Vector Properties

January-05-11
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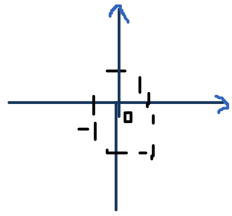
Vector in the plane
An entity with direction and magnitude. It is viewed as an arrow having a starting position and a terminating position.

Equality
Two arrows are equal if they have the same magnitude and direction

Positions vs. Vectors in $\mathbb{R}^2$
Every vector is identified with a point P so that the arrow pointing from O to P is equal to it.

Chapter 1
1.1 Introduction to Vector Spaces (Linear Spaces)

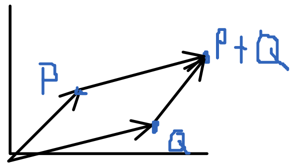


The Plane $\mathbb{R}^2$

<u>Coordinates:</u>

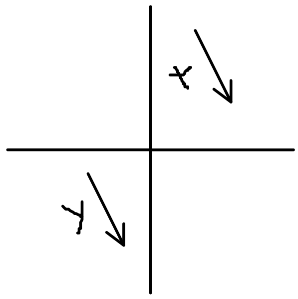
We draw a horizontal line and a vertical line intersecting a point O at right angles.
We then give the lines directions. (Arrow on the line indicates positive direction)
Further, we introduce scales. The two lines should use the same scale.
A position P on the plane (or a point) can be identified by two real quantities: its scale numbers when we draw perpendicular lines from P to the horizontal and vertical lines (coordinate axis). The numbers are represented as a tuple P = (x, y) with $x, y \in \mathbb{R}$

The plane is the set of all positions on the plane, and can be identified with the set of all pairs of real numbers.
$\mathbb{R}^2 := \{ (x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$



On $\mathbb{R}^2$ we define **addition**:
Algebraically: $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1) + (x_2 + y_2)$
Geometrically: Form a parallelogram between the two points and the origin. The 4th point is the sum.

In the diagram: x = y



Vector addition using arrows:
To add the arrows x and y, start with the arrow x from point A to point B. Then place y on the tip of x so it goes from point B to C. Then x+y is the arrow going from A to C

Scalar multiplication for the plane $\mathbb{R}^2$
Let $x = (x_1, x_2)$. Let $\lambda \in \mathbb{R}$ (a scalar)

Then $\lambda x = (\lambda x_1, \lambda x_2)$
The product λx is called the scalar multiplication of the vector x by the scalar λ
Vector addition and scalar multiplication on $\mathbb{R}^2$ satisfy 10 properties.

Properties of Vector Addition and Multiplication
(-1) $\forall\, x, y \in \mathbb{R}^2, x + y \in \mathbb{R}^2$ - Closed under addition
(0) $\forall\, \lambda \in \mathbb{R}, x \in \mathbb{R}^2, \lambda x \in \mathbb{R}^2$
(1) $x + y = y + x \;\forall\, x, y \in \mathbb{R}^2$ - Commutativity of addition
(2) $(x + y) + z = x + (y + z) \forall x, y \in \mathbb{R}^2$ - Associativity of addition
(3) $\exists 0 = (0, 0)$ so that $0 + x = x \;\forall x \in \mathbb{R}^2$ - Additive identity
(4) $\forall x \in \mathbb{R}^2, \exists y \in \mathbb{R}^2$ such that $x + y = 0$ - Additive inverse
(5) $1x = x \;\forall x \in \mathbb{R}^2$
(6) $(\lambda\mu)x = \lambda(\mu x) \;\forall \lambda, \mu \in \mathbb{R}, x \in \mathbb{R}^2$
(7) $\lambda(x + y) = \lambda x + \lambda y \;\; \forall \lambda \in \mathbb{R}, x, y \in \mathbb{R}^2$
(8) $(\lambda + \mu)x = \lambda x + \mu x \;\forall \lambda, \mu \in \mathbb{R}, x \in \mathbb{R}^2$

Vector Spaces

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Vector Space

The abstract definition of a vector space over a field.

Let V be a set (of objects) and F a field
Let there be two operations +, scalar multiplication, satisfying the ten properties of vector addition and scalar multiplication.

Uniqueness of Zero Vector

Let V be a vector space over F
Then ∃ one and only one $0 \in V$ such that $x + 0 = x$
We call the unique 0 the zero vector of V.

