Number Systems

September-14-10
12:24 AM

N - Natural Numbers
{1, 2, 3, ... }

Z - Integers
{ -∞, ..., -1, 0, 1, ..., ∞}

Q - Rational Numbers
{ $\frac{p}{q}$, p, q ϵ **R**, q ≠0}

Well Ordering Principle:
Every non-empty subset of **N** contains a least element.

Proof by Contradiction:
Assume the opposite of what you are trying to prove then derive a contradiction

Coprime:
Two numbers are said to be coprime when they have no common factors.

N is well ordered.

Proof of Well-Ordering Principle :
Let S be a non-empty subset of **N**
Pick n ϵ S

Go through all natural numbers starting at 1. If that number is in S then it is the least number and terminate. This will terminate after at most n steps.

Note: **Z** does not satisfy the WOP

Ex: **Q** = { x ϵ **Q**, x ≥ 0} does not have WOP because, for instance { x ϵ **Q**, x > 0} does not have a least element.

Q is closed under +, -, ×, ÷

Numbers that are not rational: Irrational Numbers

Eg. √2 !ϵ **Q**

Suppose √2 ϵ **Q** <- Proof by contradiction
Then $\sqrt{2} = \frac{p}{q}$ where p, q ϵ **Z** and q ≠ 0

Assume p, q are coprime
$\sqrt{2}q = p$ $\quad 2q^2 = p^2 \rightarrow p^2$ is even → p is even

Say p = 2k for some k ϵ **Z**
$2q^2 = (2k)^2 = 4k$
$q^2 = 2k^2$ → q is even

This contradicts the assumption that p and q are coprime

Mathematical Induction

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<u>Theorem:</u>
A certain and proved mathematical truth.

∀ - For all

Principle of Mathematical Induction

<u>Theorem:</u>
For each n ϵ **N** let P(n) be a statement about n.
Suppose: (hypothesis)

1. P(1) is true.
2. P(k+1) is true whenever P(k) is true

Then P(n) is true for every n ϵ **N**

Proof of Principle of Mathematical Induction

Suppose the conclusion is false (Proof by contradiction)
Then there is some n ϵ **N** so that P(n) is not true.
Let S = {k ϵ **N**: P(k) is not true}
Then S is a non-empty set of **N**
By W.O.P. S contains a least element, say n
Means P(n) is not true (because n ϵ S)
And if k ϵ **N** and k < n , then P(k) is true
Note n ≠ 1, so n -1 ϵ **N**
Hence P(n - 1) is true.

By assumption (2), P(n -1+1) = P(n) is true.
This contradicts the previous observation that P(n) is not true.
Hence our initial claim was wrong, thus P(n) is true for all n ϵ **N**

Principle of Strong Induction

Suppose P(n) is a statement for each n ϵ **N**
Assume:

1. P(1) is true
2. P(k) is true if P(j) is true for all j ϵ **N**, with j < k

Then P(n) is true for all n ϵ **N**

Proof of Principle of Strong Induction

Suppose conclusion is false.
Let S = {k ϵ **N**: P(k) is not true}
Then S is a non-empty set of **N**
By W.O.P. S contains a least element, say n
Means P(n) is not true (because n ϵ S)
And if k ϵ **N** and k < n , then P(k) is true (because n is the least element of S)
Hence P(n - 1) is true. In face, P(j) is true ∀ j < n , j ϵ **N**

By assumption (2), P(n) is true
This contradicts the previous observation that P(n) is not true.

Induction to prove sum of geometric series formula

$P(n) = 1 + r + r^2 + r^3 + \dots + r^n = \dfrac{r - r^{n+1}}{1-r}$

$P(1) = \dfrac{r - r^2}{1-r} = r$

Must show P(k + 1) is true when P(k) is true
Assume P(k) is true and look at P(k+1)
$r + r^2 + \dots + r^k + r^{k+1}$
$= \dfrac{r - r^{k+1} + r^{k+1} \times (1-r)}{1-r} = \dfrac{r - r^{k+2}}{1-r}$
So P(k+1) is true

Make sure to check base case
Eg. $P(n) \geq 100$
If P(n) is true then $n \geq 100$ and therefore $n + 1 \geq 100$
However, P(1) is false

Example:
Prove $2^n > n^2$ If $n \geq 5$

Two methods to approach:
$P(n) = 2^{n+4} > (n+4)^2$ and base case is P(1)
$P(n) = 2^n > n^2$ and base case is P(5)

Note that if P(5) is true and P(k+1) is true whenever P(k) is true then P(n) is true for all n ≥ 5.

$P(5) = 2^5 > 5^2 = 32 > 25$ is true.
$P(k) = 2^k > k^2$
$P(k+1) = 2^{k+1} > (k+1)^2$
$= 2 \times 2^k > k^2 + 2k + 1$
$P(k+1) - P(k) = 2^k > 2k + 1$

Lemma: $2^n > 2n + 1, n \geq 5$
$P(k) = 2^k > 2k + 1$
$P(k+1) = 2 \times 2^k > 2k + 3$
$P(k+1) - P(k) = 2^k > 2$

Lemma: $2^k > 2, n \geq 5$
$P(5) = 32 > 2$ is true
$2^{k+1} = 2 \times 2^k$
Therefore $2^{k+1} > 2^k$ for all integer k
Since $2^5 > 2, 2^k > 2$

Since $2^k > 2$, if $2^k > 2k + 1$is true then $2^{k+1} > 2(k+1) + 1$ is true as well.
Therefore, $2^n > 2n + 1, n \geq 5$

Since $2^n > 2n + 1$, if $2^k > k^2$ is true then $2^{k+1} > (k+1)^2$ is true as well.
Therefore $2^n > n^2, n \geq 5$

Proof by Strong Induction

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Suppose f: ℕ -> ℚ is defined by f(1) = 1, f(2) = 2, and $f(n+2) = \frac{1}{2}(f(n+1) + f(n)), n \geq 1$

Prove Range f ⊆ ℚ and 1 ≤ f(n) ≤ 2 for all n ∈ ℕ

Answer - use strong induction

Let P(n) be the statement that f(n) ∈ ℚ
True for n = 1 and n = 2
$f(k) = \frac{1}{2}(f(n-1) + f(n-2))$
Assume f(j) ∈ ℚ for all j<k in order to check f(k) ∈ ℚ
And thus is true since f(k-1) and f(k-2) ∈ ℚ and ℚ is a field.

By principle of induction, f(n) ∈ ℚ for all n ∈ ℕ

Now let P(n) be the statement 1≤ f(n) ≤ 2
True for n = 1, 2
Assume P(j) is true for j < k (and k ≥ 3)
Then $f(k) = \frac{1}{2}(f(k-1) + f(k-2))$
Since f(k) is the average of two number between 1 and 2, 1 ≤ f(k) ≤ 2
Therefore P(k) is true
So by mathematical induction, P(n) is true for all n ∈ ℕ

Inequalities

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10:54 AM

Arithmetic / Geometric mean inequality

If a, b $\geq$ 0 then $\sqrt{ab} \leq \frac{1}{2}(a+b)$

Triangle Inequality

$|a + b| \leq |a| + |b|$

$|a - b| \geq ||a| - |b||$

Definition

Define $\mathbb{R}$ by stating its properties

1. Field: can +, -, ×, ÷ and "good" properties
   a. $\mathbb{Q}$ is another field, $\mathbb{Z}$ is not a field
2. Order property
   a. There is a relation < on $\mathbb{R} \times \mathbb{R}$ so that for every x, y $\in \mathbb{R}$ either x < y or y < x or x = y and other 'good' properties

Arithmetic / Geometric mean inequality

If a, b $\geq$ 0 then $\sqrt{ab} \leq \frac{1}{2}(a+b)$

Proof :

$0 \leq (\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b$

$2\sqrt{ab} \leq a + b$

$\sqrt{ab} \leq \frac{1}{2}(a+b)$

Absolute Values

$|a| = \{a \; if \; a \geq 0, -a \; if \; a < 0\}$

Eg.

$|a|^2 = a^2$

$\sqrt{a^2} = |a|$

$|x| \leq r$ means $-r \leq x \leq r$

$|a-b| < r$ means $-r \leq a - b \leq r$

$b - r \leq a \leq b-r$

$a - r \leq b \leq a - r$

Triangle Inequality

$|a + b| \leq |a| + |b|$

Proof:

$-|a| \leq a \leq |a|$

$-|b| \leq b \leq |b|$

$-(|a| + |b|) \leq a + b \leq |a| + |b|$

Corollary: Reverse Triangle Inequality

$|a - b| \geq ||a| - |b||$

Proof:

$|a| = |(a - b) + b| \leq |a-b| + |b|$ by triangle inequality

$|a| - |b| \leq |a - b|$

Similarly, $|b| = |(b-a) + a| \leq |b-a| + |a| = |a-b| + |a|$

So $|b| - |a| \leq |a - b|$ Together this implies:

$|a - b| \geq ||a| - |b||$

Bounds

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10:31 AM

Bounded Above

A non-empty subset A of an ordered set (think of ℝ) is said to be bounded above if there is some x ∈ ordered set such that a ≤ x for every a ∈ A

Bounded Below

A non-empty subset A of an ordered set (think of ℝ) is said to be bounded above if there is some x ∈ ordered set such that a ≤ x for every a ∈ A

Bounded

We say a is bounded if it is both bounded above and bounded below.

Upper/Lower Bound

Any x with a ≤ x / a ≥ x for all a ∈ A is called an upper bound / lower bound for A.

Least Upper Bound (LUB) - Supremum or Sup

A number x is called the least upper bound (LUB) of A if:

1. x is an upper bound for A
2. If y is any other upper bound for A, then y ≥ x

Greatest Lower Bound (GLB) - Infemum or Inf

A number x is called the greatest lower bound (GLB) of A if:

1. x is a lower bound for A
2. If y is any other upper bound for A, then y ≤ x

LUB and GLB are unique.

Completeness Axiom ℝ

(Completeness Property or "No Holes Property")
Every non-empty set of real numbers that is bounded above has a LUB.

Formal Definition of ℝ

ℝ is an ordered field containing ℕ, and has the completeness axiom.

Example:

Find a number, C, such that |f(x) ≤ C| for all 2≤ x ≤ 3 when

$$f(x) = \frac{x^3 - 2x + 1}{2x - 1}$$

$$|f(x)| = |x^3 - 2x + 1| \times \frac{1}{|2x - 1|}$$

$$|x^3 - 2x + 1| \leq |x^3| + |2x| + 1$$
$$\leq 27 + 6 + 1 = 34$$

$|2x - 1| = 2x - 1$ on $2 \leq x \leq 3$

$$\geq 2 \times 2 - 1 = 3$$

$\Rightarrow |f(x)| \leq \frac{34}{3}$ so $C = \frac{34}{3}$

Bounds

ℤ - not bounded above or below
{q ∈ ℚ: q > 0} - bounded below but no x which is a lower bound belongs to the set

E.g. ℚ 0 is the GLB

E.g $A = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \right\}$
Upper bounds: 42, pi, 7
Lower bounds: -4, -100
LUB: 1 (Note 1 ∈ A)
GUB:0 (Note 0 ∉ A)
The set has a greatest element but no least element

Theorem:

x is the LUB for A ≤ R iff:

1. X is an upper bound for A
2. For every z < x, there is some a ∈ A such that z < a

<u>Proof</u>
(⇒) Assume x is the LUB of A

1. Holds directly from the definition
2. Take z < x. Then z is not an upper bound of A (property 2 of definition)
   So there must be some a ∈ A with a > z
   (Can also be written: For every ε > 0 there exists some a ∈ A such that x- ε < a)

(⇐) Assume the two properties (1) and (2) stated with the theorem hold.
Want to prove x is the LUB of A so we must verify the two parts of the definition of LUB

1. (1) clearly holds as it is property (1) of the theorem.
2. To show part 2 of definition holds, take y any other upper bound for A. Suppose y < x. By (2) of the theorem, there exists an a ∈ A such that y < a. This contradicts the fact that y is an upper bound for A. Hence we must have that y ≥ x, satisfying property (2) of the definition.

Therefore, x is the LUB

Exercise: State and prove the corresponding characterization for GLB

Completeness Axiom and ℝ

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Completeness Axiom ℝ

(Completeness Property or "No Holes Property")
Every non-empty set of real numbers that is bounded above has a LUB.

Formal Definition of ℝ

ℝ is an ordered field containing ℕ, and has the completeness axiom.

Ordered Field

An ordered field is a field with a total ordering of its elements.

Total Order

A set is totally ordered when it has the following properties:

- Antisymmetry: If $a \leq b$ and $b \leq a$ then a=b
- Transitivity: If $a \leq b$ and $b \leq c$ then $a \leq c$
- Totality: $a \leq b$ or $b \leq a$

Archimedean Property

Given any $x \in \mathbb{R}$ there is some $N \in \mathbb{N}$ such that $x \leq N$

Corollary:
GLB $\{\frac{1}{n} : n \in \mathbb{N}\} = 0$

Density of Rational Numbers

If $x, y \in \mathbb{R}$ and $x < y$, then there is some $q \in \mathbb{Q}$ such that $x < q < y$

Proof of Archimedean Property

Suppose the Archimedean property is false.
Then there is some $x \in \mathbb{R}$ with $x > N$ for every $N \in \mathbb{N}$.
This means ℕ is a set which is bounded above.
By the completeness property, ℕ has a LUB, say $z \in \mathbb{R}$.
Then z-1 is not an upper bound (UB) for ℕ hence there must be some $N \in \mathbb{N}$ which is bigger than z-1.
This means $N + 1 > z$ and since $N + 1 \in \mathbb{N}$ this contradicts the statement that z is an upper bound for ℕ.

Proof of Corollary to Archimedean Property

$S = \{\frac{1}{n} : n \in \mathbb{N}\}$

0 is a lower bound since the set consists of positive numbers.
Let $z > 0$. Then $\frac{1}{z} \in \mathbb{R}$
By the Archimedean property, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{z}$
$\Rightarrow z > \frac{1}{N}$ then z is not a lower bound for S
Therefore 0 is GLB(S)

Sketch of why $\sqrt{2} \in \mathbb{R}$
Why is there a real number r with $r > 0$ and $r^2 = 2$
Let $S = \{y \in \mathbb{R} : y^2 < 2\}$
3/2 is an upper bound so S is non-empty and bounded above.
By the completeness axiom, S has a LUB, call it $w \in \mathbb{R}$

Certainly $w > 0$.
Exercise - Verify $w^2 = 2$

Proof of Density of Rational Numbers

Do case $x \geq 0$
$y - x > 0$ so by corollary of the Archimedean principle there is some $N \in \mathbb{N}$ such that $y - x > \frac{1}{N} \Leftrightarrow Ny > 1 + Nx$
By Arch property, there is an $M \in \mathbb{N}$ with $M \geq Nx$.
Let M' be the smallest integer with this property.
(By well ordering principle of ℕ)

Then M' -1 < Nx because M'-1 < M; and is an integer.
$Nx \leq M' < Nx + 1 < Ny$
$$x \leq \frac{M'}{N} < y$$

Convergence of a Sequence

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Sequence

A sequence is an infinite list of real numbers $x_1, x_2, x_3, \ldots$
A sequence has a first element, 2nd element, etc. for each natural number.

Ex:
1. 1, 1, 1, 1, 1, ...
2. $x_n = \frac{1}{n}$, $n \in \mathbb{N}$
3. -1, 1, -1, 1, ...
4. $x_1 = 1, x_2 = 3\ x_{n+2} = \frac{1}{3}(x_{n+1} + x_n)$

Notation: $(x_n)_{n=1}^{\infty}\ or\ (x_n)$

Convergence

Say the sequence $(x_n)_{n=1}^{\infty}$ converges to a real number L provided for every $\varepsilon > 0$ there is an index $N \in \mathbb{N}$ such that $|x_n - L| < \varepsilon$ for all $n \geq N$

In this case we say L is the limit of the sequence and write
$\lim_{n \to \infty} x_n = L$
Or $x_n \to L$ as $x_n \to \infty$

Memorise this Definition

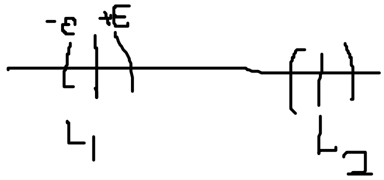
Divergence

If a sequence does not converge, it diverges.

There can only be one L that a sequence converges to

Proof

Suppose $x_n \to L_1$ and $x_n \to L_2$



Take $\varepsilon = \frac{1}{2}|L_2 - L_1|$
There is some index N_1 so $|x_n - L_1| < \varepsilon$
For all $n \geq N_1$ and there is some index N_2
So $|x_n - L_2| < \varepsilon$ for all $n \geq N_2$
If $N = \max(N_1, N_2)$ and $n \geq N$ then
Both $|x_n - L_1| < \varepsilon$ and $|x_n - L_2| < \varepsilon$
(wlog $L_1 < L_2$) - without loss of generality

$L_2 - \varepsilon < x_n < L_1 + \varepsilon$
So $L_2 - \varepsilon < L_1 + \varepsilon$
$\Rightarrow L_2 - L_1 < 2\varepsilon = L_2 - L_1$
Contradiction.

Examples

1. 1, 1, 1, 1, ...
   Converges to L= 1 since $|x_n - 1| = 0$ for all n

2. $x_n = \frac{1}{n}, L = 0$
   Prove this: Rough work - Get some $\varepsilon > 0$. Want to pick N so
   $|x_n - 0| < \varepsilon$ for all $n \geq N$
   $\left|\frac{1}{n} - 0\right| < \varepsilon$
   $\frac{1}{n} < \varepsilon\ \forall\ n \geq N$
   $n > \frac{1}{\varepsilon}$
   Take $N > \frac{1}{\varepsilon}$ (there is such an integer by the Archimedean property)

   <u>Work to hand in:</u>
   Let $\varepsilon > 0$. Take an integer $N > \frac{1}{\varepsilon}$.
   Then if $n \geq N$, $\frac{1}{n} \leq \frac{1}{N} < \varepsilon$
   Hence $\forall\ n \geq N$, $|x_n - 0| < \varepsilon$
   Therefore, $\lim_{n \to \infty} x_n = 0$

3. $x_n = \frac{(-1)^n}{n^2 + 1}$
   Rough Work - Guess L = 0

   Want
   $\left|\frac{(-1)^n}{x^2+1} - 0\right| < \varepsilon\ \forall\ n \geq N$
   Want
   $\frac{1}{n^2+1} < \varepsilon$ Notice $\frac{1}{n^2+1} < \frac{1}{n}$ so take $N \geq \frac{1}{\varepsilon}$

   <u>Answer</u>
   Let $\varepsilon > 0$. Take $N \geq \frac{1}{\varepsilon}$
   Then if $n \geq N$, $\left|\frac{(-1)^n}{n^2+1} - 0\right| = \frac{1}{n^2+1} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$
   So $\lim_{n \to \infty} x_n = 0$

4. -1, 1, -1, 1, ...
   Take $\varepsilon = \frac{1}{2}$
   <u>Proof:</u>
   Say the sequence converges to L
   Take $\varepsilon = \frac{1}{2}$ and say $|x_n - L| < \varepsilon\ \forall\ n \geq N$
   Then both $|1 - L| < \frac{1}{2}$ and $|-1 - L| < \varepsilon$
   This would imply that $|1 - (-1)| < 2\varepsilon = 1$
   False

5. $x_n = |r|^n for\ |r| < 1$
   Guess L=0
   $\frac{1}{|r|} > 1$ so $\frac{1}{|r|} - 1 = \delta > 0$
   $\frac{1}{|r|} = (1 + \delta)^n = 1 + n\delta + \ \ldots + \delta^n \geq n\delta$
   $|r^n - 0| = |r|^n < \varepsilon\ \forall n \geq N$

$$\frac{1}{N} \times \frac{1}{\delta} < \varepsilon$$
$$N > \frac{1}{\delta\varepsilon}$$

Proof:

Let ε > 0 and take $N > \frac{1}{\delta\varepsilon}$

Where $\delta = \frac{1}{|r|} - 1 > 0$

Let n ≥ N Thus $|r^n - 0| = |r|^n \leq |r|^n \leq \frac{1}{\delta\varepsilon} < \frac{\delta\varepsilon}{\varepsilon} = \varepsilon$

Therefore, $r^n \to 0$

Bounds and Convergence

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Bounded

Say the sequence (x_n) is bounded if there is some real number C such that $|x_n| \leq C$ $\forall$n $\in \mathbb{N}$
C is called a bound for the sequence.

