

Example

Power series $\sum_{n=0}^{\infty} a_n x^n$ comes from

$s_1(x) = a_0$
$s_2(x) = a_0 + a_1 x$
$\vdots$
$s_n(x) = a_0 + \cdots + a_n x^n$
$\vdots$

Problem

If

$$s(x) = \sum_{k=0}^{\infty} a_k x^k \ on\ (-R, R)$$

Does

$s_n(x) = \sum_{k=0}^{n} a_k x^k \to s(x)$ uniformly on (-R, R)?

No.

Example

$$s(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x)^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots on\ (-1,1)$$

Here, $s_{2n}(x) = 1 - x^2 + x^4 - \cdots + (-1)^n x^{2n} \nrightarrow s(x)\ uniformly\ on\ (-1,1)$

Check:

$$s_{2n}(x) = \frac{(1-(-x^2)^{n+1})}{1-(-x^2)} = \frac{1}{1+x^2} + (-1)^n \frac{x^{2n+2}}{1+x^2}$$

Thus

$$\|s - s_{2n}\|_{(-1,1)} = \left\| \left(\frac{x^{2n+2}}{1+x^2} \right) \right\|_{(-1,1)} = \frac{1}{2}\ \forall n \nrightarrow 0$$

Proof of Weierstrass M-Test

Let $s_n = \sum_{k=1}^{n} f_k$

For each $x \in I$ we have $|f_k(x)| \leq \|f_k\|_I \leq M_k$

By comparison,

$\sum_{k=1}^{\infty} |f_k(x)|$ converges since $\sum M_k$ converges

So $\sum_{k=1}^{\infty} f_k(x)$ converges to some $s(x)$.

So $\sum_{k=1}^{\infty} f_k$ converges pointwise, check if it converges uniformly

So for all $x \in I$ we have

$$|s(x) - s_n(x)| = \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \leq \sum_{k=n=1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_k\ \forall x \in I$$

So $\|s - s_n\|_I \leq \sum_{k=n+1}^{\infty} M_k$

Since $\sum_{k=n+1}^{\infty} M_k \to 0\ as\ n \to \infty$

Squeeze to see that $\|s - s_n\|_I \to 0\ as\ n\ \to \infty$

Example: Riemann Zeta Function

Take the 'p series' $(p = x)$

$$\sum_{n=1}^{\infty} \frac{1}{n^x}$$

which converges when $x > 1$

Call $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}\ for\ x > 1, x \in (1, \infty)$

Well,

$\frac{1}{n^x} = \frac{1}{e^{x \ln n}}$ are continuous on $(1, \infty)$

So $\zeta_n(x) = \sum_{k=1}^{n} \frac{1}{k^x}$ are continuous on $(1, \infty)$ for all n

We wish $\zeta_n \to \zeta$ uniformly on $(1, \infty)$
Sorry. It does not happen.

Check this:
By error estimate from integral test, for a fixed $x > 1$

$$\int_{n+1}^{\infty} \frac{dt}{t^x} \leq \zeta(x) - \zeta_n(x)$$

$$\int_{n+1}^{\infty} \frac{dt}{t^x} = \frac{1}{x-1} \frac{1}{(n+1)^{x-1}} \to \infty \; as \; x \to 1^+$$

Do this integral yourself.
So $\|\zeta - \zeta_n\|_{(0,\infty)} = \infty \nrightarrow 0$

How to rescue the situation?
Pick any $b > 1$
We will check that $\zeta_n \to \zeta$ uniformly on $[b, \infty)$
Use the M-test with $M_n = \frac{1}{n^b}$
Clearly $\frac{1}{n^x} \leq \frac{1}{n^b}\ \forall x \geq b \Rightarrow \left\|\frac{1}{n^x}\right\|_{[b,\infty)} \leq \frac{1}{n^b}$

Now $\sum_{n=1}^{\infty} \frac{1}{n^b}$ converges since $b > 1$

Thus $\zeta_n \to \zeta$ uniformly on $[b, \infty)$
Since ζ_n are continuous on $[b, \infty)$, so is ζ continuous on $[b, \infty)$
Hence, ζ is continuous on $(1, \infty)$. For every $x > 1$, there is a b such that $1 < b < x$ so ζ is continuous at x.

Power Series

March-25-11
9:34 AM

Uniform Convergence of Power Series

Let $\sum_{k=0}^{\infty} a_k x^k$
$= f(x)$ on $(-R, R)$ and $[a, b]$ is any closed interval
inside $(-R, R)$, then the series converges uniformly on $[a, b]$.