Uniqueness of Additive Inverses

Let V be a vector space over F
Then for every x ∈ V, ∃ one and only one y ∈ V such that $x + y = 0$.
This y is denoted -x, it is the additive inverse of x

Cancellation Law

If $x + y = x + z$ then $y = z$

Properties of a Vector Space

(-1) $\forall\, x, y \in V, x + y \in V$ - Closed under addition
(0) $\forall\, \lambda \in F, x \in V, \lambda x \in V$
(1) $x + y = y + x \;\forall\, x, y \in V$ - Commutativity of addition
(2) $(x + y) + z = x + (y + z) \forall x, y \in V$ - Associativity of addition
(3) $\exists 0 \in V$ so that $0 + x = x \;\forall x \in V$ - Additive identity
(4) $\forall x \in V, \exists y \in V$ such that $x + y = 0$ - Additive inverse
(5) $1\mathrm{x} = \mathrm{x} \;\forall x \in V$
(6) $(\lambda\mu)x = \lambda(\mu x) \;\forall \lambda, \mu \in F, x \in V$
(7) $\lambda(x + y) = \lambda x + \lambda y \;\; \forall \lambda \in F, x, y \in V$
(8) $(\lambda + \mu)x = \lambda x + \mu x \;\forall \lambda, \mu \in F, x \in V$

Once V (and F) are given two operations satisfying the ten properties, we call it a vector space over F

Examples:

Let S be any non-empty set. Let $V = \{f: S \to F\}$
Define + and scalar multiplication on V by
for $f, g \in V$,
$f + g: S \to F$
$(f + g)(s) = f(s) + g(s) \;\forall\, s \in S$
for all $f \in V, \lambda \in F$
$\lambda f: S \to F$
$(\lambda f)(s) = \lambda f(s) \;\forall s \in S$
Then V is a vector space over F

Proof of Uniqueness of 0

One of the ten axioms calls for the existence of a special element 0 ∈ V satisfying $x + 0 = x \;\forall\, \mathrm{x} \in \mathrm{V}$
Let $0_1, 0_2 \in V$ be two such elements.
By the properties of 0_1: $0_2 + 0_1 = 0_2$
By the properties of 0_2: $0_1 + 0_2 = 0_1$
Since addition is commutative, $0_1 + 0_2 = 0_2 + 0_1 = 0_2 = 0_1$ ■

Proof of Uniqueness of Additive Inverse

Let y_1 and y_2 be two y such that $\mathrm{x} + \mathrm{y} = 0$
$x + y_1 = 0 \Rightarrow x + y_1 + y_2 = y_2 \Rightarrow y_1 = y_2$

Proof 0x = 0

$0x + 0x = (0 + 0)x = 0x \Rightarrow 0x + 0x - 0x = 0x - 0x \Rightarrow 0x = 0$

Proof -x = (-1)x

$x + (-1)x = (1)x + (-1)x = (1 - 1)x = 0x = 0$
$x + (-1)x = 0 \Rightarrow x + (-1)x - x = 0 - x \Rightarrow (-1)x = -x$ ■

Observations

For $\mathbb{R}^2$, let $P = (x_1, x_2), Q = (y_1, y_2) \in \mathbb{R}^2$
The arrow (vector), x, starting from P, pointing and ending at Q, is equal to:
$x = Q - P$
Proof: By the parallelogram law, $P + x = Q \Rightarrow x = Q - P$
The midpoint between P and Q is $\frac{1}{2}(P + Q)$
The point along the line P, Q 1 unit away from P and 2 units away from Q is $\frac{2}{3}P + \frac{1}{3}Q$

Proof of cancellation law

$x + y = x + z \Rightarrow -x + x + y = -x + x + z \Rightarrow 0 + y = 0 + z \Rightarrow y = z$

* Set Theory

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Union

Then their union $A \cup B$ is defined by
$A \cup B \coloneqq \{x : x \in A \text{ or } x \in B\}$
Let $\{A_i : i \in I\}$ be a family of sets where the index set $I \neq \emptyset$
Then the union

$$\bigcup_{i \in I} A_i = \{x : \exists i \in I, x \in A_i\}$$

Intersection

Similarly, we can define $A \cap B$ and $\bigcap_{i \in I} A_i$
$A \cap B \coloneqq \{x : x \in A \text{ and } x \in B\}$

$$\bigcap_{i \in I} A_i \coloneqq \{x : x \in A_i \; \forall i \in I\}$$

Mapping

Let A and B be sets. A mapping $f : A \to B$ (A is called the domain & B is the co-domain of f) is a relation of A × B satisfying:

i) If (a, b_i) and (a, b_2) are in the relation, then $b_1 = b_2$
ii) ∀ a ∈ A, ∃ b ∈ B so that (a, b) is in the relation.