Ex.
$x_n = n$ - not bounded (and doesn't converge)
$x_n = \frac{(-1)^n}{n+1}$ - bounded by 1 (does converge)
$x_n = (-1)^n$ - bounded by 1 (does not converge)

Squeeze Theorem

Suppose $\forall$n $\geq N_0$
We have $x_n < y_n < z_n$
If $(x_n) \to L$ *and* $(z_n) \to L$
Then $(y_n) \to L$

Proof
Let ε > 0.
Since $(x_n) \to L$ there is some $N_1 \in \mathbb{N}$ such that $|x_{n_1} - L| < \varepsilon$
Similarly, since $(z_n) \to L$ there is some $N_2 \in \mathbb{N}$ such that $|z_{n_2} - L| < \varepsilon$ $\forall$n $\geq N_2$
Put N = max(N_1, N_2, N_0). Let n ≥ N
$L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$
This is valid because $n \geq N_0$

Ex.
$x_n = \frac{n^2}{5^n}$
Guess L = 0
Can we prove $\frac{n^2}{5^n} \leq \frac{4^n}{5^n}$
Prove $n^2 \leq 4^n$ by induction
True for n = 1
Assume $n^2 \leq 4^n$ and verify $(n+1)^2 \leq 4^{n+1}$
$\Rightarrow (n+1)^2 = n^2\left(\frac{n+1}{n}\right)^2 \leq 4^n\left(\frac{n+1}{n}\right)^2$ By induction hypothesis
Check if $\left(\frac{n+1}{n}\right)^2 \leq 4$
$$\Rightarrow \left(\frac{n+1}{n}\right)^2 = \left(1 + \frac{1}{n}\right)^2 \leq 2^2 = 4$$
Therefore $(n+1)^2 \leq 4^n \times 4 = 4^{n+1}$
By induction, $\frac{n^2}{5^n} \leq \frac{4^n}{5^n} = \left(\frac{4}{5}\right)^n$

<u>Proof</u>
Let $x_n = 0$
$\forall$n, $z_n = \left(\frac{4}{5}\right)^n$
Then $0 \leq \frac{n^2}{5^n} \leq \left(\frac{4}{5}\right)^n$ $\forall$n
$x_n \leq y_n \leq z_n$ $\forall$n $\in \mathbb{N}$
Applying the squeeze theorem, we can conclude $\frac{n^2}{5^n} \to 0$

Theorem

Every convergent sequence is bounded
Not bounded ⇒ does not converge

<u>Proof</u>
Suppose $(x_n) \to L$
Get N ∈ ℕ such that
$|x_n - L| < 1$ $\forall$n ≥ N
$L - 1 < x_n < L + 1 \Rightarrow |x_n| \leq |L| + 1$ $\forall$n $\in \mathbb{N}$
Take $C = \max(|x_1|, |x_2|, |x_3|, \ldots, |x_{N-1}|, |L| + 1) \in \mathbb{R}$
C is a bound for (x_n). Clearly $C \geq |x_j|$ *for* $j = 1, \ldots, N-1$
Furthermore, $C \geq |L| + 1 \geq |x_n|$ $\forall$n ≥ N
Therefore, (x_n) is bounded.

Limit Laws

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Increasing Sequence
Say (x_n) is increasing if $(x_{n+1} \geq x_n)$ ∀ n ∈ ℕ

Decreasing Sequence
Say (x_n) is decreasing if $(x_{n+1} \leq x_n)$ ∀ n ∈ ℕ

e.g. $x_n = \frac{1}{n}$ decreasing
$x_n = 1$ is both decreasing and increasing
$x_n = (-1)^n$ is neither increasing nor decreasing

Monotone
Say (x_n) is monotone if it is either increasing or decreasing. (Not necessarily strictly increasing/decreasing)

Monotone Convergence Theorem
Every monotonic bounded sequence converges.

Suppose $(x_n) \to L$ and $(y_n) \to K$

Addition Law
If $z_n = x_n \pm y_n$, then $(z_n) \to L \pm K$

Product Law
If $z_n = x_n y_n$, then $(z_n) \to L \times K$

Division Law
If $y_n \neq 0$ for all n and K ≠ 0
Then $(z_n) = \left(\frac{x_n}{y_n}\right) \to \frac{L}{K}$

Proof of Product Law
Take ε > 0 and look at $|x_n y_n - LK|$
$= |(x_n y_n - x_n K) + (x_n K - LK)|$
$\leq |x_n y_n - x_n K| + |x_n K - LK|$ (by triangle inequality)
$= |x_n||y_n - K| + |K||x_n - L|$

Pick N_1 so $|x_n - L| \leq \frac{\varepsilon}{2|K|}$ if n ≥ N_1
Recall, convergent sequences are bounded so there is a constant C such that $|x_n| \leq C$ ∀n
Pick N_2 so that $|y_n - K| < \frac{\varepsilon}{2C}$ ∀ n ≥ N_2
Let N = $\max(N_1, N_2)$
Then if $n \geq N$, $|x_n y_n - LK| \leq |x_n||y_n - K| + |K||x_n - L|$
$$\leq C\frac{\varepsilon}{2C} + \frac{|K|\varepsilon}{2|K|} = \varepsilon$$

Monotone Convergence Theorem
Very important, equivalent to the completeness axiom
Every monotonic bounded sequence converges.
(However, a sequence can converge even if it is not monotonic)

Proof
Assume (x_n) is increasing (the decreasing case is similar - exercise)
Look at the set of real numbers S = $\{x_1, x_2, x_3, \dots\}$
Since the sequence was bounded, S is a bounded set.

By the completeness property, S has a least upper bound, say L
Claim: $L = \lim x_n$

Let ε > 0, need to show that there is some N ∈ ℕ such that $L - \varepsilon < x_N < L + \varepsilon$
Since L is an upper bound for S, $x_n \leq L$ ∀n ∈ ℕ
Since L is the LUB, then L-ε is not an upper bound for S. So there exists some N ∈ ℕ such that $x_N > L - \varepsilon$.
But (x_n) is increasing, so $x_n \geq x_N$ if n ≥ N
$> \text{L - ε}$
Then ∀ n ≥ N, $L - \varepsilon < x_n < L + \varepsilon$
Thus (x_n) converges and $L = \lim x_n$

Note:
The proof shows that every increasing sequence that is bounded above converges to the LUB of the set $\{x_1, x_2, x_3, \dots\}$

Example
$$x_n = 1 + \frac{1}{2^2} + \dots + \frac{1}{n^2}$$
Does (x_n) converge?

$$x_{n+1} = x_n + \frac{1}{(n+1)^2}$$
So (x_n) is increasing

$$x_n = 1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}\right) + \cdots$$
$$\leq 1 + \frac{2}{4} + \frac{4}{16} + \cdots \leq 2$$
So (x_n) is a bounded sequence and by the MCT converges

Example
Recursively defined sequence
Let $a_1 = 1$
$a_{n+1} = \frac{1}{6}(2a_n + 5)$ for n ≥ 1
Does (a_n) converges and if so find the limit
$$a_2 = \frac{7}{6} > a_1$$
$$a_3 = \frac{11}{9} > a_2$$
Check if (a_n) is increasing and if $a_n \leq 2$

Prove $a_n \leq 2$ ∀n
Proceed by induction. True for a_1
Assume $a_k \leq 2$, need to prove $a_{k+1} \leq 2$

$a_{k+1} = \frac{1}{6}(2a_k + 5) \le \frac{1}{6}(2 \times 2 + 5) = \frac{9}{6} \le 2$
So by induction, $a_n \le 2$ ∀n

Prove $a_{n+1} \ge a_n$ ∀n True for n = 1
Assume $a_k \ge a_{k-1}$ and show $a_{k+1} \ge a_k$
$a_{k+1} = \frac{1}{6}(2a_k + 5) \ge \frac{1}{6}(2a_{k-1} + 5) = a_k$
By MCT, (a_n) converges, say to L

$\frac{1}{6}(2a_n + 5) \to \frac{1}{6}(2L + 5)$ as $a_{n+1} \to L$
So $L = \frac{1}{6}(2L + 5) \Rightarrow L = \frac{5}{4}$

Subsequences (B-W Theorem)

October-01-10
10:35 AM

Peak Point

Call the term x_k a peak point of our sequence if $x_k \geq x_{k+1}, x_{k+2}, \ldots$ or $x_k \geq x_n\ \forall n \geq k$

Theorem

If (x_n) is any sequence, then there is a subsequence of (x_n) which is monotonic.

Bolzano-Weierstrass Theorem

If (x_n) is a bounded sequence then it has a convergent subsequence.

Cauchy

A sequence (x_n) is called Cauchy if for every ε > 0, there is some N ∈ ℕ such that $|x_n - x_m| < \varepsilon$ $\forall n, m \geq N$

Sequence: $x_1, x_2, x_3, x_4, x_5, \ldots$
Sub-sequence: $x_2, x_5, x_6, x_8, \ldots$

Theorem

If (x_n) is any sequence, then there is a subsequence of (x_n) which is monotonic.

Let (x_n) be a sequence and suppose
$n_1 < n_2 < n_3 < \cdots$ is a list of indices
Then the sequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots = (x_{n_k})_{k=1}^{\infty}$
Is called a subsequence of (x_n)

Ex. A subsequence of $x_n = (-1)^n$ is $(1, 1, 1, 1, 1, \ldots)$ where $x_{n_k} = x_{2k}$

Proof

If (x_n) is any sequence, then there is a subsequence of (x_n) which is monotonic.

Case 1: There are infinitely many peak points
Take x_{n_1} =first peak point
x_{n_i} =ith peak point
(x_{n_k}) is a decreasing sequence and it is a subsequence of (x_n)

Case 2: There are finitely many peak points (possibly none)
Take x_{n_1} to be the first term in the sequence after the last peak point. ($x_{n_1} = x_1$ if no peak points)

Since x_{n_1} is not a peak point, there is some $n_2 > n_1$ such that $x_{n_2} > x_{n_1}$
x_{n_2} is not a peak point since it comes after the last peak point, so $\exists\, n_3 > n_2$ such that $x_{n_3} > x_{n_2}$ etc.

$(x_{n_k})_{k=1}^{\infty}$ is an increasing sequence

Bolzano-Weierstrass Theorem

(most likely on exam)

If (x_n) is a bounded sequence then it has a convergent subsequence.

By the previous proposition, (x_n) has a monotonic subsequence. It is also bounded.
By the Monotone Convergence Theorem, bounded monotone sequences converge, so this sequence converges.

Cauchy Sequences

Proposition

If (x_n) is a convergent sequence then it is a Cauchy sequence.

Proof

Let ε > 0 , we know $(x_n) \to L$
$|x_n - x_m| = |x_n - L + L - x_m| \leq |x_n - L| + |x_m - L|$ by triangle inequality
Pick N so $|x_n - L| \leq \frac{\varepsilon}{2}$ ∀n ≥ N
$|x_n - L| + |x_m - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
So (x_n) is Cauchy

Proposition

Cauchy sequences are bounded

Proof

Pick N so $|x_n - x_m| < 1$ ∀n, m ≥ N
In particular, $|x_n - x_N| < 1$
⇒ $|x_n| \leq 1 + |x_N|$ ∀n ≥ N
Take $C = \max(|x_n|, |x_2|, \ldots |x_{N-1}|, |x_N| + 1)$
This is a bound for the sequence

Convergence of Cauchy

October-01-10
11:19 AM

Theorem: Every Cauchy sequence converges

Important - equivalent to completeness property and MCT

Proof

Let (x_n) be a Cauchy sequence
Then (x_n) is a bounded sequence
By Bolzano-Weierstrass, (x_n) has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$ converges say to L.

We will prove $(x_n) \to L$
Let ε >0. Need to fine N so $|x_n - L| < \varepsilon$ if n ≥ N

2. Since (x_n) is Cauchy, we can choose N so $|x_n - x_m| < \frac{\varepsilon}{2}$ ∀n, m ≥ N

Pick N_1 so $|x_{n_k} - L| < \frac{\varepsilon}{2}$ if k ≥ N

1. Pick k ≥ N_1 and $n_k \geq$ N

Let n ≥ N
Look at $|x_n - L| \leq |x_n - x_{n_k}| + |x_{n_k} - L|$
$|x_{n_k} - L| < \dfrac{\varepsilon}{2} by\ 1.$
$|x_n - x_{n_k}| < \dfrac{\varepsilon}{2}\ by\ 2.$
$\Rightarrow |x_n - L| < \varepsilon$
Therefore, $(x_n) \to L$

Example:

Suppose (x_n) satisfies $|x_{n+1} - x_n| < \frac{1}{2^n}$ ∀n
Prove it converges
We will prove it is Cauchy
Let ε > 0
Look at $|x_n - x_m|$ (wlog m < n)
$= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots + x_{m+1} - x_m|$
$= |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|$

$$< \frac{1}{2^{n-1}} + \frac{1}{2^{n-2}} + \ \ldots + \frac{1}{2^m} = \sum_{j=m}^{n-1} \frac{1}{2^j} \leq \frac{1}{2^m} \times \frac{1}{1 - \frac{1}{2}} = \frac{2}{2^m}$$

$\frac{2}{2^m} \leq \frac{2}{2^N}$ if n>m ≥ N
Pick N so $\frac{2}{2^N} < \varepsilon$

Nice Proof:
Let ε > 0
Pick N so $\frac{2}{2^N} < \varepsilon$
If n > m ≥ N, our work shows:
$|x_n - x_m| \leq \dfrac{2}{2^m} \leq \dfrac{2}{2^N} < \varepsilon$
Hence (x_n) is Cauchy and therefore converges

Note:

It is not enough for $x_n - x_{n+1} \ \to 0$ for the sequence to be Cauchy
Example: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

Limits of Functions

October-04-10
11:02 AM

Limit L at point P

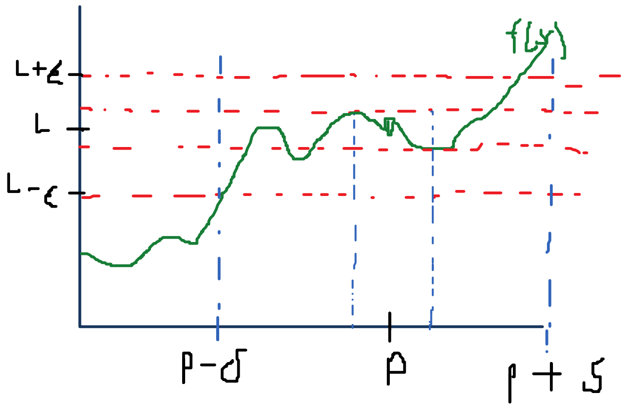
We say that f has a limit L at point p if ∀ε >0 there is some $\delta > 0$ such that whenever $0 < |x - p| < \delta$ then $|f(x) - L| < \varepsilon$

Limits of Functions

f: A -> B
Domain:
{f(x): x ∈ A} = range of f ⊆ B
Mainly A, B ⊆ R

$L - \varepsilon < f(x) < L + \varepsilon$
$p - \delta < x < p + \delta$

Say
$\lim_{x \to p} f(x) = L$
If this happens:



As ε keeps getting smaller, there will always be some δ which has the function inside that rectangle.

Example 1

f(x) = x + 2
Find $\lim_{x \to 1} f(x)$
Guess L = 3
Let ε > 0
Find δ > 0 so if $0 < |x - 1| < \delta$ then $|f(x) - 3| < \varepsilon$
$|f(x) - 3| = |x + 2 - 3| = |x - 1| < \varepsilon$
Take δ = ε

<u>Proof:</u>
Let ε > 0 and take δ = ε
If $0 < |x + 1| < \delta$ then $|x - 1| < \varepsilon$,
So $|f(x) - 3| = |x + 2 - 3| = |x - 1| < \varepsilon$

Example 2

$f(x) = x^2 + 2$ Find limit at p = 3
Guess L = 11
$\Rightarrow |f(x) - 11| = |x^2 - 9| = |x - 3||x + 3|$ and want < ε when 0 < |x-3| < δ
Take δ < 1 then 2 < x < 4
Also want $\delta < \frac{\varepsilon}{7}$

<u>Proof:</u>
Let ε > 0 and take $\delta < \min(1, \frac{\varepsilon}{7})$
If $0 < |x - 3| < \delta$, then $|x - 3| < 1 \Rightarrow 2 \le x < 4$
So $|x + 3| \le 7$
Thus $|f(x) - 11| = |x^2 + 2 - 11| = |x^2 - 9| = |(x + 3)(x - 3)| \le 7(x + 3) < 7 \times \frac{\varepsilon}{7} = \varepsilon$

Example

$$\lim_{x \to 2} \frac{(2x^2 - 8)}{x - 2} = \lim_{x \to 2} \frac{2(x^2 - 4)}{x - 2} = \frac{2(x - 2)(x + 2)}{x - 2} = 2(x + 2) = 8$$

<u>Proof:</u>
Let ε>0 and take $\delta = \frac{\varepsilon}{2}$

$$\left|\frac{2x^2 - 8}{x - 2} - 8\right| = |(2(x + 2) - 8| = |2x - 4| = 2|x - 2|$$

If $0 < |x - 2| < \delta$, then $\left|\frac{2x^2 - 8}{x - 2} - 8\right| = 2|x - 2| < 2\delta = \varepsilon$
Therefore, $\lim_{x \to 2} \frac{2x^2 - 8}{x - 2} = 8$

Example

$$\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$$

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \left|\frac{3 - x}{3x}\right| = \frac{|3 - x|}{|3x|}$$

Take δ < 1 as a start
Then |x-3| < δ ⇒ 2 < x < 4 ⇒ |3x| ≥ 6

$$\frac{|3 - x|}{|3x|} \le \frac{|x - 3|}{6} < \varepsilon$$

<u>Proof:</u>
Let ε > 0 and take $\delta = \min(6\varepsilon, 1)$
If 0 < |x-3| < δ ⇒ 2 < x < 4 , |3x| > 6
And

$$\left|\frac{1}{x}-\frac{1}{3}\right| = \frac{|x-3|}{|3x|} \le \frac{|x-3|}{6} < \frac{\delta}{6} \le \frac{6\varepsilon}{6} = \varepsilon$$

Example

Let

$$f(x) = \begin{cases} 1 \; if \; x \in \mathbb{Q} \\ 0 \; if \; x \notin \mathbb{Q} \end{cases}$$

Find

$\lim_{x \to p} f(x)$

F has no limit at any point p

<u>Proof:</u>

Suppose $\lim_{x\to p} f(x) = L$

Pick $\varepsilon = \frac{1}{2}$ and $\delta > 0$

Then then interval (p - δ , p + δ) contains

$x_1 \neq p, x_1 \in \mathbb{Q} \; and \; x_2 \neq p, x_2 \notin \mathbb{Q}$

$|(f(x_1) - f(x_2)| = |1 - 0| = 1$

$= |f(x_1) - L + L - f(x_2)| \le |f(x_1) - L| + |f(x_2) - L| < \varepsilon + \varepsilon = 1$