Continuity of Power Series

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on $(-R, R)$
then f is continuous on $(-R, R)$

Derived & Integrated Series

Given $f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$
on $(-R, R)$

Derived Series:

$$\sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots$$

Integrated Series:

$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} = a_0 x + \frac{a_1}{2} x^2 + \cdots + \frac{a_n}{n+1} x^{n+1} + \cdots$$

Radii of Derived Series

If $\sum_{k=0}^{\infty} a_k x^k$ has radius R, then so does $\sum_{k=1}^{\infty} k a_k x^{k-1}$

Proof of Uniform Convergence of Power Series

Let $c = \max\{|a|, |b|\} \in [0, R)$
For all $x \in [a, b]$ we have
$|x| \leq c$ so $|a_n x^n| \leq |a_n||x|^n \leq |a_n| c^n = |a_n c^n|$

Now, $\sum_{k=0}^{\infty} |a_k c^k|$ converges since $c < radius\ R$
Also $\|a_n x^n\|_{[a,b]} \leq |a_n c^n|$
By the M-test $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly on $[a, b]$

Proof of Continuity of Power Series

Pick $p \in (-R, R)$. Want f continuous at p.
Enclose p by some $[a, b] \in (-R, R)$
Now, $s_n(x) = \sum_{k=0}^{n} a_k x^k$ converges uniformly on [a, b] to $f(x) = \sum_{k=0}^{\infty} a_k x^k$
Since s_n are continuous on [a, b], f is continuous at p. ■

Proof of Radii of Derived Series

The series $\sum_{k=1}^{\infty} k a_k x^k$ has the same radius of converges as derived series $\sum_{k=1}^{\infty} k a_k x^{k-1}$

Differentiation an Integration Theorem

March-28-11
9:33 AM

If $s(x) = \sum_{k=0}^{\infty} a_x x^n \; for\; x \in (-R, R)$, the summs $s_n(x) = \sum_{k=0}^{n} a_k x^k$ converge uniformly on every $[a, b] \subseteq (-R, R)$, but not necessarily on $(-R, R)$.
Thus $s(x)$ is continuous on $(-R, R)$

Derived series

$$\sum_{k=1}^{\infty} k a_k x^{k-1}$$

Integrated Series

$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$$

Proposition

$\sum_{k=1}^{\infty} a_k x^k$ & $\sum_{k=1}^{\infty} k a_k x^{k-1}$ have the same radius

Corollary

$\sum a_k x^k$ & $\sum \frac{a_k}{k+1} x^{k+1}$ have the same radius too

(Since the derived series of the integrated series is the beginning again.

Integrated Series Formula

If $s(x) = \sum_{k=0}^{\infty} a_k x^k \; on\; (-R, R)$ & $[a, b] \subseteq (-R, R)\; then$

$$\int_a^b s(t)dt = \sum_{k=0}^{\infty} a_k \int_a^b t^k dt$$

Special Case

Pick any $x \in (-R, R)$. Use $[a, b] = [0, x]$

Get $\int_0^x s(t)dt = \sum_{k=0}^{\infty} a_k \int_0^x t^k dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$

Derived Series Formula

If $f(x) = \sum_{k=0}^{\infty} a_k x^k \; on\; (-R, R)$

then f is differentiable and

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} \; \forall x \in (-R, R)$$

Proof of Proposition

Let $R =$ radius for $\sum_{k=0}^{\infty} a_k x^k$

For $x \in (-R, R)$ pick t such that $|x| < t < R$

Know $\sum_{k=0}^{\infty} |a_k t^k|$ converges

To get that $\sum_{k=1}^{\infty} |k a_k x^{k-1}|$ converges, we will show

$\sum_{k=0}^{\infty} |k a_k x^k|$ converges.

Do limit comparison of

$\sum_{k=0}^{\infty} |k a_k x^k|$ with $\sum_{k=0}^{\infty} |a_k t^k|$

Look at

$$\left|\frac{k a_k x^k}{a_k t^k}\right| = k \left|\frac{x}{t}\right|^k . \left|\frac{x}{t}\right| < 1 \; so\; k \left|\frac{x}{t}\right|^k \to 0 \; as\; k \to \infty$$

Thus $\frac{|k a_k x^k|}{|a_k t^k|} < 1 \; eventually\; with\; k\; so\; eventually$

$|k a_k x^k| < |a_k t^k|$

Since $\sum a_k t^k$ converges, so does $\sum |k a_k x^k|$ by comparison.

Furthermore, if $|x| > R$ then $|a_n x^n| \nrightarrow 0$ hence $|n a_n x^n| = n|a_n x^n| \nrightarrow 0$
So $\sum k a_k x^k$ diverges.