The unique b for the given a is marked f(a)

Let A and B be sets.

Union

Example: Let

$$\left\{ \left(\frac{1}{n}, \infty\right) : n \in \mathbb{N} \right\}$$

Then

$$\bigcup_{n \in \mathbb{N}} \left(\frac{1}{n}, \infty\right) = (0, \infty)$$

Need to show

$$\bigcup_{n \in \mathbb{N}} \left(\frac{1}{n}, \infty\right) \subseteq (0, \infty) \text{ and } (0, \infty) \subseteq \bigcup_{n \in \mathbb{N}} \left(\frac{1}{n}, \infty\right)$$

As for the first inclusion, we see that for each n ∈ ℕ, $\left(\frac{1}{n}, \infty\right) \subseteq (0, \infty)$, therefore their union, $\cup_{n \in \mathbb{N}} \left(\frac{1}{n}, \infty\right)$ is contained in $(0, \infty)$
For the second inclusion: Let $x \in (0, \infty)$ be given. $x > 0$ *and* $x \in \mathbb{R}$. Then ∃ n ∈ ℕ so that $\frac{1}{n} < x$. In which case $x \in (\frac{1}{n}, \infty)$ so $x \in \cup_{n \in \mathbb{N}} (\frac{1}{n}, \infty)$
So

$$\bigcup_{n \in \mathbb{N}} \left(\frac{1}{n}, \infty\right) = (0, \infty)$$

The Axiom of Choice

Let I be a non-empty (index) set.
Let $\{X_i : i \in I\}$ be a family of non-empty sets.
Consider the set

$$\bigcup_{i \in I} X_i$$

The there exists a mapping

$$f : I \to \bigcup_{i \in I} X_i$$

satisfying $f(i) \in X_i$

Accepting the axiom of choice leads to :
Every vector space has a basis

Subspaces

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Subspace

Let V be a vector space over F. A subset W ⊆ V is called a subspace of V if when the operations (addition, scalar multiplication) on V are restricted to W, W is again a vector space (over F).

Proposition

A subset W ⊆ V is a subspace iff

i. 0 of V is in W
ii. $w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$
iii. $\lambda w \in W \ \forall \lambda \in F, w \in W$

Note: i is sometimes replaced by $W \neq \emptyset$

Theorem

Let V be a vector space.
Let $\{W_i : i \in I\}$ be a family of subspaces of V, when $I \neq \emptyset$. Then

$$\bigcap_{i \in I} W_i$$

is again a subspace (of V)

Example

Let $V = \mathbb{R}^2$ and let $W = \{(x, 0) | x \in \mathbb{R}\}$
Then all 10 axioms are satisfied by W, so W is a subspace of $\mathbb{R}^2$

The subset
$S = \{(x, y) | \ x > 0, y > 0\}$
is not a vector space under the operations of $\mathbb{R}^2$ because there is no 0 and no additive inverse for any element.

Example

Let the space be $\mathcal{F}((-2,3), \mathbb{R})$
The set of all functions $f : (-2,3) \to \mathbb{R}$
Let W bet the subset of all the continuous functions.

i. $0 : f(x) = 0$
ii. If f and g are continuous, then f+g is continuous.
iii. If f is continuous, then λf is continuous for $\lambda \in \mathbb{R}$

Let S be the set of all functions of $\mathcal{F}((-2,3), \mathbb{R})$ which vanish at $-1 \ and \ 1$
i.e. $f \in \mathcal{F}((-2,3), \mathbb{R})$ and $f(-1) = 0, f(1) = 0$
Then S is a subspace
0: $f(x) = 0$, if $f, g(-1) = f, g(1) = 0$ then $f + g(-1) = f + g(1) = 0$ and $\lambda f(-1) = \lambda f(1) = 0$

Proof of Theorem

i. ∀ i ∈ I, because W_i is a subspace, $0 \in W_i$. So $0 \in \bigcap_{i \in I} W_i$
ii. Suppose $w_1, w_2 \in \bigcap_{i \in I} W_i$ are given.
Consider $w_1 + w_2$. $\forall \ i \in I, w_1 \in W_i \ and \ w_2 \in W_i \ so \ w_1 + w_2 \in W_i$
So $w_1 + w_2 \in W_i \ \forall i, so \ w_1 + w_2 \in \bigcap_{i \in I} W_i$
iii. Suppose $w \in \bigcap_{i \in I} W_i$ and $\lambda \in F$
Consider λw. $\forall i \in I, w \in W_i \ so \ \lambda w \in W_i$

$$So \ \lambda w \in W_i \ \forall \ i, so \ \lambda w \in \bigcap_{i \in I} W_i$$

Linear Combinations

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Linear Combination

Let S ⊆ V. Suppose S ≠ ∅.