(If delta worked in the definition of limit of f(x))

Thus contradiction showing δ cannot work which proves there is no limit a p

Limit Laws

October-06-10
10:31 AM

$\lim_{x \to p^+} f(x)$ *or* $\lim_{x \to p^-} f(x)$
$0 < x - p < \delta \Rightarrow p < x < p + \delta$
$or\ -\delta < x - p < 0 \Rightarrow p - \delta < x < p$

Limit Laws

If
$\lim_{x \to p} f = L, \lim_{x \to p} g = K$
Then

1. $\lim_{x - p} f \pm g = L \pm K$
2. $\lim_{x \to p} fg = fg$
3. $\lim_{x - p} \frac{f}{g} = \frac{L}{K}\ if\ K \neq 0$

   Since $\lim_{x \to p} g = K \neq 0$
   Then for small δ, g(x) ≠ 0
   If $0 < |x - p| < \delta$

Squeeze Theorem

$If\ f(x) \leq g(x) \leq h(x)$ ∀x ≠ p and
$\lim_{x \to p} f = L = \lim_{(x - p)} h$
Then,
$\lim_{x - p} g = L$

Exercise

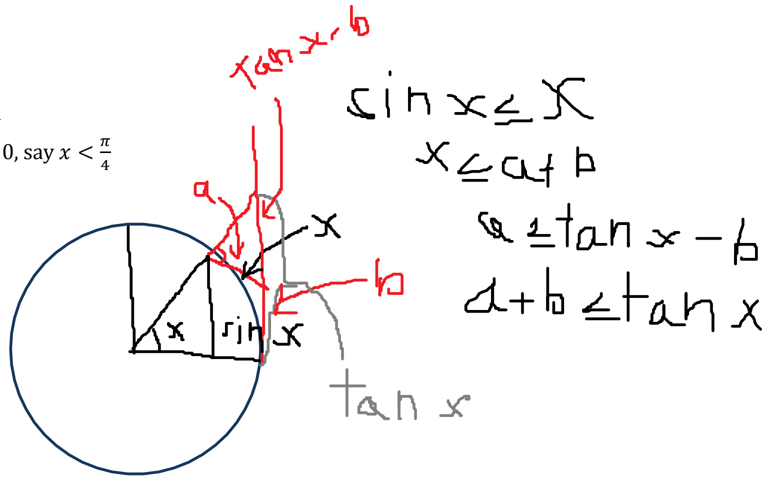
$\lim_{x \to p} f(x) = L$
If and only if
$\lim_{x \to p^+} f(x) = L\ and\ \lim_{x \to p^-} f(x) = L$

Proof of Squeeze Theorem

Given ε > 0, choose δ > 0 such that $|f(x) - L| < \varepsilon$ and $|h(x) - L| < \varepsilon$ if $0 < |x - p| < \delta$
$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$
$\Rightarrow \lim_{x \to p} g = L$

Example

$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$
First, take x > 0, say $x < \frac{\pi}{4}$



$\sin(x) < x \leq a + b \leq tanx$
$\frac{\sin x}{x} \leq 1$
$x \leq \tan x = \sin x / \cos x$
$\cos x \leq \frac{\sin x}{x}$
$\cos x \leq \frac{\sin x}{x} \leq 1\ for\ x > 0$
Take, instead, x < 0
$\cos x = \cos|x|$
$\sin x = -\sin|x|$
$x = -|x|$

$\frac{\sin x}{x} = -\frac{\sin|x|}{-|x|} = \frac{\sin|x|}{|x|}$
Therefore,
$\cos x \leq \frac{\sin x}{x} \leq 1\ \forall x \neq 0$
So by squeeze theorem,
$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

$0 \leq \sin x \leq x$ for x > 0
$\lim_{x \to 0^-} \sin x = 0$
$-x < \sin x < 0$ if x < 0
$\lim_{x \to 0^+} \sin x = 0$

Continuous Functions

October-08-10
10:32 AM

Function

$f \cdot A \subseteq \mathbb{R} \to \mathbb{R}$

(A is the domain of f)

Continuous at a

Say f is continuous at $a \in A$ if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that if $|x - a| < \delta$ and $x \in A$ then $|f(x) - f(a)| < \varepsilon$

If $A = (c, d) = \{x : c < x < d\}$ then to say f is continuous at a is the same as saying
$\lim_{x \to a} f(x) = f(a)$

Continuous

Say f is continuous if f is continuous at each $a \in A$

Proposition:

f is continuous at $x = a$ if and only if whenever (x_n) is a sequence in A and $(x_n) \to a$, then $(f(x_n)) \to f(a)$

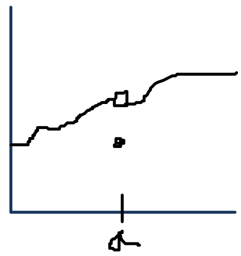
Proposition

If f is continuous at a and $f(a) > 0$ then there is an interval I containing a with $f(x) > 0 \ \forall x \in I$

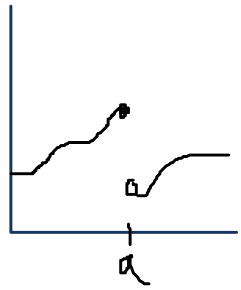
Theorem

If f, g are continuous at a then so are $f \pm g$, $f \times g$, cf for a c constant,
f/g as long as $g(a) \neq 0$

Examples of continuity or discontinuity



$\lim_{x \to a} f(x) = L \neq f(a)$



Jump discontinuity - no limit at x =a

Look at

$$y = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\frac{1}{x} = 2\pi k$$

$$x = \frac{1}{2\pi k}$$

This function has no limit at $x = 0$ so it is discontinuous

Proposition:

f is continuous at $x = a$ if and only if whenever (x_n) is a sequence in A and $(x_n) \to a$, then $(f(x_n)) \to f(a)$

Proof

"$\Rightarrow$"
Assume f is continuous at a
Take (x_n) a sequence in A with $(x_n) \to a$
RTP $(f(x_n)) \to f(a)$
RTP $\forall \varepsilon > 0$ there exists N such that $|f(x_n) - f(a)| < \varepsilon$ if $n \geq N$
Since f is continuous at a, $\exists \delta > 0$ s.t. $|f(x) - f(a)| < \varepsilon$ if $|x - a| < \delta$ and $a \in A$

Since $(x_n) \to a$, we know there is some index N so $\forall n \geq N$ $|x_n - a| < \delta$

Take this choice of N. If $n \geq N$ then $|x_n - a| < \delta$ and so by the continuity and the choice of delta, we have
$|f(x_n) - f(a)| < \varepsilon \Rightarrow (f(x_n)) \to f(a)$

"$\Leftarrow$"
Suppose f is not continuous at a
There is some $\varepsilon > 0$ so no δ will "work"
This means for each choice $\delta > 0$, there is a "bad" x, meaning
$|x - a| < \delta$ but $|f(x) - f(a)| \geq \varepsilon$
Do this for each $\delta = 1/n$, $n \in \mathbb{N}$
For each δ, get "bad" x and call it x_n
(x_n) is a sequence from A and $|x_n - a| < \frac{1}{n}$
So $(x_n) \to a$
So we also know that $|f(x_n) - f(a)| \geq \varepsilon$
Therefore, $(f(x_n))\,! \to f(x)$
This contradicts the second statement.

Examples of continuity/discontinuity

$$f(x) = \begin{cases} 1 \ if \ x \in \mathbb{Q} \\ 0 \ if \ x \notin \mathbb{Q} \end{cases}$$

Not continuous at any point because it has no limit at any point

Ex

If $g(x) = 1$ at every $x \in \mathbb{Q}$ and g is continuous, then $g(x) = 1$ for every $x \in \mathbb{R}$

Proof:
If a is rational then take $(x_n) \to a$ with all $x_n \in \mathbb{Q}$
Say g is continuous at a, by proposition
$g(x_n) \to g(a)$
But $g(x_n) = 1$ ∀n
So $g(a) = 1$ ∀a

Example Continuous on Irrationals and Discontinuous on Rationals

$$f(x) = \begin{cases} \frac{1}{n}\ if\ x \in \mathbb{Q} \\ 0\ if\ x \notin \mathbb{Q}, x = 0 \end{cases} \quad and\ x = \frac{m}{n} where\ n \in \mathbb{N}\ \gcd(m, n) = 1$$

F is discontinuous at any a ∈ $\mathbb{Q} \setminus \{0\}$
Why?
Take ε < 1/n. Then $\left|f(x) - \frac{1}{n}\right| = \left|0 - \frac{1}{n}\right| > \varepsilon$

If $a \notin \mathbb{Q}$ then there exists $x \in (a - \delta, a + \delta)$ for any δ < 0
Take $a \notin \mathbb{Q}, f(a) = 0$
Want $|f(x) - f(a)| < \varepsilon\ \forall x \in (a - \delta, a + \delta)$
Take N ∈ ℕ so $\frac{1}{N} < \varepsilon$
If $x = \frac{m}{n} with\ n \geq N, then\ 0 < f(x) = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$
So $|f(x) - 0| < \varepsilon$

Temporarily take δ = 1 and consider $(a - 1, a + 1)$
There are only finitely many rations of the form $\frac{m}{n}$ with n ≤ N in the interval (a-1, a+1)
Now take δ < 1 so (a - δ , a + δ) misses all of these finitely many points $\frac{m}{n}$ with n ≤ N
So if x ∈ (a - δ, a + δ) either x ∉ ℚ so f(x) = 0 = f(a) or $x = \frac{m}{n}$ with n ≥ N and then
$f(x) = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$
Either way $|f(x) - f(a)| < \varepsilon\ \forall x \in (a - \delta, a + \delta)$
Thus if is continuous at a ∉ ℚ

Comment

Is there a function continuous on rationals and discontinuous on irrationals?
No, but very difficult to show.

Proposition

If f is continuous at a and f(a) > 0 then there is an interval I containing a with f(x) > 0 ∀ x ∈ I

Proof

Take $\varepsilon = f(a) > 0$.
Get δ > 0 so |x - a| < δ implies |f(x) - f(a)| < ε ⇔ f(a) - ε < f(x) < f(a) + ε
⇒ f(x) > 0 ∀ x ∈ (a -δ, a + δ)

Theorem

If f, g are continuous at a then so are f± g, f × g, cf for a c constant,
f/g as long as g(a) ≠ 0

Proof

Just use limit laws for sequences in functions

Ex:

Polynomials are continuous functions.
To see this, note p(x) = x is continuous
Then $p(x) = x^n$ is continuous ∀ n ∈ ℕ
And $p(x) = c_n x^n$ is continuous ∀ n ∈ ℕ
Sum of continuous functions are continuous so p(x) = $a_n x^n + \cdots a_1 x + a_0$ is cont.
Rational functions = $\frac{poly\ p(x)}{poly\ q(x)}$
Continuous on its domain, or at all a ∈ ℝ except where q(a) = 0

Ex.

$$f(x) = \begin{cases} 3x^2 + 1 if\ x > 0 \\ 1 - x\ if\ x \leq 0 \end{cases}$$

$f(a) = 1$
$\lim_{x \to a^+} f = \lim_{x \to a^+} 3x^2 + 1 = 1$
$\lim_{x \to a^-} f = \lim_{x \to a^-} 1 - x = 1$
Continuous everywhere.
The case at x = 0 is cont. because $3x^2, 1 - x$ are cont. everywhere
And $\lim_{x \to 0} f(x) = f(a)$ as shown

Composition of Continuous Functions

f: $A \subseteq \mathbb{R} \to \mathbb{R}$ range $f = B \subseteq \mathbb{R}$
g: $B \subseteq \mathbb{R} \to \mathbb{R}$

$g \circ f: A \to \mathbb{R}$
$g \circ f(x) = g(f(x))$

Theorem

If f is continuous at a and g is continuous at f(a), then $g \circ f$ is continuous at a.

Proof

Equivalent to prove if $(x_n) \to a$ then $(g \circ f(x_n)) \to g \circ f(a)$
Since $(x_n) \to a$ and f is continuous at a, $(f(x_n)) \to f(a)$
But g is continuous at a so whenever $(y_n) \to f(a) then\ (g(y_n)) \to g(f(a))$

- Apply with $y_n = f(x_n)$
- So $(g \circ f(x_n)) = (g(f(x_n)) \to g(f(a)) = g \circ f(a)$

Alternate Proof

Let $\varepsilon > 0$ and find $\delta > 0$ so $|x - a| < \delta$ implies $|g \circ f(x) - g \circ f(a)| < \varepsilon$
$\Rightarrow |g(f(x)) - g(f(a))| < \varepsilon$
Know g is continuous at f(a) so there exists $\delta_1 > 0$ such that $|g(y) - g(f(a))| <$
$\varepsilon\ if\ |y - f(a)| < \delta$
Apply this with $y = f(x)$
Since f is continuous at a, there will be some $\delta_2 > 0$ such that $|f(x) - f(a)| < \delta_1$ when
$|x - a| < \delta_2$

Take $\delta = \delta_2$ Then $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \delta_1 \Rightarrow |g(f(a)) - g(f(x))| < \varepsilon$
Therefore, $g \circ$ f is continuous at a.

Intermediate Value Theorem

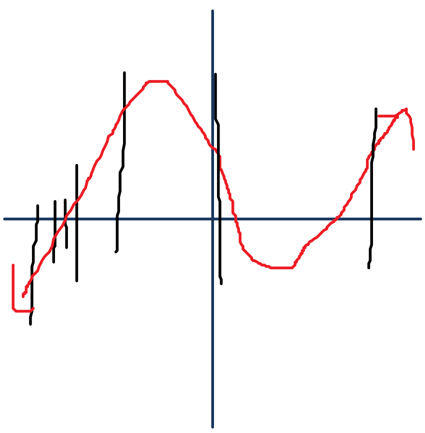
October-15-10
10:32 AM

Intermediate Value Theorem

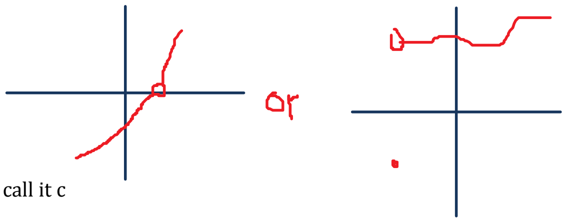
Suppose f · [a, b] → $\mathbb{R}$ is continuous
And f(a) < 0 and f(b) > 0 then there is some c ∈ [a, b] with f(c)=0

Corollary

If f ·[a, b] → $\mathbb{R}$ is continuous and f(a) < f(b) then for every z with f(a) < z < f(b) then there is some c ∈ [a, b] such that z = f(c)



Continuity condition is essential:



Proof of Intermediate Value Theorem

Let A = {x ∈ [a, b]: f(x) < 0}
a ∈ A so A is non-empty
A ⊆ [a,b] so A is bounded
By completeness property, A has a LUB, call it c
$c \geq a$ as c is an UB for A and a ∈ A
$c \leq b$ because b is also an UB for A and c = LUB(A)
So c ∈ [a, b]
$c - \frac{1}{n} < c$ so it is not an UB for A
So $\exists\, x_n \in A$ with $c - \frac{1}{n} < x_n < c$
Of course, $f(x_n) < 0$
$$|x_n - c| < \frac{1}{n} \to 0 \ as\ n \to \infty$$
Hence $(x_n) \to c$
Since f is continuous at c, this implies that
$(f(x_n)) \to f(c)$
Since $f(x_n) < 0\ \forall n \Rightarrow f(c) \leq 0$

This shows c ≠ b since f(b) > 0
And so $c + \frac{1}{N} < b$ for large enough N
So $c + \frac{1}{N} \in [a, b]\ \forall n \geq N$
$c + \frac{1}{n} > c$ so $c + \frac{1}{n} \notin A$
Hence $f\left(c + \frac{1}{n}\right) \geq 0\ \forall n \geq N$
$$\left(c + \frac{1}{n}\right)_{n=N}^{\infty} \to c$$
By continuity of f,
$$0 \leq f\left(c + \frac{1}{n}\right) \to f(c)$$
So $f(c) \geq 0$

Since $f(c) \leq 0\ and\ f(c) \geq 0\ then\ f(c) = 0$
■

Corollary

If f· [a, b] → $\mathbb{R}$ is continuous and f(a) < f(b) then for every z with f(a) < z < f(b) then there is some c ∈ [a, b] such that z = f(c)

Proof

Let g(x) = f(x) - z
g is continuous
g(a) = f(a) - z < 0
g(b) = f(b) - z > 0
By I.V.T there is some c ∈ [a, b] with g(c) = 0 = f(c) - z, so f(c) = z

Applications:

Any odd degree polynomial has at least one real root.

Proof

$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, where\ n\ is\ odd$
WE want to prove there is some c such that $p(c) = 0$
Wlog assume a_n is 1
$$p(x) = x^n\left(1 + \frac{a_{n+1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)$$
Pick N so large that $\left|\frac{a_j}{x^{n-j}}\right| < \frac{1}{2n}\ \forall$ j = 0, ..., n-1
$$\left|\frac{a_{n+1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right| \leq \left|\frac{a_{n+1}}{x}\right| + \cdots + \left|\frac{a_1}{x^{n-1}}\right| + \left|\frac{a_0}{x^n}\right| \leq \frac{1}{2n} n\ if\ |x| \geq N$$
$$p(N) = N^n\left(1 + \frac{a_{n+1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \geq N^n\left(\frac{1}{2}\right) = \frac{N^n}{2} > 0$$
$$p(-N) = (-N)^n\left(1 + \frac{a_{n+1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \leq -\frac{1}{2}N^n < 0$$
p is continuous on [-N, N]
So by I.V.T p has a root in [-N, N]

Bisection Method of Finding Roots

Take a function on [a, b] where a < 0 and b > 0
Keep splitting the domain and taking the half where the sign of the two bounds are opposite.

Bounded Functions + EVT

October-18-10
10:29 AM

Bounded

Say f is bounded above if there exists M such that
$f(x) \leq M \; \forall x$
Say f is bounded if it is both bounded above and below

Extreme Value Theorem

Suppose $f[a, b] \to \mathbb{R}$ is continuous
Then there are $c, d \in [a, b]$ such that
$f(c) \leq f(x) \leq f(d) \; \forall x \in [a, b]$

In particular, f is bounded and f achieves minimum and maximum values.

Examples

$f: \mathbb{R} \to \mathbb{R}$
$f(x) = x$
- Not bounded either above or below

$f: (0, 1] \to \mathbb{R}$
$f(x) = \frac{1}{x}$
- Bounded below but not above
- Has a minimum

$f: (1, \infty) \to \mathbb{R}$
$f(x) = \frac{1}{x}$
-Bounded, however no minimum or maximum

Extreme Value Theorem

Suppose $f[a, b] \to \mathbb{R}$ is continuous
Then there are $c, d \in [a, b]$ such that
$f(c) \leq f(x) \leq f(d) \; \forall x \in [a, b]$

In particular, f is bounded and f achieves minimum and maximum values.