Proof of Integrated Series Formula

$s_n(x) = \sum_{k=0}^{n} a_k x^k \to s(x)$ uniformly on $[a, b]$

Hence $\int_a^b s_n(t)dt \to \int_a^b s(t)\,dt$

$$i.e. \int_a^b \left(\sum_{k=0}^{n} a_n t^k\right) dt = \sum_{k=0}^{n} a_k \int_a^b t^k dt \to \int_a^b s(t)dt$$

Proof of Derived Series Formula

$Let\; g(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$

Note g is continuous on $(-R, R)$, since it is a power series.

Just saw

$$\int_0^x g(t)dt = \sum_{k=1}^{\infty} \frac{k a_k}{k} x^k = \sum_{k=1}^{\infty} a_k x^k \; \forall x \in (-R, R)$$

So $f(x) = a_0 + \int_0^x g(t)dt$

By FTCII get

$f'(x) = g(x)$

Application

Prove

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$R = \infty$, check with ratio test

$Let\; f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

Want $f(x) = e^x$

$$Notice\; f'(x) = \sum_{k=1}^{\infty} \frac{1}{k!} k x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = f(x)$$

Now find

$$\left(\frac{f(x)}{e^x}\right)' = \frac{e^x f'(x) - f(x)(e^x)'}{e^{2x}} = 0$$

$So\; \frac{f(x)}{e^x} = C \Rightarrow f(x) = Ce^x$

$f(0) = 1 = 1 \times e^0$ so

$f(x) = e^x$

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Lifting Principle for Integration

If $f(x) \le g(x)$ on $|a, \infty)$
then

$$\int_a^x f(t)dt \le \int_a^x g(t)dt$$

Fun Stuff with Power Series

Getting power series for known function. Good to memorise these expansions

$$\text{We did } e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \ \textit{on all of } \mathbb{R}$$

Know $\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots \ \textit{for } |x| < 1$

Integrate

$$\ln(1+x) = \int_0^x \frac{dt}{1+t} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \textit{for } |x| < 1$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

Integrate

$$\arctan x = \int_0^x \frac{dt}{1+x^2} = 1 - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots \textit{for } |x| < 1$$

Estimate this integral using power series, with error < 10^{-5}

$$\int_0^{\frac{1}{2}} e^{-x^2} dx$$

Know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

So $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!} + \cdots$

Integrate

$$\int_0^x e^{-t^2} dt = x - \frac{x^3}{3} + \frac{x^5}{5 \times 2!} - \frac{x^7}{7 \times 3!} + \frac{x^9}{9 \times n!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1) \times n!} + \cdots$$

Plug in $x = \frac{1}{2}$ and get

$$\int_0^{\frac{1}{2}} e^{-t^2} dt = \frac{1}{2} - \frac{1}{2^3 \times 3} + \frac{1}{2^5 \times 5 \times 2!} - \frac{1}{2^7 \times 7 \times 3!} + \frac{1}{2^9 \times 9 \times 4!} + \cdots \quad x \in \mathbb{R}$$

By error formula in AST we know

$$\int_0^{\frac{1}{2}} e^{-t^2} dt \approx \frac{1}{2} - \frac{1}{2^3 \times 3} + \frac{1}{2^5 \times 5 \times 2!} - \frac{1}{2^7 \times 7 \times 3!} = \frac{12399}{26880}$$

With error $\le \frac{1}{2^9 \times 9 \times 4!} = \frac{1}{110529} \le 10^{-5}$

Power series for sin and cos

Start with $\cos x \le 1 \ \textit{on} \ |0, \infty)$

Lift 1:

$$\int_0^x \cos t \ dt \le \int_0^t dt \ \textit{ on } |0, \infty) \Rightarrow \sin x \le x$$

Lift 2:

$$\int_0^x \sin t \, dt \le \int_0^x t dt$$

$$-\cos x + 1 \le \frac{x^2}{2}$$

Lift 3:

$$\int_0^x \left(1 - \frac{t^2}{2}\right) dt \le \int_0^x \cos t \, dt$$

$$x - \frac{x^3}{x!} \le \sin x$$

Lift 4

$$\int_0^x \left(t - \frac{t^3}{3!}\right) dt \le \int_0^x \sin t \, dt$$

$$\frac{x^2}{2!} - \frac{x^4}{4!} \le -\cos x + 1$$

$$\cos x \le 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ \textit{on} \ |0, \infty)$$

Lift 5

$$\sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

By extending the pattern we learn sin x is always between

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n-1}}{(2n-1)!} \ \textit{and}$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n-1}}{(2n-1)!} + (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!}$$

Thus for $x \ge 0$

$$\left|\sin x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n-1}}{(2n-1)!}\right)\right| \le \left|\frac{x^{2n+1}}{(2n+1)!}\right|$$

But this is good for x≤ 0 too since all the functions are odd.