A vector $v \in V$ is said to be a linear combination of S if there exist finitely many vectors of S, say $s_1, s_2, s_3 \dots s_n \in$ and scalars $\lambda_1, \lambda 2, \dots, \lambda n \in F$ so that:
$v = \lambda_1 s_1 + \lambda_2 s_2 + \cdots + \lambda_n s_n$

Span

$Span(S) = \{v \in V \mid v \text{ is a lin. comb. of vectors of } S\}$
$Span(\emptyset) = \{0\}$ by convention

Notation: Matrices

$M_{n\times m}(F)$ means an n by m matrix with elements in F

Proposition

Let V be a vector space, S ⊆ V and S ≠ ∅
Let $Span(S) = \{v \in V \mid v \text{ is a lin. comb. of vectors of } S\}$

$$= \left\{ \sum_{i=1}^{n} \lambda_i s_i \mid s_i \in S, \lambda_i \in F, n \in \mathbb{N} \right\}$$

Then Span(S) is the subspace of V generated by S

If V is a vector space and $S \subseteq V$, then there exists a unique smallest subspace of V containing S, say

$$W = \bigcap_{i\in I} W_i$$

Where $\{W_i | i \in I\}$ is the set of all subspaces of V containing S.
We call W the subspace **generated** by S.

(Unique smallest because intersection of all subspaces containing S)

Example

For $M_{2\times 2}(\mathbb{R})$ and $S = \left\{ \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \right\}$

Then $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \in M_{2\times 2}(\mathbb{R})$ is not a linear combination of vectors in S because

$\lambda_1 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \lambda_2 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \lambda_3 \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{vmatrix} \neq \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

as that would require $\lambda_2 = 2$ and $\lambda_2 = 3$

Whereas $\begin{vmatrix} 1 & 10 \\ 10 & 7 \end{vmatrix}$ is a linear combination of vectors of S

Proof of Proposition (outline)

1. Show that Span(S) is truly a subspace of V
   e.g. to show that it is closed under addition:
   Let $v_1, v_2 \in Span(S)$ be given.
   Consider $v_1 + v_2$
   $v_1 = \lambda_1 s_1 + \lambda_2 s_2 + \cdots + \lambda_n s_n$
   $v_2 = \lambda_{n+1} s_1 + \cdots + \lambda_{n+m-1} s_{n+m-1} + \lambda_{n+m} s_{n+m}$
   For some $s_1, \dots, s_{n+m} \in S$ and $\lambda_1, \dots, \lambda_{n+m} \in F$
   $$v_1 + v_2 = \sum_{i=1}^{n+m} \lambda_i s_i \in Span(S)$$
2. Observe that Span(S) ⊇ S
   Proof: Let s ∈ S be given. Then s = 1s
3. Let W_0 be any given subspace of V which contains S. We shall show $W_0 \supseteq Span(S)$
   Proof:
   For that purpose, let $v \in Span(S)$ be given.
   Then by definition, there exists vectors $S_i \in S, \lambda_i \in F$ so that
   $v = \lambda_1 s_1 + \cdots + \lambda_n s_n$. Now because $S \subseteq W_0$, $s_1, \dots s_n \in W_0$
   Since W_0 is closed under scalar multiplication and vector addition,
   $v = \lambda_1 s_1 + \cdots + \lambda_n s_n \in W_0$

Example

Let the space be $\mathcal{P}(\mathbb{C})$ - polynomials with complex coefficients, and let
$S = \{1, x^2, x^4, x^6, \dots, x^{2k}, \dots\}$
Then Span(S) = the space of all polynomials with even terms.
$Span(1, x, x^2, x^3\} = \mathcal{P}_3(\mathbb{C})$

Remark

Let V be a vector space. If S is a subspace of V, then Span(S) = S

Linear Dependence/Span

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Linear Dependence

Let V be a vector space.
Let $v_1, v_2, \dots, v_n$ be a finite list of vectors of V
We say the list is linearly dependent if one of the following two equivalent statements is satisfied:

1. There is a v_{i_0} which is in the span $\{v_i | i \neq i_0\}$
2. $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$ for some list of scalars $a_1, a_2, \dots, a_n$ not all 0

Linear Dependence on Subsets

A subset S of a vector space V is linearly dependent if for some distinct finite list of vectors extracted from S, the list is linearly dependent.