Proof of Extreme Value Theorem

Uses Bolzano-Weierstrass Theorem (any bounded sequence has a convergent subsequence)
Fact - If $(x_n) \to L$ then every subsequence of $(x_n) \to L$)

1. First show that f is bounded

Suppose f is not bounded above.
Then $\forall$ n $\in \mathbb{N}$, there is some $x_n \in [a, b]$ with $f(x_n) > n$. Consider the sequence (x_n). It is bounded.
By B-W Theorem, there is a convergent subsequence $(x_{n_k}) \to L \in [a, b]$
By continuity of f, $f(x_{n_k}) \to f(L)$
By construction sequence, $f(x_{n_k}) > n_k$ *so* $f(x_{n_k})$ is unbounded and therefore cannot be converging because every convergent sequence is bounded.
This is a contradiction, so f is bounded above.
Similarly, we can prove f is bounded below $\Rightarrow$ f is bounded

2. Look at S = {f(x) : x ∈ [a, b]}

This is a non-empty set, and a bounded set by 1.
So S has a LUB and a GLB. Call LUB(S) = z
Then $f(x) \leq z \; \forall x \; [a, b]$
And $\forall n \in \mathbb{N}, x \in [a, b]$ with $f(x_n) > z - \frac{1}{n}$
$|f(x_n) - z| \leq \frac{1}{n} \to 0 \text{ as } n \to \infty$
So $(f(x_n)) \to z$

Sequence (x_n) is a bounded sequence, so by B-W Theorem it has a convergent subsequence. Say $(x_{n_k}) \to d \in [a, b]$
By continuity of f at d, $f(x_{n_k}) \to f(d)$
Since $\left(f(x_{n_k})\right)$ is a subsequence of $(f(x_n))$ which converges to z $\Rightarrow \left(f(x_{n_k})\right) \to z$
But limits are unique, therefore $z = f(d)$
In other words, $f(d) \geq f(x) \; \forall x \in [a, b]$

Showing minimum value is left as an exercise.

Inverse Functions

October-18-10
11:01 AM

One-to-one Functions (Injections)

Say f is 1-1 if whenever $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

In other words, pass the horizontal line test.

Increasing (or Strictly Increasing)

Say f is (strictly) increasing if whenever $x_2 > x_1$ then $f(x_2)\ (>) \geq f(x_1)$

Theorem

If $f: |a, b| \to \mathbb{R}$ is continuous and invertible, then f is either strictly increasing or strictly decrasing.

Examples

$y = x^2$, $not\ 1-1$
$y = \sin x$, $not\ 1-1$
$y = x^3 + 1$, $yes\ 1-1$

One-to-One Functions have inverses
$y \in Range\ f$
Define $f^{-1}(y) = x$ when $f(x) = y$ (unique choice of x)

Usually we write $f^{-1}(x) = y$ when $f(y) = x$
$fof^{-1}(x) = f(y) = x$
$f^{-1}of(x) = x$
$\Rightarrow f\ of^{-1} = f^{-1}of = Identity\ Function$

Ex. $x = y^3 + 1 \Rightarrow y = \sqrt[3]{x-1}$

$Range\ f = Domain\ f^{-1}$
$Range\ f^{-1} = Domain\ f$

Theorem

If $f: |a, b| \to \mathbb{R}$ is continuous and invertible, then f is either strictly increasing or strictly decrasing.

Proof

Notice f(a) ≠ f(b) since f is 1-1
Assume f(a) < f(b) (Leave f(a) > f(b) as exercise)
And we will show f is strictly increasing.

Assume f is not strictly increasing.
Then there is some y > x, but f(y) ≤ f(x)
Case 1: $x \neq a$
Clearly f(x) ≠ f(a) because otherwise the function would not be 1-1 and therefore not be invertible

1. f(x) > f(a)

By I. V. T on [a, x] f takes on every value in [f(a), f(x)]
Similarly on [x, y] f takes on every value in [f(y), f(x)]
These intervals [f(a), f(x)] and [f(y), f(x)] overlap.
So values in overlap are taken on at least twice. Contradicts that f is 1-1

2. f(x) < f(a) same thing

Case 2: x = a
There is some y > x such that $f(y) \leq f(x) = f(a)$
y ∈ (a, b] since x ∈ [a,b], and clearly y ≠ b since f(y)≤ f(a)<f(b)
so y ∈ (a, b)
By I.V.T on [a, y] f takes on every value in [f(y), f(a)]
Similarly, on [y, b] f takes on every value in [f(y), f(b)]
These intervals overlap on [f(a), f(b)], contradicts that f is 1-1

Consequence:

If $f: |a, b| \to \mathbb{R}$ is 1-1 and continuous then Range f = [c, d]

Proof:

Either f is strictly increasing or strictly decreasing. Say f is increasing.
Then Range f ⊆ [f(a), f(b)] and we get the entire interval on the range by the Intermediate Value Theorem

Continuity of f^{-1}

October-20-10
10:34 AM

Theorem

If $f: |a,b| \to \mathbb{R}$ is continuous and 1-1
Then Range f= [c,d] for some c, d and $f^{-1}|c,d| \to [a,b]$ is continuous

Theorem

If $f: |a,b| \to \mathbb{R}$ is continuous and 1-1
Then Range f= [c,d] for some c, d and $f^{-1}|c,d| \to [a,b]$ is continuous

Proof

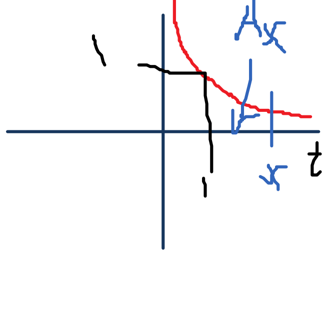
Suppose $(x_n) \to x_0$ where $x_n \in |c,d|$
Want to prove $f^{-1}(x_n) \to f^{-1}(x_0)$
$Let\ y_n = f^{-1}(x_n), y_0 = f^{-1}(x_0)$
We know $f(y_n) = x_n$ and $f(y_0) = x_0$
Also, $y_n, y \in |a,b|$
We proceed by contradiction and suppose $y_n \nrightarrow y_0$
This means there exists some $\varepsilon > 0$ such that for every N there is some $n \geq N$ with $|y_n - y_0| \geq \varepsilon$
Pick n_1 so $|y_{n_1} - y_0| \geq \varepsilon$
Think of $N = n_1 + 1$. Pick $n_2 \geq N = n_1 + 1$ so $|y_{n_2} - y_0| \geq \varepsilon$
Having picked $y_{n_1}, \dots y_{n_k}$, put $N = n_k + 1$ And pick $n_{k+1} \geq N$ so $|y_{n_{k+1}} - y_0| \geq \varepsilon$
Gives a subsequence (y_{n_k}) with the property that $|y_{n_k} - y_0| \geq \varepsilon\ \forall k$
All $y_{n_k} \in |a,b|$ so (y_{n_k}) is a bounded sequence.
By the Bolzano-Weierstrass Theorem this has a convergent subsequence call it $\left(y_{n_{k_j}}\right)$ with limit t.
$t \neq y_0$ because of the construction of (y_{n_k})

$f\left(y_{n_{k_j}}\right) = x_{n_{k_j}} \to x_0$
By continuity of f, $f\left(y_{n_{k_j}}\right) \to f(t)$
By uniqueness of limits, $f(t) = x_0$
But $x_0 = f(y_0) \Rightarrow f(t) = f(y_0)$ and since f is 1-1 $t = y_0$
This is a contradiction, proving $y_n \to y_0 \Rightarrow f^{-1}(x_n) \to f^{-1}(x_0)$
So f^{-1} is continuous.

"Inverse" Trig Functions

Sin(x) is not invertible but sin(x) restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is invertible.
The inverse of this restriction is $arc\sin(x)$
$\arcsin(x) = \theta \in \left|-\frac{\pi}{2}, \frac{\pi}{2}\right|, x \in |-1,1|\ with \sin(\theta) = x$
$\arccos(x)$is the inverse of cos restricted to $|0,\pi|$

$\tan(x) = \frac{\sin(x)}{\cos(x)}$ so tan(x) is periodic every π
$\arctan(x)$ is the inverse of tan restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$
Domain of arctan(x) is $\mathbb{R}$ and Range of arctan(x) = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$


1
Ax
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t
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Natural Logarithm

November-17-10
10:45 AM

Properties of Logarithm Function

1. ln ab = ln a + ln b
2. $\ln\frac{1}{a} = -\ln a$
3. $\ln(x^r) = r\ln x$

Logarithm Function

Consider $y = \frac{1}{t}$ for t ≥ 0
For x > 0, let $A_x = area\ bounded\ by\ the\ curve\ y = \frac{1}{t}, t\ axis\ and\ the\ vertical\ lines\ t = 1, t = x$
Define $\ln x = \begin{cases} A_x if\ x \geq 1 \\ -A_x if\ x < 1 \end{cases}$
$\ln 1 = 0$
$\ln x \geq 0$ if x ≥ 1
$\ln x < 0$ if x < 1
$\ln x$ is strictly increasing and so it is invertible.

Properties

1. $\ln(x^r) = r\ln x\ if\ r \in \mathbb{Q}$
   a. $\ln\left(\frac{1}{a}\right) = -\ln a$
2. $\ln x \to \infty$ as $x \to \infty$
   ∀ M ∈ ℕ there is some N so that if x ≥ N then $\ln x \geq M$

Proof of 2.

Consider $\ln 2^n = n\ln 2$
If $x > 2^n$, then $\ln x > \ln 2^n$, so if $\ln 2^n > M$ then $\forall x \geq 2^n, \ln x \geq \ln 2^n > M$

$\exp(x)$ is the inverse function of $y = \ln x$

Range of $\ln = (-\infty, \infty) =$ Domain of exp

$\ln x \to -\infty\ as\ x \to 0^+$

Domain $\ln = (0, \infty) =$ Range of exp

$\exp 0 = 1\ since\ \ln 1 = 0$
$\ln(\exp x) = x = \exp(\ln x)$

<u>Fact</u>
$\exp(xr) = (\exp x)^r$ for $r \in \mathbb{Q}$
Proof:
Let y = LHS = exp(xr)
$\ln y = \ln(\exp xr) = xr$

RHS= $(\exp x)^r$
$\ln RHS = \ln((\exp x)^r) = r\ln(\exp x) = rx$
So ln LHS = ln RHS
But ln is 1-1 so LHS = RHS

Take $x = 1$. Gives $\exp r = (\exp 1)^r = e^r$ ∀ r ∈ ℚ
Call e = exp 1
For any $x \in \mathbb{R}$,
Define $e^x = \exp(x)$
This is consistent when $x \in \mathbb{Q}$
This gives us a definition for an irrational power

Define a^x for any a > 0, x ∈ ℝ
Set $a^x = e^{x\ln a} = \exp(x\ln a)$
Consistent with what we know a^x is when $x \in \mathbb{Q}$

Proof of Properties

Uses the future

Proof of 1

Let $f(x) = \ln xb - \ln x - \ln b$
Notice $f(1) = 0$
$f'(x) = \frac{1}{xb}b - \frac{1}{x} = 0$
So by the corollary to the Mean Value Theorem, f is constant
Hence $f(x) = f(1) = 0\ \forall x$

Proof of 3

Let $g(x) = \ln x^r - r\ln x$
Notice $g(1) = 0$
$g'(x) = \frac{1}{x^r}rx^{r-1} - \frac{r}{x} = 0$
So $g(x) = 0\ \forall x$

2 follows from 3

Inequalities

$\ln x \leq x - 1 \; \forall x > 0$
Let $f(x) = \ln x - (x - 1)$
$f(1) = 0$
$f'(x) = \frac{1}{x} - 1 = \frac{1-x}{x}$

$(0, 1]$	$[1, \infty)$
f' > 0	f' < 0

Analysis of f' shows x=1 is the maximum value of f.
So $f(x) \leq f'(x) = 0 \; \forall x$

$(1+x)^{\frac{1}{p}} < 1 + \frac{1}{p}x \; \forall x > 0, p > 1$
Let $f(x) = (1+x)^{\frac{1}{p}} - 1 - \frac{1}{p}x$
$f(0) = 1 - 1 - 0 = 0$
$f'(x) = \frac{1}{p}(1+x)^{\frac{1}{p}-1} - \frac{1}{p} = \frac{1}{p}\left(\frac{1}{(1+x)^{1-\frac{1}{p}}} - 1\right) > 0 \; for \; x > 0, p > 1$
since $(1+x)^{1-\frac{1}{p}} > 1$
By the Increasing Function Theorem, f is strictly decreasing on $[0, \infty)$
Therefore $f(0) = 0 > f(x) \; \forall x > 0$

Differentiation

October-25-10
10:29 AM

Differentiable at a

Say f is differentiable at a (or x=a at (a, f(a))) if
$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$
exists.

Alternate definition of differentiability

Sometimes we put $x = a + h$ Then $h \to 0$ is the same as $x \to a$ so we can write
$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a)$$

Derivative

When
$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists we denote this by $f'(a)$

$f'(a)$ is called the derivative of f at a.

This defines a function f', called the derivative of f, which is defined on all the points at which f is differentiable.

Differentiable

Say f is differentiable if it is differentiable at every point in its domain.

Tangent Line

This is the line through the point $(a, f(a))$ with slope $f'(a)$
Equation:
$y - f(a) = f'(a)(x-a)$

Theorem

If f is differentiable at a, then f is continuous at a.

Find slope with secant line on f through f(a), f(a+h)

The slope of the secant line is $\frac{f(a+h)-f(a)}{h}$ average rate over [a, a+h]

Examples

$f(x) = mx + b$
$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{m(a+h)+b-ma-b}{h} = \lim_{h\to 0}\frac{mh}{h} = m = f'(a)$$

$f(x) = x^3$
$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{(a+h)^3-a^3}{h} = \lim_{h\to 0}\frac{a^3+3a^2h+3ah^2+h^3-a^3}{h} = \lim_{h\to 0}3a^2+3ah+h^2$$
$= 3a^3 = f'(a)$

$f(x) = |x|$
$$f'(a) = \lim_{h\to 0}\frac{|a+h|-|a|}{h}$$

1. $a > 0$
$$\lim_{h\to 0}\frac{a+h-a}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$
2. $a < 0$
$$\lim_{h\to 0}\frac{-a\ -h+a}{h} = \lim_{h\to 0}-\frac{h}{h} = -1$$
3. $a = 0$
$$\lim_{h\to 0}\frac{|0+h|-|0|}{h} = \lim_{h\to 0}\frac{|h|}{h}$$
$$\lim_{h\to 0^+}\frac{|h|}{h} = 1,\ \lim_{h\to 0^-}\frac{|h|}{h} = -1$$
The limit does not exist, so f is not differentiable at 0.

$f(x) = |x|$ is an example of a function that is continuous but not differentiable.

Theorem

If f is differentiable at a, then f is continuous at a.

Proof

RTP
$\mathit{lim}_{x\to a} f(x) = f(a)$
Equivalently, prove
$\lim_{x\to a}(f(x)-f(a)) = 0$
$$f(x)-f(a) = \frac{f(x)-f(a)}{x-a}(x-a)$$
Since both $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ *and* $\lim_{x\to a}(x-a)$ exist, by the product rule for limits
$\lim_{x\to a} f(x)-f(a) = f'(a)\times 0 = 0$
So f is continuous at a

Only one way, examples even exist of functions that are continuous at every point but differentiable at no point.

Examples

Consider
$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), if\ x \neq 0 \\ 0\ if\ x = 0\end{cases}$$
Does $f'(0)$ exist?
$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{\left(h\sin\left(\frac{1}{h}\right)-0\right)}{h} = \lim_{h\to 0}\sin\left(\frac{1}{h}\right)$$
Which has no limit

$$g(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right), if\ x \neq 0 \\ 0\ if\ x = 0\end{cases}$$
$$g'(0) = \lim_{h\to 0}\frac{\left(h^2\sin\left(\frac{1}{h}\right)-0\right)}{h} = \lim_{h\to 0}h\sin\frac{1}{h}$$
Cannot apply product law of limits
$$|-h| \le \left|h\sin\left(\frac{1}{h}\right)\right| \le |h|$$
$$\lim_{h\to 0}\pm|h| = 0$$
So by the squeeze theorem for functions, $\lim_{h\to 0}h\sin\left(\frac{1}{h}\right) = 0$
So $g'(0) = 0$

Squeeze of Absolute Values

Say $|F(x)| \le |G(x)|$ *and*
$\lim_{x\to a}G(x) = 0 \Rightarrow \lim_{x\to a}|G(x)| = 0$
Means given any $\varepsilon > 0$
$\exists\delta > 0$ such that for $|x-a| < \delta$ then $|G(x)-0| < \varepsilon$

$|F(x)| \leq |G(x)| < \varepsilon$ if $|x-a| < \delta$
So $\lim_{x \to a} |F(x)| = 0$

Exercise

$$f(x) = \begin{cases} x^{\alpha} \sin\left(\frac{1}{x^{\beta}}\right), x \neq 0 \\ 0, x = 0 \end{cases}$$

When is f(x) differentiable?

Common Derivatives

October-27-10
10:34 AM

sin x

$$\frac{d}{dx}\sin x = \cos x$$

sin x is continuous

ln x

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

ln x is continuous
exp x is continuous

Derive Derivative of Sine Function

$$f'(a) = \frac{d}{dx}\sin x\Big|_{x=a} = \lim_{h\to 0}\frac{\sin(a+h)-\sin(a)}{h} = \lim_{h\to 0}\frac{\sin a\cos h + \sin h\cos a - \sin a}{h}$$

$$= \lim_{h\to 0}\frac{\sin a\,(\cos h - 1)}{h} + \frac{\sin h\cos a}{h}$$

$$\lim_{h\to 0}\frac{\sin h}{h}\cos a = \cos a$$

$$\lim_{h\to 0}\frac{\cos h - 1}{h}\times\frac{\cos h + 1}{\cos h + 1} = \lim_{h\to 0}\frac{\cos^2 h - 1}{h\cos h} = \lim_{h\to 0} -\frac{\sin^2 h}{h\,(\cos h + 1)}$$

$$\left|-\frac{\sin h}{h}\times\sin h\times\frac{1}{\cos h + 1}\right|$$

Pick $\delta > 0$ so $\left|\frac{\sin h}{h}\right| \le 2$ if $|h| < \delta$

$\Rightarrow |\sin h| \le 2|h|$ if $|h| < \delta$

$$\cos h \ge 0 \text{ if } h \in \left|-\frac{\pi}{2},\frac{\pi}{2}\right|$$

So $\cos h + 1 \ge 1 \text{ if } h \in \left|-\frac{\pi}{2},\frac{\pi}{2}\right|$

$$\Rightarrow \left|\frac{1}{\cos h + 1}\right| \le 1 \text{ if } h \in \left|-\frac{\pi}{2},\frac{\pi}{2}\right|$$

$$0 \le \left|\frac{\sin h}{h}\times\sin h\times\frac{1}{\cos h + 1}\right| \le 2\times 2|h|$$

If $|h| \le \delta$ and $2\times 2|h| \to 0$ as $h \to 0$

By squeeze theorem,

$$-\frac{\sin^2 h}{h}\times\frac{1}{\cos h + 1} \to 0 \text{ as } h \to 0$$

Therefore,

$$\frac{d}{dx}\sin x\Big|_{x=a} = \lim_{h\to 0}\frac{\sin a\,(\cos h - 1)}{h} + \frac{\sin h\cos a}{h} = \cos a$$

Since both terms have limits, so the addition rule of limits applies

So sin x is differentiable and $\frac{d}{dx}\sin x = \cos x$

Corollary

sin x is a continuous function.