But regardless of x

$$\left|\frac{x^{2n+1}}{(2n+1)!}\right| \to 0 \; as \; n \to \infty$$

Thus $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

Differentiating gives

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$$

A cool function

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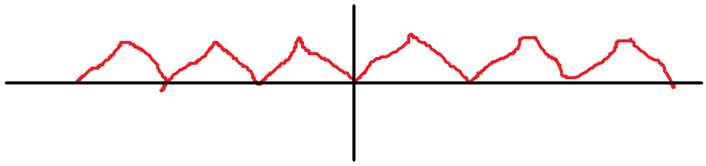
$Let\ \varphi(x) = |x|$ on $|-1,1|$ and extend to $\mathbb{R}$ by the rule $\varphi(x+2) = \varphi(x)$



Note:

1. $|\varphi(x) - \varphi(y)| \le |x - y|$ because slope from $(x, \varphi(x))$ to $(y, \varphi(y))$ is ≤ 1
2. If x, y have no integer strictly between them, then $\varphi(x) - \varphi(y) = \pm(x - y)$

Now take the series

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x) = \varphi(x) + \frac{3}{4}\varphi(4x) + \frac{9}{16}\varphi(16x) + \frac{27}{64}\varphi(64x) + \cdots$$

Observe that

$$\left\| \left(\frac{3}{4}\right)^n \varphi(4^n x) \right\|_{\mathbb{R}} \le \left(\frac{3}{4}\right)^n \ \& \ \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \text{ converges, by M test so does w}$$

So series converges uniformly on $\mathbb{R}$ by M-test and since each $\left(\frac{3}{4}\right)^n$ is continuous so $f(x)$ is continuous on $\mathbb{R}$ as well.

This f, which is all teeth is nowhere differentiable on $\mathbb{R}$

Let $x \in \mathbb{R}$

We will find a sequence where $t_m \to 0$ while

$$\left| \frac{f(x+t_m) - f(x)}{t_m} \right| \to \infty\ as\ n \to \infty$$

Note

$$\frac{(f(x+t_m) - f(x))}{t_m} = \sum_{n=0}^{\infty} \frac{\left(\frac{3}{4}\right)^n (\varphi(4^n(x+t_m)) - \varphi(4^n x))}{t_m}$$

For each $m = 1, 2, 3, \ldots$ there is no integer strictly between $4^m x$ and $4^m x \pm \frac{1}{2}$

$$\text{Put } t_m = \begin{cases} \dfrac{1}{2 \times 4^m}\ if\ no\ integer\ in\ \left(4^m x, 4^m x + \dfrac{1}{2}\right) \\ -\dfrac{1}{2 \times 4^m}\ if\ no\ integer\ in\ \left(4^m x - \dfrac{1}{2}, 4^m x\right) \end{cases}$$

Clearly $t_m \to 0$ as $m \to \infty$

These t_m were chosen so that

$$4^m x \ \& \ 4^m(x + t_m) = 4^m x + 4^m t_m = 4^m x \pm \frac{1}{2}$$

have no integer between them

Now look at $\dfrac{f(x+t_m) - f(x)}{t_m}$

For $n > m$ we get

$4^m(x + t_m) = 4^n x + 4^n t_m = 4^n x + even\ integer$

So $\varphi(4^n(x+t_m)) = \varphi(4^n x)$

$$\text{Thus } \frac{(f(x+t_m) - f(x))}{t_m} = \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \left(\frac{(\varphi(4^n(x+t_m)) - \varphi(4^n x))}{t_m}\right)$$

When $n = m$ we get

$$\left(\frac{3}{4}\right)^m \frac{\varphi(4^m(x - t_m)) - \varphi(4^m x)}{t_m} = \pm\left(\frac{3}{4}\right)^m \frac{4^m x - 4^m t_m - 4^m x}{t_m} = \pm 3^m$$

Since no integer between the two

$$\text{So } \left| \frac{f(x+t_m) - f(x)}{t_m} \right| = \left| \pm 3 + \sum_{n=0}^{m-1} \left(\frac{3}{4}\right) \frac{(\varphi(4^n(x+t_m)) - \varphi(4^n x))}{t_m} \right|$$

$$\ge 3^m - \sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n \left| \frac{\varphi(4^n(x+t_m)) - \varphi(4^n x)}{t_m} \right| \ge 3^m - \sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n \left| \frac{4^n(x+t_m) - 4^n x}{t_m} \right| = 3^m - \sum_{n=0}^{m-1} 3^n$$

$$= 3^m - \left(\frac{3^m - 1}{2}\right) = \frac{3^m}{2} \to \infty$$

Estimating π

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Know $arctan\, x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots\, for\, |x| < 1$
Saw for $x = 1, that\, \frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$
However, this is too slow.