Derive Derivative of Log Function

Definition of ln: The area A_x from t=1 to t=x under y=1/t

$$\ln x = \begin{cases} A_x \text{ if } x \ge 1 \\ -A_x \text{ if } x < 1\end{cases}$$

Case x > 1

$$\lim_{h\to 0}\frac{\ln(x+a) - \ln x}{h}$$

wlog, $x + h > 1$

Case a: $h > 0$

$$\lim_{h\to 0}\frac{A_{x+h} - A_x}{h} = \frac{\text{area under } y = \frac{1}{t} \text{ between } x \text{ and } x+h}{h}$$

$$\frac{1}{x+h}\times h \le \text{area} \le \frac{1}{x}\times h$$

$$\frac{1}{x+h} \le \frac{\text{area}}{h} \le \frac{1}{x}$$

By squeeze theorem, $\frac{\text{area}}{h} \to \frac{1}{x}$ as $h \to 0$

$$\lim_{h\to 0^+}\frac{(\ln(x+h) - \ln x)}{h} = \frac{1}{x}$$

Case b: h <0
Left as exercise but the same thing

Case x < 1

Left as exercise

Corollary

ln x is a continuous function

Corollary
exp is a continuous function

This is also the proof of the fundamental theorem of calculus

Rules for Differentiation

October-27-10
11:02 AM

Suppose f and g are differentiable at a

1. $f \pm g$ are differentiable at a and $(f+g)'(a) = f'(a) \pm g'(a)$
   Proof left as exercise

2. Product Rule

$f \cdot g$ is differentiable at a and $(fg)'(a) = f'(a)g(a) + g'(a)f(a)$

Proof

$$(fg)'(a) = \lim_{h\to 0} \frac{(f \cdot g\,(a+h) - f \cdot g(a))}{h}$$
$$= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)}{h}$$
$$= \lim_{h\to 0} g(a+h) \times \frac{f(a+h)-f(a)}{h} + f(a) \times \frac{g(a+h)-g(a)}{h}$$
$$= g(a)f'(a) + f(a)g'(a)$$

g is continuous so (g(a+h)) → g(a) as (a+h) → a

Corollary

$Let\ f(x) = x^n$. Then $f'(x) = nx^{n-1}$

Proof

$g(x) = x \qquad g'(x) = 1\ \forall x$

This f is differentiable since it is the product of differentiable g

Proceed by induction on n. Have result for n = 1

Assume $\frac{d}{dx}x^{n-1} = (n-1)x^{n-2}$

$$\frac{d}{dx}x^n = \frac{d}{dx}x \times x^{n-1} = 1 \times x^{n-1} + (n-1)x^{n-2}x = x^{n-1}(1 + x - 1) = nx^{n-1}$$

Corollary

If p(x) is a polynomial then p is differentiable. (Since polynomials are just linear combinations of x^n for various n

3. If g is differentiable at a and g(a) ≠ 0 then 1/g is differentiable at a and

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{(g(a))^2}$$

Corollary

$$\frac{d}{dx}x^{-n} = -nx^{-n-1}\ for\ n \in \mathbb{N}$$

Proof: Write $x^{-n} = \frac{1}{x^n}$

4. Quotient Rule:

$\frac{f}{g}$ is differentiable at a if f, g are differentiable at a and g(a) = 0

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - g'(a)f(a)}{(g(a))^2}$$

Proof

$$\left(f \cdot \frac{1}{g}\right)' = f'(a) \times \frac{1}{g(a)} + f(a)\left(\frac{1}{g}\right)'(a) = f'(a)\frac{1}{g(a)} + f(a)\frac{-g'(a)}{(g(a))^2}$$
$$= \frac{f'(a)g(a) - g'(a)f(a)}{(g(a))^2}$$

5. Chain Rule

Let $f \cdot A \to \mathbb{R}$ and $g \cdot B\ \to \mathbb{R}$

Suppose f is differentiable at a and g is differentiable at f(g(a)), then gof is differentiable at a and $(g\ of)' = g'(f(a))f'(a)$

Proof

To prove this we need to look at

$$\lim_{x\to a} \frac{gof\ (x) - gof(a)}{x-a} = \lim_{x\to a} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \times \frac{f(x)-f(a)}{x-a}$$

however, it is possible for $f(x) - f(a)$ to be zero where $x \neq a$

Use Coratheodory Theorem

Coratheorory Theorem

If F is differentiable at a, then there is a function ϕ which is continuous at a, satisfies $F(x) - F(a) = \phi(x)(x-a)$ for all x and $\phi(a) = F'(a)$

Proof

Define

$$\phi(x) = \begin{cases} \frac{F(x)-F(a)}{x-a}, if\ x \neq a \\ F'(a), if\ x = a \end{cases}$$

$\lim_{x\to a} \phi(x) = \lim_{x\to a} \frac{F(x)-F(a)}{x-a} = F'(a) = \phi(a)$
So ϕ is continuous

Proof of Chain Rule Cont.

Since f is differentiable at a, there is a function ϕ, continuous at a, satisfying
$f(x) - f(a) = \phi(x)(x-a)$ and $\phi(a) = f'(a)$

Similarly, since g is differentiable at f(a), there is a function ψ, which is continuous at f(a), satisfying $g(z) - g(f(a)) = \psi(z)(z - f(a))$ and $\psi(f(a)) = g'(f(a))$

Take $z = f(x)$. This gives $g(f(x)) - g(f(a)) = \psi(f(x))(f(x) - f(a))$
$g(f(x)) - g(f(a)) = \psi(f(x))\phi(x)(x-a)$

Calculate if possible
$gof'(a) = \lim_{x\to a} \frac{g(f(x)) - g(f(a))}{x-a} = \lim_{x\to a} \frac{\psi(f(x))\phi(x)(x-a)}{x-a} = \lim_{x\to a} \psi(f(x))\phi(x)$
$\lim_{x\to a} \phi(x) \to \phi(a)$ so the limit of ϕ at a exists.

Since f is continuous at a, $f(x) \to f(a)$
So since ψ is continuous at f(a) ψof is continuous at a and therefore
$\lim_{x\to a} \psi of(x) = \psi of(a) = \psi(f(a))$

Thus gof is differentiable at a and $(gof)'(a) = \psi(f(a))\phi(a) = g'(f(a))f'(a)$

Example:
$y = \cos\left(\frac{1}{x}\right)$
differentiable everywhere on its domain

$y' = -\sin\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = \frac{\sin\left(\frac{1}{x}\right)}{x^2}$

Derivatives of Inverse Functions

October-29-10
11:02 AM

Theorem

Let f be a continuous one-to-one function defined on an open interval (c, d). Suppose that f is differentiable at the point a ∈ (c, d) and f'(a) ≠ 0. Then f' is differentiable at f(a) and

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}$$

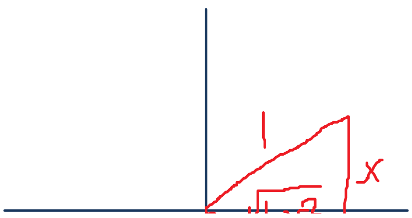
Notation

f^n means the nth derivative of f

Inverse Trig

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\arctan x = \frac{1}{x^2+1}$$

Since f^{-1} is continuous, (by continuity of f^{-1} theorem),
$\lim_{h\to 0} f^{-1}(b+h) = f^{-1}(b) = a$



If starting from $f^{-1}of = x$ we need to know apriori that f^{-1} is differentiable at f(a) and f is differentiable at a.

Ex. $y = x^3 = f(a)\ f^{-1}(a) = x^{\frac{1}{3}}$, which is not differentiable at 0, despite f being differentiable everywhere. Problem: $f'(0) = 0$

Whenever $f'(a) = 0$, f^{-1} is not differentiable at f(a).

If $f'(a) \neq 0$ then
$f^{-1}of(x) = x$
$(f^{-1})'(f(a)) \times f'(a) = 1$
$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}$$

Theorem

Let f be a continuous one-to-one function defined on an open interval (c, d). Suppose that f is differentiable at the point a ∈ (c, d) and f'(a) ≠ 0. Then f' is differentiable at f(a) and

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}$$

Proof

Write b = f(a)
$$\lim_{h\to 0}\frac{f^{-1}(b+h) - f^{-1}(b)}{h}$$
$f^{-1}(b) = a$
$f^{-1}(b+h) = z\ where\ f(z) = b+h$
Write $z = a + (z-a) = a + k$
$$\lim_{h\to 0}\frac{f^{-1}(b+h) - f^{-1}(b)}{h} = \lim_{h\to 0}\frac{a+k-a}{f(z)-b} = \lim_{h\to 0}\frac{k}{f(a+k)-f(a)} = \lim_{h\to 0}\frac{1}{\frac{f(a+k)-f(a)}{k}}$$

$k = f^{-1}(b+h) - a = f^{-1}(b+h) - f^{-1}(b)$
As $h \to 0, f^{-1}(b+h) \to f^{-1}(b)$ since f^{-1} is continuous (by continuity of $f^{-1} theorem$)
$so\ as\ h \to 0, k \to 0$

$h = f(a+k) - f(a)$
As $k \to 0$, the continuity of f gives that $f(a+k) \to f(a)$ therefore $h \to 0$

Hence,
$$\lim_{h\to 0}\frac{f^{-1}(b+h) - f^{-1}(b)}{h} = \lim_{k\to 0}\frac{1}{\frac{f(a+b)-f(a)}{k}} = \frac{1}{f'(a)}$$
by using the differentiability of f at a and the quotient rule for limits, which can be applied since $f'(a) \neq 0$
Thus f^{-1} is differentiable at b=f(a) and $(f^{-1})'(b) = \frac{1}{f'(a)}$
ie.
$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}$$
or
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Examples

$f(x) = x^{\frac{1}{n}}, n \in \mathbb{N}$
$f = g^{-1}\ where\ g(x) = x^n, n \in \mathbb{N}$

$g'(x) = nx^{n-1}$ and $g'(x) = 0\ iff\ x = 0$
By the theorem, f is differentiable at g(x) except for those x where g'(x) = 0 i.e. x = 0

$$f'(x) = \frac{1}{g'(f(x))} = \frac{1}{n(f(x))^{n-1}} = \frac{1}{n\left(x^{\frac{1}{n}}\right)^{n-1}} = \frac{1}{nx^{1-\frac{1}{n}}} = \frac{x^{\frac{1}{n}-1}}{n}$$

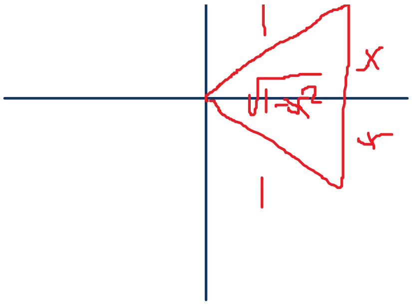
$f(x) = \exp(x), f = g^{-1}\ where\ g(x) = \ln x$
$$g'(x) = \frac{1}{x} \neq 0\ for\ any\ x$$
$$f'(x) = \frac{1}{g'(f(x))} = \frac{1}{\frac{1}{f(x)}} = f(x)$$
That is $\frac{d}{dx}\exp(x) = \exp(x)$

Definition of x^r

$y = x^r, r \in \mathbb{R}$
$y = \exp(r \ln x) = \exp(\ln x)^r = x^r$

$$y' = \exp(r\ln x)\frac{r}{x} = x^r \times \frac{r}{x} = rx^{r-1}$$

$y = x^x = \exp(x \ln x)$
$$y' = \exp(x\ln x)\left(1\ln x + \frac{1}{x}x\right) = \exp(x\ln x)(\ln x + 1) = x^x(\ln x + 1)$$



$y = x^x = \exp(x \ln x)$

$y' = \exp(x \ln x)\left(1 \ln x + \frac{1}{x} x\right) = \exp(x \ln x)(\ln x + 1) = x^x(\ln x + 1)$

Inverse Trig Functions

$y = \arcsin x = f(x)$

$f(x)$ is the inverse of sin restricted to $\left|-\frac{\pi}{2}, \frac{\pi}{2}\right|$

$f'(x) = \frac{1}{\sin'(f(x))} = \frac{1}{\cos(\arcsin x)}$

Except if there is a zero in the denominator

Suppose $\arcsin(x) = \theta$, means $\sin(\theta) = x$ and $\theta \in \left|-\frac{\pi}{2}, \frac{\pi}{2}\right|$

So except where $x \pm 1$

$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}$

Similarly,

$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1 - x^2}}$

$\frac{d}{dx} \arctan x = \frac{1}{\sec^2(\arctan x)}$

Find $\sec^2 \theta$ where $\tan \theta = x$

$\tan^2 \theta + 1 = \frac{(\sin^2 \theta + \cos^2 \theta)}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} = \sec^2 \theta$

$\frac{d}{dx} \arctan x = \frac{1}{x^2 + 1}$

Optimization Problems

November-03-10
10:30 AM

Local Maximum

A point x is a local maximum for the function f if there exists a $\delta > 0$ so that for every point $y \in (x - \delta, x + \delta)$, y ∈ Domain f, we have $f(y) \leq f(x)$.

Local Minimum

A point x is a local maximum for the function f if there exists a $\delta > 0$ so that for every point $y \in (x - \delta, x + \delta)$, y ∈ Domain f, we have $f(y) \geq f(x)$.

Global Maximum (Maximum)

A point x is a global maximum of f if $f(y) \leq f(x)$ for all y ∈ Domain f

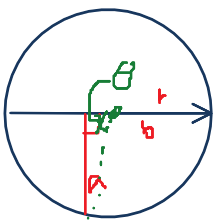
Global Maximum ⇒ Local Maximum, but the converse is not true.

Critical Points Theorem

If f has a local maximum or minimum at some point x ∈ (a, b) ⊆ Domain f, and if f is differentiable at x, then f'(x)=0

Critical Point

Call x a critical point of f if $f'(x) = 0$



Critical Points Theorem

If f has a local maximum or minimum at some point x ∈ (a, b) ⊆ Domain f, and if f is differentiable at x, then f'(x)=0

Proof

Look at $\frac{f(x+h)-f(x)}{h}$ for h < δ where δ is as in the definition for local maximum

Hence $f(x+h) \leq f(x)$

If h > 0, then $\frac{f(x+h)-f(x)}{h} \leq 0$

If h < 0 then $\frac{f(x+h)-f(x)}{h} \geq 0$

Since the $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists (since f is differentiable at x), the right and left hand limits exist.

$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}$$

But

$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \geq 0$$

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \leq 0$$

And since the two sides are equal,

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 0 = f'(x)$$

Note

A point can be a critical point but not a local maximum/minimum. Example: x=0 at $f(x) = x^3$

Finding Maximums/Minimums

Suppose $f: [a, b] \to \mathbb{R}$ which is continuous.

By the Extreme Value Theorem f has a global maximum and minimum.

The global max & min must also be a local max or min (respectively), and hence the theorem tells us the can only occur at:

1. a, b (Endpoints of [a,b])
2. at a point x where f is not differentiable (singular point)
3. at a critical point

Generally there are only finitely many points in 1, 2, and 3, allowing you to evaluate f at each of them and take the largest as the global maximum and the smallest as the local minimum.

Example

$f(x) = x - x^{\frac{2}{3}}\ on\ [-1, 8]$

$f'(x) = 1 - \frac{2}{3}x^{-\frac{1}{3}}, diff\ except\ at\ 0$

f is continuous

E.V.T implies there is a global maximum and minimum

<u>Candidates for max+min</u>

1. -1, 8
2. SP at 0
3. CP at $\frac{8}{27}$

$f(-1) = 2, f(0) = 0, f\left(\frac{8}{27}\right) = -\frac{4}{27}, f(8) = 4$

So the global max at $x = 8$ and the global min at $x = -1$

Problem

A right angle is moved along the diameter of a circle of radius r as shown.

Maximize the sum of length a+b.

Clearly, b ≥ r

$a + b = r\sin\theta + r\cos\theta + r, \theta \in \left[0, \frac{\pi}{2}\right]$

(a+b) is differentiable everywhere and is continuous

Possible candidates:

1. $\theta = 0, \frac{\pi}{2}$
3. $\theta = \frac{\pi}{4}$

$(a+b)(0) = (a+b)\left(\frac{\pi}{2}\right) = 2r$

$(a+b)\left(\frac{\pi}{4}\right) = r \times \frac{1}{\sqrt{2}} \times 2 + r = r(\sqrt{2}+1)$

So our maximum is at $\theta = \frac{\pi}{4}$ and the largest possible value for a+b is $r(\sqrt{2}+1)$

Mean Value Theorem

November-05-10
10:28 AM

Mean Value Theorem

If f is continuous on [a, b] and differentiable on (a, b) then there is a c ∈ (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Corollary

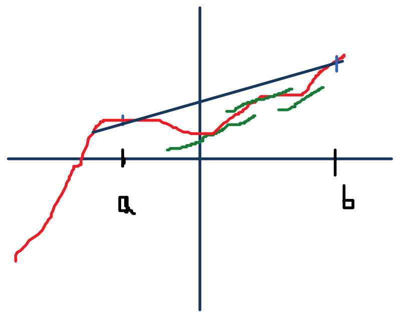
If $f'(x) = 0$ at every $x \in I$ interval then f is constant over that interval.

Rolle's Theorem

Suppose f is continuous on [a, b] and differentiable on (a, b). In addition, assume $f(a) = f(b)$. Then there is some $c \in (a, b)$ such that $f'(c) = 0$.

Corollary to MVT (Increasing Function Theorem)

If f is continuous on [a, b] and $f'(x) > 0$ for all $x \in (a, b)$ then f is strictly increasing on b]



There exists a tangent somewhere on the function between a and b which is equal to the secant between f(a) and f(b)

Proof of Corollary to Mean Value Theorem

Let a < b, a, b ∈ I. Then f is continuous and differentiable on [a, b]. By MVT, there is some $c \in (a, b) \subseteq$ I with

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow f(b) = f(a)$$

So f is constant.

Note the importance of $f'(x) = 0$ at *every* x on an interval.
The function

$$f(x) = \begin{cases} +1 \; if \; x > 0 \\ -1 \; if \; x < 0 \end{cases}$$

Is continuous and differentiable on it's domain, and $f'(x) = 0$ for all x on its domain, but f is not constant.

To prove MVT, we

Proof of Rolle's Theorem

By E.V.T, f has a maximum and minimum on [a, b].
If either the maximum or minimum occurs at $c \in (a, b)$ then by the critical points theorem, $f'(c) = 0$
Otherwise, both the maximum and minimum occur at the endpoints a, b. But $f(a) = f(b)$ so the maximum and minimum must be the same. So f is constant on [a, b]. Hence $f'(x) = 0$ at every $x \in (a, b)$. So c could be any point in (a, b) in this case.

Proof of the Mean Value Theorem

Let L(x) = secant line joining (a, f(a)) to (b, f(b)) and g(x) = f(x) -L(x)

$$g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right)$$

g is continuous on [a, b] because f is a y=L(x) is continuous everywhere and g is the difference of continuous functions. Similarly, g is differentiable on (a, b) since both f and y are and g is the difference of differentiable functions.
Furthermore, $g(a) = 0 = g(b)$, so Rolle's Theorem applies
So $g'(c) = 0$ for some $c \in (a, b)$

$$0 = g'(c) = f'(c) - L'(c) = f'(c) - \left(\frac{f(b) - f(a)}{b - a}\right)$$

Therefore

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Increasing and Decreasing Functions

Proof of Corollary to MVT (f is increasing on the interval where f' > 0)

Take $a \leq x < y \leq b$
MVT applies to f, so there is some $c \in (x, y)$ with $f'(c) = \frac{f(y)-f(x)}{y-x} > 0$, by the assumption $y - x > 0$ so $f(y) - f(x) > 0$ so f is strictly increasing.