Here's an identity about arctan that helps
$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right), when\ 0 \le xy < 1, \qquad x, y \ge 0$

Proof of arctan identity

Pick any $y > 0$ and x such that $0 \le x < \frac{1}{y}$
Let $f(x) = \arctan x$
$g(x) = \arctan\left(\frac{x+y}{1-xy}\right)$
For $x \in \left|0, \frac{1}{y}\right)$ we have
$f'(x) = \frac{1}{1+x^2}$
$g'(x) = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \times \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2+(x+y)^2}$
$= \frac{1+y^2}{1-2xy+x^2y^2+x^2+y^2+2xy} = \frac{1+y^2}{1+x^2y^2+x^2+y^2} = \frac{1+y^2}{(1+x^2)(1+y^2)} = \frac{1}{1+x^2}$

So $g(x) = f(x) + C$
Put x = 0, get $g(0) = \arctan y = f(0) + c = c$
Hence $\arctan\left(\frac{x+y}{1-xy}\right) = \arctan x + \arctan y$

Example

$$4\arctan\frac{1}{5} = 2\left(\arctan\frac{1}{5} + \arctan\frac{1}{5}\right) = 2\arctan\left(\frac{\frac{2}{5}}{1-\frac{1}{25}}\right) = 2\arctan\left(\frac{5}{12}\right)$$

$$= \arctan\left(\frac{5}{12}\right) + \arctan\left(\frac{5}{12}\right) = \arctan\left(\frac{\frac{10}{12}}{1-\frac{25}{144}}\right) = \arctan\left(\frac{120}{119}\right)$$

Example

$$\arctan 1 + \arctan\left(\frac{1}{239}\right) = \arctan\left(\frac{1+\frac{1}{239}}{1-\frac{1}{239}}\right) = \arctan\left(\frac{240}{238}\right) = \arctan\left(\frac{120}{119}\right)$$

$$\text{Thus } \frac{\pi}{4} + \arctan\left(\frac{1}{239}\right) = 4\arctan\left(\frac{1}{5}\right) \Rightarrow \pi = 16\arctan\left(\frac{1}{5}\right) - 4\arctan\left(\frac{1}{239}\right)$$

Now,
$\arctan\frac{1}{239} = \frac{1}{239} - \frac{1}{3\times 239^3} + \cdots$
$a = \frac{1}{239} \approx \arctan\frac{1}{239}\ with\ error \le \frac{1}{3\times 239^2} =$
$4a = \frac{4}{239} \approx 4\arctan\frac{1}{239}\ with\ error \le \frac{4}{3\times 239^2}$
and
$\arctan\frac{1}{5} = \frac{1}{5} - \frac{1}{3\times 5^3} + \frac{1}{5\times 5^5} - \frac{1}{7\times 5^7} + \frac{1}{9\times 5^9} - \frac{1}{11\times 5^{11}} + \cdots$
So b=$\frac{1}{5} - \frac{1}{3\times 5^3} + \frac{1}{5\times 5^5} - \frac{1}{7\times 5^7} + \frac{1}{9\times 5^9} \approx \arctan\left(\frac{1}{5}\right)\ with\ error \le \frac{1}{11\times 5^{11}}$
$16b \approx 16\arctan\left(\frac{1}{5}\right) with\ error \le \frac{16}{11\times 5^{11}}$

Errors

$If\ a_1 \approx b_1\ with\ error \le c_1$
$and\ a_2 \approx b_2\ with\ error \le c_2$
$then\ a_1 - a_2 \approx b_1 - b_2\ with\ error \le c_1 + c_2$

So
$16b - 4a \approx \pi\ with\ error \le \frac{4}{3\times 239^3} + \frac{16}{11\times 5^{11}} \le 1.3\times 10^{-7}$
Well,

$$16b - 4a = \frac{92388592868}{29408203125} \approx 3.14159258473906$$

$$\pi \approx 3.141592654$$

$$\pi - (16b - 4a) = 6.9 \times 10^{-8}$$