NOTE: The Converse is not true

Can have f strictly increasing and differentiable everywhere but $f'(x) > 0$ is not true for all x
$y = x^3 = f(x)$, in which case $f(0) = 0$

Non-Decreasing

If $f'(x) \geq 0$ on (a, b) and continuous on [a, b] then f is increasing on [a, b]

Converse of Non-Decreasing case is true

If f is differentiable on (a, b) and increasing on (a, b), then $f'(x) \geq 0$ for all x ∈ (a, b)

<u>Proof:</u>

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$If \; h > 0, then \; f(x + h) \geq f(x)$$

$$If \; h < 0, then \; f(x + h) \leq f(x)$$

$$Hence \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \geq 0$$

Derivative Tests

November-08-10
10:40 AM

First Derivative Test

Assume f is continuous on $[x_1, x_2]$ and $c \in (x_1, x_2)$ is either a CP or a SP

1. If $f' > 0$ on (x_1, c) and $f' < 0$ on (c, x_2)
   Then c is a local maximum
2. If $f' < 0$ on (x_1, c) and $f' > 0$ on (c, x_2)
   Then c is a local minimum
3. If f' has the same sign on both sides of c, then c is neither a local minimum or maximum.

Vertical Asymptote

A point on a function is an asymptote if either the left or right hand limits at that point go to infinity.

Oblique Asymptote

The function approaches a line of non-zero slope as x $\to \pm \infty$

Horizontal Asymptote

$y = b \; where \lim_{x \to +or -\infty} f(x) = b$

Asymptotes on Polynomial Functions

deg $Q >$ deg P we get H.A. y = 0
deg Q = deg P we get H.A. y = b, b $\neq$ 0
deg Q + 1= deg P we get O.A $y = mx + b$

Concave Up

Say f is concave up on interval I if $f'(x)$ increases on I

Concave Down

Say f is concave down on interval I if $f'(x)$ decreases on I

Inflection Point

Call c an inflection point if $f'(c)$ exists and the concavity of f changes at c

Second Derivative Theorem

1. If $f'' > 0$ on I then f is concave up on I
2. If $f'' < 0$ on I then f is concave down on I
3. If f has an IP(Inflection Point) at c and $f''(c)$ exists, then $f''(c) = 0$

Second Derivative Test

Suppose $f'(c) = 0$. If $f''(c) > 0$ then f has a local min at c. If $f''(c) < 0$, then f has a local max at c. If $f''(c) = 0$ then we do not know.

Example

$f(x) = x^3 - 12x + 1$
$f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$
CP $= \pm 2$
Sign of $f'(x)$

(-∞ , -2)	(-2, 2)	(2, ∞)
+	-	+

f is strictly increasing on (-∞ , -2] ∪ [2, ∞) and strictly decreasing on [-2, 2]
-2 is a local max, 2 is a local min

Example

Maximize, if possible, $y = xe^{-x}$ $on \,[0, \infty)$
$y' = e^{-x} - xe^{-x} = e^{-x}(1-x)$
$e^{-x} > 0 \; for \; all \; x$
CP at $x = 1$

[0,1]	[1, ∞)
-	+

1 is a local maximum, but it is also a global maximum since it is greater than every other value on the range of f
So the maximum value of y is $\frac{1}{e}$

Example

Analyse the function

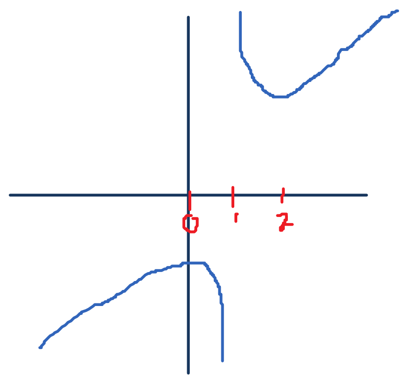
$f(x) = \dfrac{x^2 - 2x + 2}{x - 1}$

$Domain \; f = \mathbb{R} \setminus \{1\}$
Continuous and differentiable on its domain

$f'(x) = \dfrac{x(x-2)}{(x-1)^2}$

CP on 0, 2
Can only change signs on 0, 1, 2
Sign of f'

(-∞, 0]	[0, 1)	(1, 2]	[2, ∞)
+	-	-	+



Vertical asymptote at x = 1

$\lim_{x \to 1^+} \dfrac{x^2 - 2x + 2}{x - 1} = \lim_{x \to 1^+} x - 1 + \dfrac{1}{x-1} = +\infty$

$\lim_{x \to 1^-} x - 1 + \dfrac{1}{x-1} = -\infty$

Local minimum at 2, local maximum at 0, asymptote at 1

$\lim_{x \to \pm\infty} (f(x) - (x - 1)) = \lim_{x \to \pm\infty} \dfrac{1}{x-1} = 0$

This is called an oblique asymptote.

Example of Inflection Point

$y = x^3$
Has an inflection point at x = 0
$y' = 3x^2$
$y'' = 6x$
y'' is negative when x < 0 and positive when x > 0 so the concavity changes at x = 0

Proof of Second Derivative Theorem

1 and 2 are exercise, come from increasing function theorem.

3. Assume f' increases on (x_1, c) and f' decreases on (c, x_2) so c is a local maximum of f'. Since f' is differentiable, $f''(c) = 0$ by the critical points theorem.

Second Derivative Test

If $f'(c) = f''(c) = 0$
Ex $f(x) = x^3$ inflection point at 0
$f(x) = x^4$ local min at 0
$f(x) = -x^4$ local max at 0
Impossible to tell from just $f''(c) and \; f'(c)$

Proof of other statements:
Case $f''(c) > 0$

$f''(c) = \lim_{h \to 0} \dfrac{f'(c+h) - f'(c)}{h} > 0$

$f'(c) = 0 \; so \lim_{h \to 0} \dfrac{f'(c+h)}{h} > 0$

$So \; f'(c+h) > 0$ if h >0 and small
and $f'(c+h)$< 0 if h < 0 and small
By increasing function theorem, f is decreasing to left of c and decreasing to the right of

c. By the first derivative test c is a local minimum.

L'Hôpital's Rule and CMVT

November-10-10
10:40 AM

Limit of Infinity

Write $\lim_{x\to a} f(x) = \infty$ if for every $N \in \mathbb{N}$ there exists $\delta > 0$ such that if $|x - a| < \delta$ then $f(x) > N$

L'Hôpital's Rule

Assume f, g are differentiable on $I = |a - \delta, a + \delta|$ except possibly at a. Suppose
$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \; or \pm \infty$
Suppose $g(x) \neq 0$ for any x in I (except possibly a)
If
$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \text{ then } \lim_{x\to a} \frac{f(x)}{g(x)} = L$$

Cauchy Mean Value Theorem

If f, g are continuous on [a, b] and differentiable on (a, b).
Then there is some $c \in (a, b)$ such that
$(f(b) - f(a))g'(c) = g'(c)(g(b) - g(a))$

Intuitive Idea of L'Hôpital's Rule

Case:
$f(a) = 0 = g(a)$
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \times \frac{x - a}{g(x) - g(a)}$$
expect that to equal $\frac{f'(a)}{g'(a)}$

$$\lim_{x\to a} \frac{f'}{g'} = \frac{f'(a)}{g'(a)}$$
True if f', g' are continuous at a and $g'(a) \neq 0$

By the Mean Value Theorem, $f(x) - f(a) = f'(c_x)(x - a)$
If f is continuous on [a, x] and f is differentiable on (a, x)
$g(x) - g(a) = g'(d_x)(x - a)$
If g is continuous on [a, x] and g is differentiable on (a, x)
$c_x, d_x \in (a, x)$

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c_x)(x - a)}{g'(d_x)(x - a)} = \frac{f'(c_x)}{g'(d_x)}$$
$As\ x \to a, \qquad c_x, d_x \to a$
Suppose we get really lucky and $c_x = d_x$
$$\lim_{x\to a} \frac{f'(c_x)}{g'(d_x)} = \lim_{x\to a} \frac{f'}{g'}(c_x)$$

Recall

$\lim_{x\to a} F(x) = L$ if and only if whenever $(c_n) \to a$, then $\lim_{n\to\infty} F(c_n) = L$

Then
$$\lim_{x\to a} \frac{f'(c_x)}{g'(d_x)} = \lim_{x\to a} \frac{f'}{g'}(c_x) = \lim_{x\to a} \frac{f'}{g'}$$

Cauchy Mean Value Theorem

If f, g are continuous on [a, b] and differentiable on (a, b). Then there is some $c \in$ (a, b) such that
$(f(b) - f(a))g'(c) = f'(c)(g(b) - g(a))$

So if there arises no division by zero trouble, that means
$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Proof of CMVT

Define
$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$
h is continuous on [a, b] and differentiable on (a, b)
$h(a) = f(a)(g(b) - g(a)) - g(a)(f(b) - f(a))$
$= f(a)g(b) - f(a)g(a) - g(a)f(b) + g(a)f(a) = f(a)g(b) - g(a)f(b)$
$h(b) = g(b)f(a) - f(b)g(a)$
$h(a) = h(b)$
By the Mean Value Theorem (Rolle's Theorem) there is some $c \in (a, b)$ such that $h'(c) = 0$
$0 = f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a))$
So
$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$

Examples

$$\lim_{x\to 1} \frac{\log x}{\sin(\pi x)}$$
Being continuous, $\lim_{x\to 1} \log x = 0$, $\lim_{x\to 1} \sin(\pi x) = 0$
By L'Hopital's rule, study
$$\lim_{x\to 1} \frac{(\log x)'}{(\sin \pi x)'} = \lim_{x\to 1} \frac{\frac{1}{x}}{\pi \cos \pi x} = -\frac{1}{\pi} = \lim_{x\to 1} \frac{\log x}{\sin \pi x}$$

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x} = \lim_{x\to 0} \frac{(\sin x - x)}{x \sin x}$$
$\lim_{x\to 0} \sin x - x = 0$
$\lim_{x\to 0} x \sin x = 0$

By L'Hopital's Rule
$$= \lim_{x\to 0} \frac{\cos x - 1}{\sin x + x \cos x}$$
Again

$\lim_{x \to 0} \cos x - 1 = 0$
$\lim_{x \to 0} \sin x + x \cos x = 0$

By L'Hopital's Rule again

$= \lim_{x \to 0} -\frac{\sin x}{\cos x + \cos x - x \sin x} = \frac{0}{2} = 0$

So

$\lim_{x \to 0} \frac{1}{x} - \frac{1}{\sin x} = 0$

L'Hôpital's Proof

November-12-10
10:31 AM

Cauchy Mean Value Theorem

If f, g are continuous on [a, b] and differentiable on (a, b) then there is some $c \in (a,b)$ such that
$(f(b)-f(a))g'(c) = f'(c)(g(b)-g(a))$

Remarks on L'Hôpital's Rule

1. If $\lim_{x\to a^+} \frac{f'}{g'} = L$ then $\lim_{x\to a^+} \frac{f}{g} = L$
   (with all other assumptions)
2. In cases where $\lim_{x\to a} g(x) = \infty$ $(or\ -\infty)$, the behaviour of f does not matter, f does not need to go to infinity.
   a. Of course, if $|f| \le C$ then automatically $\lim_{x\to a} \frac{f}{g} = 0$ if $\lim_{x\to a} g(x) = \infty$
3. L'Hopital's rule is valid if $a = \pm\infty$
4. L'Hopital's rule is valid if L = $\pm\infty$

The non-existence of $\lim_{x\to a} \frac{f'}{g'}$ does not imply the non-existence of $\lim_{x\to a} \frac{f}{g}$

Assumptions

1. f and g are differentiable on $I = |a-\delta, a+\delta|$ but not necessarily a
2. $g(x), g'(x) \neq 0$ on I except at a
3. Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0\ or \pm\infty$
4. $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$

Want to prove:
$\lim_{x\to a} \frac{f(x)}{g(x)} = L$

Cauchy Mean Value Theorem

If f, g are continuous on [a, b] and differentiable on (a, b) then there is some $c \in (a,b)$ such that
$(f(b)-f(a))g'(c) = f'(c)(g(b)-g(a))$

Remember:
$\lim_{x\to a} F(x) = L$ if and only if whenever $(x_n) \to a$ then $(F(x_n)) \to L$

Proof Of L'Hôpital's Rule

Case 1

$\lim_{x\to a} f = \lim_{x\to a} g = 0$

Define (or redefine) f and g at a by setting $f(a) = 0 = g(a)$
This makes the functions continuous on $[a-\delta, a+\delta]$ since $\lim_{x\to a} f = 0 = f(a)$ and same for g
f and g are still differentiable on $|a-\delta, a) \cup (a, a+\delta|$

Enough to prove $\lim_{x\to a^+} \frac{f}{g} = L = \lim_{x\to a^-} \frac{f}{g}$

To prove $\lim_{x\to a^+} \frac{f}{g} = L$, it is enough to prove that whenever $(x_n) \to a, x_n > a, then\ \left(\frac{f}{g}(x_n)\right) \to L$

Take $(x_n) \to a,\ x_n > a. Wlog\ x_n \le a+\delta$
f, g are continuous on $[a, a+\delta]$ and differentiable on $(a, a+\delta)$
Therefore continuous on $[a, x_n]$ and differentiable on (a, x_n)

By the CMVT, there is some $c_n \in (a, x_n)$ with $(f(x_n)-f(a))g'(c_n) = f'(c_n)(g(x_n)-g(a))$
By assumption, $g'(c_n) \neq 0$
Also, $g(x_n) - g(a) \neq 0$, because by assumption $g(x_n) \neq 0$ but $g(a) = 0$ (or because by MVT $g(x_n) - g(a) = g'(t_n)(x_n - a) for\ some\ t_n \in (a, x_n). But\ g'(t_n) \neq 0$)

Divide to get
$$\frac{f(x_n)-f(a)}{g(x_n)-g(a)} = \frac{f'(c_n)}{g'(c_n)} = \frac{f'}{g'}(c_n)$$
But $f(a), g(a) = 0$, so
$$\frac{f}{g}(x_n) = \frac{f'}{g'}(c_n)$$
As $n\to\infty, (c_n) \to a\ since\ (x_n) \to a$. So since $\lim_{x\to a} \frac{f'}{g'} = L$ by recalled fact concerning limits and sequences,
$$\lim_{n\to\infty} \frac{f'}{g'}(c_n) = L$$
Therefore,
$$\lim_{n\to\infty} \frac{f}{g}(x_n) = L$$
So by the recalled fact again,
$$\lim_{x\to a} \frac{f}{g} = L$$

Case 2

$\lim_{x\to a} f = \lim_{x\to a} g = \infty$

Recall Definition:
$\lim_{x\to a} F = \infty$ means that ∀ N ∈ ℕ there is some δ > 0 such that if $0 < |x-a| < \delta$ then $F(x) > N$

Suffices to prove whenever $(x_n) \to a$, then $\frac{f}{g}(x_n) \to L$
Take such a sequence with $x_n \in (a, a+\delta|$
Consider each pair $x_j, x_n, j < n$
f and g are continuous and differentiable on $[x_n, x_j]$ (or $[x_j, x_n]$)
Apply CMVT to get c_{jn} between x_j and x_n
$\left(f(x_n) - f(x_j)\right) g'(c_{jn}) = f'(c_{jn}) \left(g(x_n) - g(x_j)\right)$
$g' \neq 0$ so
$$f(x_n) - f(x_j) = \frac{f'(c_{jn})\left(g(x_n)-g(x_j)\right)}{g'(c_{jn})}$$

Look at
$$\frac{f(x_n)}{g(x_n)} - \frac{f(x_j)}{g(x_n)} = \frac{f(x_n)-f(x_j)}{g(x_n)}$$

$$= \frac{f'(c_{jn})\left(g(x_n) - g(x_j)\right)}{g'(c_{jn})g(x_n)} = \frac{f'(c_{jn})}{g'(c_{jn})}\left(1 - \frac{g(x_j)}{g(x_n)}\right)$$

So

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_j)}{g(x_n)} + \frac{f'(c_{jn})}{g'(c_{jn})} - \frac{f'(c_{jn})g(x_n)}{g'(c_{jn})g(x_n)}$$

Know $\lim_{x \to a} g(x) = \infty \Rightarrow \lim_{n \to \infty} g(x_n) = \infty$

$$\Rightarrow \lim_{n \to \infty} \frac{1}{g(x_n)} = 0$$

Want to prove

$$\frac{f}{g}(x_n) \to L$$

So

For every $\varepsilon > 0, \exists\ \mathrm{N} \in \mathbb{N}$ so $\left|\frac{f}{g}(x_n) - L\right| < \varepsilon\ \forall n \geq N$

Fix $\varepsilon > 0$, $(\varepsilon < 1)$

Since

$$\lim_{x \to a} \frac{f'}{g'} = L$$

there exists a $\delta > 0$ so that if $|x - a| < \delta$ then $\left|\frac{f'}{g'}(x) - L\right| < \frac{\varepsilon}{3}$

Pick N_1 so that $|x_m - a| < \delta$ if $m \geq N_1$

If $n, j \geq N_1$ then $|x_n - a| < \delta$, and $|x_j - a| < \delta$

Since $x_n < c_{jn} < x_j\ or\ x_j < c_{jn} < x_n$ then $|c_{jn} - a| < \delta\ \forall j, n \geq N_1$

Take $j = N_1$ so $|c_{N_1 n} - a| < \delta\ \forall n > N_1$

Hence

$$\left|\frac{f'}{g'}(c_{N_1 n}) - L\right| < \frac{\varepsilon}{3}\ \forall n > N_1$$

In particular,

$$\left|\frac{f'}{g'}(c_{N_1 n})\right| \leq |L| + \frac{\varepsilon}{3} \leq |L| + 1$$

By assumption

$$(g(x_n)) \to \infty\ so\ \frac{1}{g(x_n)} \to 0$$

Pick $N \geq N_1$ such that for all $n > N$

$$\frac{1}{|g(x_n)|} \leq \frac{\varepsilon}{3(|L| + 1)|g(x_{N_1})|}$$

and

$$\leq \frac{\varepsilon}{3\left(f(x_{N_1})\right)}\ if\ f(x_{N_1}) \neq 0$$

$$\left|\frac{f}{g}(x_n) - L\right| \leq \left|\frac{f(x_{N_1})}{g(x_n)}\right| + \left|\frac{g(x_{N_1})}{g(x_n)} \times \frac{f'}{g'}(c_{N_1 n})\right| + \left|\frac{f'}{g'}(c_{N_1 n}) - L\right|$$

Let $n > N$

$$\left|\frac{f(x_{N_1})}{g(x_n)}\right| \leq \frac{|f(x_{N_1})|\varepsilon}{3|f(x_{N_1})|} = \frac{\varepsilon}{3}$$

$$\left|\frac{g(x_{N_1})}{g(x_n)} \times \frac{g'}{f'}(c_{N_1 n})\right| \leq \frac{|g(x_{N_1})|}{|g(x_n)|}(|L| + 1) \leq \frac{\varepsilon}{3(|L| + 1)|g(x_{N_1})|}|g(x_{N_1})|(|L| + 1) = \frac{\varepsilon}{3}$$

$$\left|\frac{f'}{g'}(c_{N_1 n}) - L\right| \leq \frac{\varepsilon}{3}$$

Therefore

$$\left|\frac{f}{g}(x_n) - L\right| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

So

$$\frac{f}{g}(x_n) \to L\ as\ n \to \infty$$

Therefore

$$\lim_{x \to a^+} \frac{f}{g} = L$$

And the same for the left hand limit, so

$$\lim_{x \to a} \frac{f}{g} = L$$

■

Remarks

1. If $\lim_{x \to a^+} \frac{f'}{g'} = L$ then $\lim_{x \to a^+} \frac{f}{g} = L$
   (with all other assumptions)
2. In cases where $\lim_{x \to a} g(x) = \infty\ (or\ -\infty)$, the behaviour of f does not matter, f does not need to go to infinity.
   a. Of course, if $|f| \leq C$ then automatically $\lim_{x \to a} \frac{f}{g} = 0$ if $\lim_{x \to a} g(x) = \infty$
3. L'Hopital's rule is valid if $a = \pm\infty$
4. L'Hopital's rule is valid if $L = \pm\infty$

Examples

Failure:

$$\lim_{x\to 1}\frac{x-1}{x}=0, not \lim_{x\to 1}\frac{1}{1}=1$$

The non-existence of $\lim_{x\to a}\frac{f'}{g'}$ does not imply the non-existence of $\lim_{x\to a}\frac{f}{g}$

$$\lim_{x\to\infty}\frac{x+\sin x}{x}=1$$

but $\lim_{x\to\infty}\frac{1+\cos x}{1}$ does not exist

Successful Examples

$$\lim_{x\to\infty} xe^{-x}=\lim_{x\to\infty}\frac{x}{e^x}$$

$$\lim_{x\to\infty}\frac{1}{e^x}=0, so \lim_{x\to\infty}\frac{x}{e^x}=0$$

Exercise: Prove

$$\lim_{x\to\infty} x^n e^{-x}=0\ \forall n\in\mathbb{N}$$

$$\lim_{x\to 0^+} x^a\log x, for\ a>0=\lim_{x\to 0^+}\frac{\log x}{x^{-a}}$$

$$\lim_{x\to 0^+}\frac{\left(\frac{1}{x}\right)}{-ax^{-a-1}}=\lim_{x\to 0^+}-\frac{x^a}{a}=0$$

So

$$\lim_{x\to 0^+} x^a\log x=0$$

$$\lim_{x\to 0^+} x^x=\lim_{x\to 0^+}\exp(\ln x^x)$$

$$=\exp\left(\lim_{x\to 0^+} x\ln x\right)$$

Justification: Define

$$F(x)=\begin{cases} x\log x\ if\ x>0\\ 0\ if\ x=0\end{cases}$$

$$\lim_{x\to 0^+} F(x)=0=F(0)$$

so F is continuous at 0

Asking for

$$\lim_{x\to 0^+}\exp(\mathrm{F(x)})=\exp(F(0))=\exp 0=1$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

Look at

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\lim_{\mathrm{x}\to\infty}\exp\left(x\log\left(1+\frac{1}{x}\right)\right)$$

Let $y=\frac{1}{x}$

$As\ x\ \to\infty\ then\ y\to 0^+$

$$\lim_{y\to 0^+}\exp\left(\frac{1}{y}\log(1+y)\right)=\exp\left(\lim_{y\to 0^+}\frac{\log(1+y)}{y}\right)$$

$$\lim_{y\to 0^+}\frac{\frac{1}{1+y}}{1}=1$$

$$so \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\exp(1)=e$$

Limits and Continuity

November-17-10
10:26 AM

Theorem

If $\lim_{x\to\infty} f(x) = L$ and g is continuous at L, then
$\lim_{x\to\infty} gof(x) = g(L) = g(\lim_{x\to\infty} f(x))$

$$\lim_{x\to\infty} \exp\left(x\log\left(1+\frac{1}{x}\right)\right) = \exp\left(\lim_{x\to\infty} x\log\left(1+\frac{1}{x}\right)\right)$$

Justification

Seen before that

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right) = 1$$

exp is continuous at 1, so this should hold true

Proof of Theorem

RTP that $\forall \varepsilon > 0$ there is a number N so that if $x > N$ then $|g(f(x)) - g(L)| < \varepsilon$

Know, given any $\varepsilon' > 0$ there is N' so that if $x > N'$ then $|f(x) - L| < \varepsilon'$
Know, given any $\varepsilon > 0$ there is some $\delta > 0$ so that if $|z - L| < \delta$ then $|g(z) - g(L)| < \varepsilon$

Fix $\varepsilon > 0$. Take N so $|f(x) - L| < \delta$ when $x > N$ where δ comes from the definition of continuity of g at L. Let $x > n$, then $|f(x) - L| < \delta$ and $|g(f(x)) - g(L)| < \varepsilon$
So $\lim_{x\to 0} g(f(x)) = g(L)$ ■

Taylor Polynomials

November-17-10
11:00 AM

Taylor Polynomial

The Taylor Polynomials of degree n at a for the function f is the polynomial:
$y = a_0 + a_1(x-a) + \cdots + a_n(x-a)^n = P_{n,a}(x)$
where
$a_k = \dfrac{f^{(k)}(a)}{k!}$

Types of Taylor Polynomials

$P_{0,a}(a) = a_o = f(a)$

$P_{1,a}(x) = f(a) + f'(a)(x-a) = tangent\ line\ to\ f\ at\ a$
$P_{2,a}(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2}(x-a)^2$

If f is a polynomial of degree n then $f(x) = P_{n,a}(x)$
Exercise: Write $f = b_0 + b_1(x-a) + \cdots + b_n(x-a)^n$
Take derivatives to see $b_k = a_k$

The Taylor Polynomials are often good approximations of the function but not always

Example

$f(x) = \sin x\ \ at\ x = 0$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$f'''(x) = -\cos x$
$f''''(x) = \sin x$

$f(0) = 0$
$f'(0) = 1$
$f''(0) = 0$
$f'''(0) = -1$
…

$a_0 = 0$
$a_1 = 1$
$a_2 = 0$
$a_3 = -\dfrac{1}{3!}$
$a_4 = 0$
$a_5 = \dfrac{1}{5!}$

Taylor polynomial
$P_{n,0} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots ends\ at\ n$

Taylor Polynomials Accuracy

November-19-10
10:28 AM

Taylor Polynomial

f - n times differentiable at a

$$P_{n,a} = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Theorem

If f is n times differentiable at then

$$\lim_{x\to a} \frac{f(x) - P_{n,a}(x)}{(x-a)^n} = 0$$

Application of Theorem to Second Derivative Test

Suppose

$f'(a) = 0 = f''(a) = \cdots = f^{n-1}(a)$

$f^n(a) \neq 0$

1. If n is even and $f^n(a) > 0$, then f has a local minimum at a
2. If n is even and $f^n(a) < 0$, then f has a local maximum at a
3. If n is odd, then f has neither a local minimum nor a local maximum at a

Comment

Can have $f^{(n)}(a) = 0$ for every n without f being constant in which the above theorem does not give any information.

Example:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, x \neq 0 \\ 0, x = 0 \end{cases}$$

Local minimum at x=0 deispite the fact that $f^{(n)}(0) = 0$ for all n

Proof of Theorem

Look at

$$\lim_{x\to a} \frac{f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}{(x-a)^n} = \lim_{x\to a} \frac{f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k}{(x-a)^n} - \frac{\frac{f^{(n)}(a)}{n!}(x-a)^n}{(x-a)^n}$$

So RTP

$$\lim_{x\to a} \frac{f(x) - Q_n(x)}{(x-a)^n} = \frac{f^{(n)}(a)}{n!}$$

$$Q_n(x) = \sum_{k=0}^{n-1} \frac{f^k(a)}{k!}(x-a)^k = P_{n-1,a}(x)$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{n-1}(a)}{(n-1)!}(x-a)^{n-1}$$

$Q_n(a) = f(a)$

Q_n is a polynomial, so continuous function, so $\lim_{x\to a} Q(x) = Q(a)$

Hence

$$\lim_{x\to a} f(x) - Q_n(x) = 0 = \lim_{x\to a} (x-a)^n$$

Apply L'Hopital's Rule

Look at

$$\lim_{x\to a} \frac{f'(x) - Q_n'(x)}{n(x-a)^{n-1}}$$

$$\lim_{x\to a} Q_n'(x) = Q_n'(a)$$

$$Q_n'(x) = f'(a) + \frac{2f'(a)}{2}(x-a) + \cdots + \frac{(n-1)f^{n-1}(a)(x-a)^{n-2}}{(n-1)!}$$

$Q_n'(a) = f'(a)$

So

$$\lim_{x\to a} f'(x) - Q_n'(x) = f'(a) - Q_n'(a) = 0$$

Apply L'Hopital's Rule again

$$\lim_{x\to a} \frac{f''(x) - Q_n''(x)}{n(n-1)(x-a)^{n-2}}$$

Keep applying L'Hopital's Rule

$Q_n^{(k)}(a) = f^{(k)}(a)\ \forall k = 1, \dots, n-1$

$$\lim_{x\to a} \frac{f(x) - Q_n(x)}{(x-a)^n} = \lim_{x\to a} \frac{f^{n-1}(x) - Q_n^{n-1}(x)}{n!\,(x-a)} = \lim_{x\to a} \frac{f^n(x) - Q_n^n(x)}{n!}$$

Q_n is a polynomial of degree n-1 so $Q_n^{(n)} = 0$

$$= \lim_{x\to a} \frac{f^{(n)}(x)}{n!}$$

But don't know that $f^n(x)$ is continuous

Notice Q_n^{n-1} is constant since Q_n is a degree n-1 polynomial

So $Q_n^{n-1}(x) = Q_n^{n-1}(a) = f^{n-1}(a)$

Hence

$$\lim_{x\to a} \frac{f^{n-1}(x) - Q_n^{n-1}(x)}{n!\,(x-a)} = \lim_{x\to a} \frac{f^{n-1}(x) - f^{n-1}(a)}{n!\,(x-a)} = \frac{1}{n!} f^n(a)$$

By the definition of the derivative of f^{n-1}

Therefore

$$\lim_{x\to a} \frac{f(x) - Q_n(x)}{(x-a)^n} = \frac{f^n(a)}{n!}$$

Application of Theorem to Second Derivative Test

Suppose

$f'(a) = 0 = f''(a) = \cdots = f^{n-1}(a)$

$f^n(a) \neq 0$

1. If n is even and $f^n(a) > 0$, then f has a local minimum at a
2. If n is even and $f^n(a) < 0$, then f has a local maximum at a
3. If n is odd, then f has neither a local minimum nor a local maximum at a

Proof

If $f(a) \neq 0$, replace f by $f(x) - f(a)$

Subtracting a constant does not change any derivative or local extremum location.

So wlog we can assume $f(a) = 0$

$$P_{n,a}(x) = \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k = \frac{f^n(a)}{n!}(x-a)^n$$

Theorem said

$$\lim_{x\to a} \frac{f(x) - P_{n,a}(x)}{(x-a)^n} = 0 = \lim_{x\to a} \frac{f(x) - \frac{f^n(a)}{n!}(x-a)^n}{(x-a)^n} = \lim_{x\to a} \frac{f(x)}{(x-a)^n} - \frac{f^n(a)}{n!}$$

Therefore

$$\lim_{x\to a} \frac{f(x)}{(x-a)^n} = \frac{f^n(a)}{n!}$$

Case n is even

1. $f^n(a) > 0$

   $\frac{f^n(a)}{n!}$ is positive
   $(x-a)^n$ is positive
   so $f(x) > 0 = f(a)$ as $x \to a$
   So a is a local min

2. $f^n(a) < 0$

   $\frac{f^n(a)}{n!}$ is negative
   $(x-a)^n$ is positive
   so $f(x) < 0 = f(a)\ as\ x \to a$
   So a is a local max

Case n is odd

$(x-a)^n$ is positive when x > a and negative when x < a
$\frac{f^n(a)}{n!}$ is either positive or negative, but is constant
So f(x) > 0 = f(a) as x approaches a from one side, and f(x) < 0 = f(a) as x approaches a from the other side.
So $f(a)$ is neither a local maximum or minimum.

Taylor's Theorem

November-22-10
10:31 AM

Taylor's Theorem

Suppose $f, f', \ldots, f^{(n+1)}$ are defined on [a, x]
Then

$$f(x) - P_{n,a}(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$$

For some $c \in (a, x)$
Similar statement for a < x

Example: Tangent Line

$$f(x) - P_{1,a}(x) = \frac{f''(c)(x-a)^2}{2}$$

$$|f(x) - P_{1,a}| \le \sup_{c \in |a,x|} \left|\frac{f''(c)(x-a)^2}{2}\right|$$
$$\le \frac{M(x-a)^2}{2}$$

If f is a continuous function on [a, x]

Example: Sin x

$f(x) = \sin x$
$a = 0$
$f^{(n)} = \pm \sin x, \pm \cos x$
$|f^{(n)}(c)| \le 1 \; \forall$ c, $\forall$ n

So

$$|f(x) - P_{n,a}(x)| \le \frac{1(x-a)^{n+1}}{(n+1)!} \to 0 \; as \; n \to \infty$$

No loss of generality in assuming $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
so $\frac{(x-a)^{n+1}}{(n+1)!} \le \frac{2^{n+1}}{(n+1)!}$ which is accurate to 12 digits when n = 20

Proof of Taylor's Theorem

Think of x as fixed.
For each $t \in [a, x]$ write

$$f(x) = f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(t)(x-t)^k}{k!} + R(t)$$

Defines a function R(t) on $|x, a|$

$$R(t) = f(x) - f(t) - \sum_{k=1}^{n} \frac{f^{(k)}(t)(x-t)^k}{k!}$$

$t = a$

$$R(a) = f(x) - f(a) - \sum_{k=1}^{n} \frac{f^{(k)}(a)(x-a)^k}{k!} = f(x) - P_{n,a}(x)$$

$t = x$

$$R(x) = f(x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)(x-x)^k}{k!} = 0$$

Define

$$F(t) = \frac{(x-t)^{n+1}}{(n+1)!}$$

$$F(a) = \frac{(x-a)^{n+1}}{(n+1)!}$$

$F(x) = 0$

Want to show

$$R(a) = f(x) - P_{n,a} = \frac{f^{n+1}(c)(x-a)^{n+1}}{(n+1)!} = f^{n+1}(c)F(a)$$

or $R(a) - R(x) = f^{(n+1)}(c)\big(F(a) - F(x)\big)$

Want to apply Cauchy Mean Value Theorem to R and F
R, F are differentiable on [a, x] (because $f^{(n+1)}$ exists on [a, x]

$$F'(t) = \frac{(n+1)(x-t)^n(-1)}{(n+1)!} = -\frac{(x-t)^n}{n!}$$

$$\frac{d}{dt}\frac{f^{(k)}(x-t)^k}{k!} = \frac{f^{k+1}(t)(x-t)^k}{k!} + -\frac{k(x-t)^{k-1}f^{(k)}(t)}{k!} = \frac{f^{(k+1)}(t)(x-t)^k}{k!} - \frac{(x-t)^{k-1}f^{(k)}(t)}{(k-1)!}$$

$$R'(t) = -f'(t) - \sum_{k=1}^{n} \frac{f^{(k+1)}(t)(x-t)^k}{k!} - \frac{f^{(k)}(t)(x-t)^{k-1}}{(k-1)!}$$

$$= -f'(t) - \left(\frac{f^{(n+1)}(t)(x-t)^n}{n!} - f'(t)\right) = -\frac{f^{(n+1)}(t)(x-t)^n}{n!}$$

By CMVT
$\big(R(a) - R(x)\big)F'(c) = R'(c)\big(F(a) - F(x)\big)$
for some $c \in (a, x)$

$$\big(R(a) - R(x)\big)\left(-\frac{(x-c)^n}{n!}\right) = -\frac{f^{(n+1)}(c)(x-c)^n}{n!}\big(F(a) - F(x)\big)$$

$R(a) - R(x) = f^{(n+1)}(c)\big(F(a) - F(x)\big)$
Which is what we wanted to prove. ■

So

$$f(x) - P_{n,a}(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$$

$$\frac{(x-a)^{n+1}}{(n+1)!} \to 0\ as\ n \to \infty$$

So $f(x) - P_{n,a}(x) \to 0$ if $f^{(n+1)}(c)$ does not grow too quickly as $n \to \infty$

Irrationality of e

November-24-10
10:31 AM

$f(x) = e^x$
$f^{(n)}(x) = e^x$
$f^{(n)}(0) = 1\ \forall n$
$$P_{n,0} = f(0) + \sum_{k=1}^{n} \frac{f^{(k)}(0)(x-a)^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$
$$f(x) - P_{n,0} = \frac{e^c x^{n+1}}{(n+1)!}\ for\ some\ c \in (0, x)$$
Take x = 1
$$e\ - \left(1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) = \frac{e^c}{(n+1)!} \le \frac{3}{(n+1)!}$$
$c \in (0, 1)$
Suppose $e = \frac{p}{q}\ for\ p, q \in N$
Then
$$\frac{p}{q} = 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n!} + \frac{e^c}{(n+1)!}$$
Take $n > \max(q, 3)$
$$\frac{p}{q} n! = n! + n! + \frac{n!}{2} + \frac{n!}{3!} + \cdots + \frac{n!}{n!} + \frac{n!\, e^c}{(n+1)!}$$
$n \ge q\ so \frac{p}{q} n\ is\ an\ integer$
$Every\ term\ n!, \frac{n!}{2}, \ldots, \frac{n!}{n!}\ are\ integers$
$$0 < \frac{e^c}{n+1} < \frac{e}{n+1} < \frac{3}{n+1} < 1$$
This is impossible, so e is irrational

e is in fact transcendental

Newton's Method

November-24-10
10:45 AM

Theorem

Suppose $f: |a,b| \to \mathbb{R}, f, f', f'' continuous,$
$f(a) < 0 < f(b)\ and\ f'and\ f'' > 0\ on\ [a,b]$
Suppose $f(c) = 0\ for\ c \in (a,b)$
Define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\ for\ n = 0, 1, 2, \ldots$$

where $x_0 \in [c,b]$ then (x_n) is well defined and x_n converges to c.

Accuracy

Let $M_1 = \max\{f''(x): x \in |a,b|\}$
(M_1 exists because f'' is continuous and use E.V.T)
Let $M_2 = f'(a)\ (= \min\{f'(x): x \in |a,b|\})$
Let $M \geq \frac{M_1}{M_2}$

Then

$$|x_n - c| \leq \frac{1}{M}\left(M(x_0 - c)\right)^{2^n}$$

(Can use bisection method to bring x_0 close enough to c that $M(x_0 - c)$ is < 1

Finding Roots

Say f is continuous on [a, b] and f(a) < 0 and f (b) > 0, by the intermediate value theorem there is a root c ∈ [a, b] with f(c) = 0

Bisection Method

Need only a continuous function, keep cutting interval in half and checking whether the midpoint is above or below zero.

Newton's Method

Suppose f' > 0 on [a, b]
Then f is strictly increasing so the root c is unique.

Algorithm

Pick $x_0 \in [c,b]$ (for example, pick $x_0 = b$)
Inductively define

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This sequence is called the Newton Iterates

Tangent line to f at x_n
$y = f(x_n) + f'(x_n)(x - x_n)$
Crosses x axis at
$0 = f(x_n) - f'(x_n)(x - x_n)$

$$-\frac{f(x_n)}{f'(x_n)} = x - x_n \Rightarrow x = x_n - \frac{f(x_n)}{f'(x_n)}$$

Proof of Theorem

Check that $c \leq x_1 \leq x_0 \leq b$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$f(x_0) \geq f(c) = 0$
Of course $f'(x_0) > 0$
Therefore $x_1 \leq x_0$

$$\frac{f(x_0) - f(c)}{x_0 - c} = f'(t_0)$$

for some $t_0 \in (c, x_0)$ by MVT
$f(x_0) = f'(t_0)(x_0 - c)$

$$\frac{f(x_0)}{f'(t_0)} = (x_0 - c) \Rightarrow c = x_0 - \frac{f(x_0)}{f'(t_0)}$$

$t_0 < x_0$ and f' is strictly increasing so
$f'(t_0) < f'(x_0)$

$$\frac{1}{f'(t_0)} > \frac{1}{f'(x_0)}$$

$$-\frac{f(x_0)}{f'(t_0)} < -\frac{f(x_0)}{f'(x_0)}$$

So

$$c = x_0 - \frac{f(x_0)}{f'(t_0)} \leq x_0 - \frac{f(x_0)}{f'(x_0)} = x_1$$

Proceed inductive and assume
$b \geq x_0 \geq x_1 \geq x_2 \geq \cdots \geq x_n \geq c$
Check $x_0 \geq x_1 \geq \cdots \geq x_n \geq x_{n+1} \geq c$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} < x_n$$

By MVT

$$\frac{f(x_n) - f(c)}{x_n - c} = f'(t_n)\ for\ t_n \in (c, x_n)$$

Get

$$c = x_n - \frac{f(x_n)}{f'(t_n)}$$

$t_n < x_n \Rightarrow f'(t_n) < f'(x_n)$

$$\leq x_n - \frac{f(x_n)}{f'(x_n)}$$

Hence x_n is a decreasing sequence which is bounded by low (by c) By MCT,

$x_n \to p \ \ with\ c \le p \le b$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Since f and f' are continuous at p, $f(x_n) \to f(p) and\ f'(x_n) \to f'(p) \neq 0$

By passing to the limit we see

$$p = p - \frac{f(p)}{f'(p)} \Rightarrow f(p) = 0$$

So p is a root, but c was the only root of f in [a, b]

So $(x_n) \to c$

Newton's Method Accuracy

November-26-10
10:35 AM

Newton's Method

$f:[a,b] \to \mathbb{R}, \qquad f, f', f'' \text{ are continuous}$
$f', f'' > 0 \text{ on } [a,b]$
$f(a) < 0 < f(b) \text{ and } f(c) = 0 \text{ for } c \in [a,b]$

Newton Iterates

$$Define\ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, x_0 \in [c,b]$$

$Then\ x_n \to c$

Accuracy

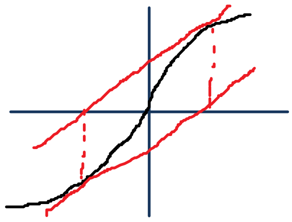
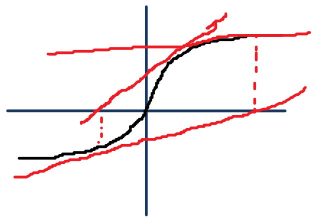
Let $M_1 = \max\{f''(x) : x \in [a,b]\}$
$M_2 = f'(a) = \min\{f'(x): x \in [a,b]\}$
$Put\ M = M_1/M_2$
$$Then\ |x_n - c| \le \frac{1}{M}(M|x_0 - c|)^{2^n}$$

Proof of Accuracy

Already have seen that
$$c = x_n - \frac{f(x_n)}{f'(t_n)}$$
for some $t_n \in (c, x_n)$

$$|x_{n+1} - c| = \left|x_n - \frac{f(x_n)}{f'(x_n)} - \left(x_n - \frac{f(x_n)}{f'(t_n)}\right)\right| = \left|\frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)}{f'(t_n)}\right| = \left|\frac{f'(t_n)f(x_n) - f(x_n)f'(x_n)}{f'(x_n)f'(t_n)}\right|$$
$$= \left|\frac{f(x_n)}{f'(t_n)}\right| \frac{|f'(t_n) - f'(x_n)|}{|f'(x_n)|} = |x_n - c| \frac{|f''(u_n)(t_n - x_n)|}{|f'(x_n)|}$$
For some $u_n \in (t_n, x_n)$ by MVT
$$|x_n - c| \frac{|f''(u_n)(t_n - x_n)|}{|f'(x_n)|} \le |x_n - c|^2 \frac{M_1}{M_2} = M|x_n - c|^2$$
Write as $M|x_{n+1} - c| \le (M|x_n - c|)^2 \le \left((M|x_{n-1} - c|)^2\right)^2 \le (M|x_0 - c|)^{2^{n+1}}$
■

But things can go wrong if not all the hypothesises are satisfied.

These examples need property that (x_n) does not converge
$If\ (x_n) \to p$, must $f(p) = 0$?
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$x_{n+1} \to p$
$x_n \to p$
so $\frac{f(x_n)}{f'(x_n)} \to 0$
Assuming f is continuous at p
$f(x_n) \to f(p)\ as\ x_n \to p$
Assuming f' is continuous at p
$f'(x_n) \to f'(p)$, a real number
so
$$\frac{f(x_n)}{f'(x_n)} \to 0 \Rightarrow f(x_n) \to 0 \Rightarrow f(p) = 0$$

Example of Failure

$x_n \to 0$, f continuous and differentiable everywhere
but $f(0) \ne 0$

Step 1:
Claim for each $n \in \mathbb{N}$, there is a polynomial P_n with
1. $P_n(2^{-n}) = -2\pi n$
2. $P_n(2^{-(n+1)}) = -2\pi(n+1)$
3. $P_n'(2^{-n}) = 2^{3n+1}$
4. $P_n'(2^{-(n+1)}) = 2^{3(n+1)+1}$

Define $g: (0, \infty) \to \mathbb{R}$ by
$$g(x) = \begin{cases} P_n(x)\ if\ x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], n \ge 0 \\ 2x - 2\ if\ x > 1 \end{cases}$$
$\lim_{x\to 1^-} P_0 = P_0(1) = 0 = g(1) = \lim_{x \to 1^+} g$
$P_n(2^{-(n+1)}) = -2\pi(n+1)$
$P_{n+1}(2^{-(n+1)}) = -2\pi(n+1)$
$\lim_{x \to 2^{-(n+1)}} g = g(2^{-(n+1)})$
as RH and LH limits both equal $g(2^{-(n+1)}) = P_{n+1}(2^{-(n+1)})$
This shows g is continuous on $(0, \infty)$
g is even differentiable on $(0, \infty)$
Similar argument looking at RH and LH Newton quotients at 2^{-1}

Step 2:
Define
$$f(x) = \begin{cases} x^2 \sin(g(x)) + 1\ , x > 0 \\ 1\ , \qquad x = 0 \\ -x^2 + 1, \qquad x < 0 \end{cases}$$
Clearly f is continuous everywhere and differentiable everywhere except zero. "Where the clearly statement does not apply, because it's not clear."

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$\lim_{h\to 0^-}\frac{-h^2+1-1}{h}=0$$
$$\lim_{h\to 0^+}\frac{h^2\sin(g(h))+1-1}{h}=\lim_{h\to 0} h\sin g(h)=0$$
So $f'(0)=0$

$f(2^{-n})=(2^{-n})^2\sin g(h)+1=1\ \forall n$
$f'(2^{-n})=(2^{-n})^2 2^{3n+1}=2^{n+1}$
Take $x_0=1$
$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=1-\frac{1}{2^1}=\frac{1}{2}$$
$$x_2=x_1-\frac{f(x_0)}{f'(x_0)}=\frac{1}{2}-\frac{1}{2^2}=\frac{1}{4}$$
Could prove by induction
$$x_n=\frac{1}{2^n}\ \forall n$$
Therefore $x_n\to 0$ but $f(0)\neq 0$

* Implicit Differentiation

November-29-10
10:49 AM

Start with
$y = -(x^5 + x)$
y is strictly decreasing so it is invertible
$x = -(y^5 + y)$
To take the derivative:
$1 = -(5y^4y' + y')$

What about
$x^7y + y^3 + x^2\sqrt{y} + x = 1$
Don't know if it's differentiable or a function

Can use the implicit differentiation theorem, but don't know it.

Cardinality

November-29-10
10:56 AM

Bijection
A function that is 1-1 and onto.

Finite Set
A set E is finite if there is a bijection
$f: E \to \{1, 2, \dots, n\}$
for some unique n.

The cardinality of E = n.

Countable
Say a set E is countable if there is a bijection
$f: E \to \mathbb{N}$ ($or\ f^{-1}: \mathbb{N} \to E$)

Any two countable sets have the same cardinality
If E, F are countable, there is a bijection
$g: E \to F$

Uncountable
Say E is uncountable if E is not countable or finite.

Say E is a finite set of n elements. Then E ↔ {1, 2, … , n}
i.e. there is a bijection (1-1, onto function)
    f : E → {1, 2, …, n}

Example of countable sets
$\mathbb{N}$
$2\mathbb{N}$
$\mathbb{Z}$
$\mathbb{N} \times \mathbb{N}, so\ is\ \mathbb{Q}$

Countable sets are those which can be put in an ordered list because if E is countable, then there is a bijection $f: \mathbb{N} \to E$ so $E = \{f(n)\}_{n=1}^{\infty}$
Conversely, if $E = \{e_n\}_{n=1}^{\infty}$ then there is a bijection
$f: E \to \mathbb{N}$
  $e_n \mapsto n$

Irrationals

December-01-10
10:40 AM

Fact
A union of two countable sets is countable.

Theorem
E is either countable or finite iff there is a map $g: \mathbb{N} \to E$ that is onto

Corollary
If $h: E \to \mathbb{N}$ is 1-1 then E is either countable or finite

Corollary
If $A \subseteq B$ and B is countable then A is either countable or finite
If $A \subseteq B$ and A is uncountable then B is uncountable

Countable set E: There is a bijection
$f: \mathbb{N} \to E$
$E = \{e_j\}_{j=1}^{\infty}$ were $e_j = f(j)$
Eg. $E = \mathbb{Q}$

Suppose $Q = \{r_j\}_{j=1}^{\infty}$
$Let\ I_j = \left(r_j - \frac{\varepsilon}{2^j}, r + \frac{\varepsilon}{e^j}\right)$
$X = \bigcup_{j=1}^{\infty} I_j = \{x: x \in I_i\ for\ some\ j\}$
Notice $r_j \in X\ \ for\ each\ j$
So $\mathbb{Q} \subseteq X$
$\sum_{j=1}^{\infty} length\ I_j = \sum_{j=1}^{\infty} \frac{2\varepsilon}{2^j} = 2\varepsilon$
R \ X has no intervals since every interval has to contain a rational number

Show (0, 1) is uncountable

Cantor Diagonal Argument
Suppose (0, 1) is countable, say $(0,1) = \{a_j\}_{j=1}^{\infty}$
Write out decimal expansion for each number, pick the expansion terminating with all 9's if there is a choice
$a_1 = 0.a_{1,1}, a_{1,2}, a_{1,3} \ldots$
$a_2 = 0.a_{2,1}, a_{2,2}, a_{2,3} \ldots$
$a_3 = 0.a_{3,1}, a_{3,2}, a_{3,3} \ldots$

Now define $r \in (0,1)$ as follows:
$r = 0.r_1 r_2 r_3$
$r_j = \begin{cases} 5\ if\ a_{j,j} \neq 5 \\ 4\ if\ a_{j,j} = 5 \end{cases}$

$r \notin \{a_j\}_{j=1}^{\infty}$
Therefore $(0,1) \neq \{a_j\}_{j=1}^{\infty}$
So (0, 1) is not countable

Unaccountability of R
If $\mathbb{R}$ is countable there is a bijection
$f: \mathbb{R} \to \mathbb{N}$

There is a bijection: $\mathbb{R} \to (0,1)$
$g(x) = \arctan(x)$
$g: \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\ bijection$
$h(x) = \frac{1}{\pi} \arctan x + \frac{1}{2}$
$h: \mathbb{R} \to (0,1)\ is\ a\ bijection$
$h\ o\ f^{-1}: \mathbb{N} \to (0,1) is\ a\ bijection$
This contradicts the fact that (0, 1) is uncountable
So $\mathbb{R}$ is uncountable.

Proof of Fact (Union of two countable sets is countable)
If A, B are countable, then
$A = \{a_j\}_{j=1}^{\infty}$
Look at $A \cup B = A \cup (B \backslash A)$
If B\A is finite, then AUB is an exercise
(just start counting a after counting all the elements of (B\A)
If B\A is not a finite set then say $B \backslash A = \{c_j\}_{j=1}^{\infty}$
Define bijection:
$f: A \cup (B \backslash A) \to \mathbb{N}$
$a_j \mapsto 2j - 1$
$c_j \mapsto 2j$

Corollary
The irrationals are uncountable. Since rationals are countable and R is uncountable.

Proof of Theorem (Countable/Finite iff onto from N)
$\Rightarrow$
Follows directly from the definition of countable/finite
$\Leftarrow$
Proof in textbook (take a function which is the onto function with every term that is a duplicate removed)

Corollary
If $h : E \to \mathbb{N}$ is 1-1 then E is either countable or finite
Proof:
Define $g: \mathbb{N} \to E$ as follows:
$If\ n \in Range\ h$, there is a unique $e \in E$ with $h(e) = \text{n}$ because h is 1-1
Then define $\text{g(n)} = \text{e}$

If $n \notin Range\ h$, then pick $e^* \in E$ and define $g(n) = e^*$
So g is an onto map, and therefor E is either countable or finite.

Corollary
1. If $A \subseteq B$ and B is countable then A is either countable or finite
2. If $A \subseteq B$ and A is uncountable then B is uncountable

Proof
1. B countable there is a bijection $f : \mathbb{N} \to B$
Define $g : \mathbb{N} \to A$ by $g(n) = f(n)$ if $f(n) \in A$, and if $f(n) \notin A$ then define $g(n) = A^* \in A$
$g: \mathbb{N} \to A$ is onto therefore, A is countable or finite

2. If B is countable, then A is countable or finite but A is not countable or finite, so B is uncountable.

Cardinality and Unions

December-03-10
10:30 AM

Theorem

E is either countable or finite iff there is a map $f: \mathbb{N} \to E$, which is onto.

Corollary

A countable union of countable or finite sets is either countable or finite.

In other words: If $A_j, j = 1, 2, 3, \dots$ are either countable or finite, then

$$A = \bigcup_{j=1}^{\infty} A_j = \{x \in A_j \; for \; some \; j = 1, 2, 3, \dots\}$$

Then A is either countable or finite
or more generally,
if $A_\alpha, \alpha \in I$ are either countable or finite and I is countable then

$$\bigcup_{\alpha \in I} A_\alpha = \{x \in A_\alpha \; for \; some \; \alpha \in I\}$$

Proof of Corollary

Each A_j is either countable or finite, so there is an onto map $f_j: \mathbb{N} \to A_j$
Define

$$h: \mathbb{N} \times \mathbb{N} \to \bigcup_{j=1}^{\infty} A_j$$

$h(j, k) = f_j(k) \in A_j \subseteq A$
h is onto, because if $a \in A$, then $a \in A_j$ for some j and since $f_j: \mathbb{N} \to A_j$ is onto there is some k $\in \mathbb{N}$ with $f_j(k) = a \Rightarrow h(j, k) = a$
Let $g: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijection
Take $h \; o \; g: \mathbb{N} \to A$, which is onto
So the union is either countable or finite.

Example

Algebraic numbers are countable, and therefore Transcendental numbers are uncountable

Proof

Algebraic numbers are numbers which satisfy polynomials with integer coefficients and by the minimal polynomial of an algebraic number, we mean the polynomial of minimal degree, with GCD of the coefficients equal to 1, and a positive leading coefficient.
$p(x) = a_n x^r + \dots + a_1 x + a_0, \qquad a_j \in \mathbb{Z}, a_n > 0, \gcd(a_1, \dots, a_n) = 1$

A_n =algebraic numbers whose minimal polynomials has degree n

$$Alg \; Numbers = \bigcup_{n=1}^{\infty} A_n$$

It's enough to prove each A_n is countable

$$A_n \subseteq \{all \; roots \; of \; integer \; polynomials \; of \; degree \; n\} = \bigcup_{p \in P_n} R_p$$

Where P_n = all integer polynomials of degree n and R_p = roots of polynomial p
Notice each R_p is a finite set of at most n elements.
So it's enough to prove each P_n is countable, because then each A_n will be contained in a countable union of finite sets.

Define a map $F_n: P_n \to \mathbb{Z}^{n+1}$ by $p = a_n x^n + \dots + a_1 x + a_0 \mapsto (a_n, \dots, a_1, a_0) \in \mathbb{Z}^{n+1}$

Finally, have to prove $\mathbb{Z}^{n+1}$ is countable. By induction on n.
$\mathbb{Z} \times \mathbb{Z}$ is countable, so $\mathbb{Z}^{n+1}$ is countable for n = 1

Assume $\mathbb{Z}^k$ is countable and prove $\mathbb{Z}^{n+1}$
$\mathbb{Z}^{n+1} = \mathbb{Z}^n \times \mathbb{Z}$
Let $f: \mathbb{Z}^k \to \mathbb{N}$ be a bijection
and $g: \mathbb{Z} \to \mathbb{N}$ be a bijection
Define $h : \mathbb{Z}^{k+1} \to \mathbb{N} \times \mathbb{N}$ by $h(w, z) = (f(w), g(z)), \; w \in \mathbb{Z}^k, x \in \mathbb{Z}$
h is a bijection,
Let $H: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection then $H \; o \; h : \mathbb{Z}^{n+1} \to \mathbb{N}$ is a bijection
This proves $\mathbb{Z}^{n+1}$
So P_n is countable
So A_n is countable
So the algebraic numbers are countable.

$\mathbb{Z}^n, \mathbb{N}^n$ are countable
What about "$\mathbb{Z}^\infty$"?
Now look at $\mathbb{N}^\mathbb{N} = \{(a_1, a_2, a_3, \dots): a_j \in \mathbb{N}\} = \{f: \mathbb{N} \to \mathbb{N}\}$
The set of all functions from $\mathbb{N}$ to $\mathbb{N}$

$\{0, 1\}^\mathbb{N} \subseteq \mathbb{N}^\mathbb{N}$
$\{0, 1\}^\mathbb{N} = \left\{(a_1, a_2, \dots) : a_j \in \{0, 1\}\right\}$

$$(a_1, a_2, a_3, \dots) \in \{0, 1\}^\mathbb{N} \leftrightarrow \sum_{j=1}^{\infty} a_j 2^{-j}$$

So $\{0, 1\}^\mathbb{N} \leftrightarrow [0, 1]$
So $\{0, 1\}^\mathbb{N}$ is uncountable.

Furthermore, $\{0, 1\}^\mathbb{N} \leftrightarrow all \; subsets \; of \; \mathbb{N}$
$(a_1, a_2, a_3, \dots) \leftrightarrow A \; where \; j \in A \; iff \; a_j = 1$
When there are finite elements in a set: $\{1, 2, \dots, n\}$ has 2^n elements
So $2^\mathbb{N} = \{0, 1\}^\mathbb{N}$
So $2^\mathbb{N}$ is the next cardinality up, in fact the cardinality of $\mathbb{R}$

Exam

December-06-10
10:44 AM

Assignments 20
Midterm 30
Final 50

Office Hours
Monday Dec 20 2:30-4:30
Friday Dec 17 1-3
or send an email to make an appointment

Previous Exam
2 proofs out of notes
4 definitions

Do some derivatives
Define BWT and EVT
Define Differentiability
Deal with a function defined differently on min/max
Find global and local extrema
Can you make a function diff at 0
State MVT
Increasing/Decreasing Concavity + Sketch graph
Invertability of Function and derivatives (application)

Suppose
$\lim_{x \to a} g(x) = 0$
and $|h(x)| \leq M \; \forall x$
prove
$\lim_{x \to a} g(x)h(x) = 0$

$Find\ f'(0) if$

$$f'(x) = \begin{cases} \frac{k(x)}{x} if\ x \neq 0 \\ 0\ if\ x = 0 \end{cases}$$

and $k(0) = k'(0) = 0, k''(0) = 17$