Corollary to Span({}) = {0}

In a vector space, any subset S which has 0 in it is linearly dependent.

Example

Is (1, 2, 3) ∈ Span{(1, 0, 0), (0, 1, 1), (0, 1, 2)} in $\mathbb{R}^3$? Yes

$(1,2,3) = x_1(1,0,0) + x_2(0,1,1) + x_3(0,1,2)$
System of equations:
$x_1 = 1$
$x_2 + x_3 = 2$
$x_2 + 2x_3 = 3$
Solving the above, we first bring it to the reduced system
$x_1 = 1$
$x_2 + x_3 = 2$
$x_3 = 1$
From that we read the solutions in reverse order
$x_3 = 1$
$x_2 = 2 - x_3 = 2 - 1 = 1$
$x_1 = 1$
So there is a solution, $x_1 = x_2 = x_3 = 1$

Example

Is it true that Span{(1, 0, 0), (2, 1, 0), (3, 1, 0)} = $\mathbb{Z}^3$?
Ans: Equivalently we are asking: Is every given $(a, b, c) \in \mathbb{R}^3$ in $Span\{(1,0,0), (2,1,0), (3,1,0)\}$?
We solve:
$(a,b,c) = x_1(1,0,0) + x_2(2,1,0) + x_3(3,1,0)$ for all possible $x_1, x_2, x_3 \in \mathbb{Z}_5$
$x_1 + 2x_2 + 3x_3 = a$
$x_2 + x_3 = b$
$0 = c$
Clearly, when c ≠ 0, there is no solution

Example

Consider the space of differentiable functions from ℝ to ℝ. Those satisfying the differentiable equation $f'' = 0$ are given by $f(x) = ax + b$ where a, b, are constants.
Using the language of span, the set of all solutions is Span{x, 1}

The solutions to $f'' = -f$ is $span\{\sin x, \cos x\}$

Proof of Equivalence of Linear Dependence definition

Suppose that 2 holds true.
Then there are scalars $a_1, \dots, a_n$ not all zero so that

$$\sum_{i=1}^{n} a_i v_i = 0$$

Say that $a_{i_0} \neq 0$ Now have

$$a_{i_0} v_{i_0} + \sum_{\substack{i=1 \\ i \neq i_0}}^{n} a_i v_i = 0 \Rightarrow v_{i_0} = a_{i_0}^{-1}\left(-\sum_{\substack{i=1 \\ i \neq i_0}}^{n} a_i v_i\right) = \sum_{\substack{i=1 \\ i \neq i_0}}^{n} \left(-\frac{a_i}{a_{i_0}}\right) v_i$$

So $v_{i_0} \in span\{v_i | i \neq i_0\}$

Suppose statement 1 holds true. Show 2 as an exercise.

Example

The list of vectors $\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix}, \begin{vmatrix}0 & 1\\0 & 0\end{vmatrix}, \begin{vmatrix}0 & 0\\1 & 0\end{vmatrix}, \begin{vmatrix}0 & 0\\0 & 1\end{vmatrix}, \begin{vmatrix}1 & 2\\3 & 4\end{vmatrix}$ in $M_{2\times 2}(\mathbb{R})$ is linearly dependent.
Because (using statement 1 with $i_0 = 5$)

$$\begin{vmatrix}1 & 2\\3 & 4\end{vmatrix} = 1\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix} + 2\begin{vmatrix}0 & 1\\0 & 0\end{vmatrix} + 3\begin{vmatrix}0 & 0\\1 & 0\end{vmatrix} + 4\begin{vmatrix}0 & 0\\0 & 1\end{vmatrix}$$

or

$$1\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix} + 2\begin{vmatrix}0 & 1\\0 & 0\end{vmatrix} + 3\begin{vmatrix}0 & 0\\1 & 0\end{vmatrix} + 4\begin{vmatrix}0 & 0\\0 & 1\end{vmatrix} + (-1)\begin{vmatrix}1 & 2\\3 & 4\end{vmatrix} = \begin{vmatrix}0 & 0\\0 & 0\end{vmatrix} = 0$$

Where $a_5 \neq 0b$

Example

Let the space be $\mathcal{P}(\mathbb{R})$ and let S be the set of all even polynomials. (even means $p(-x) = p(x)$)
It is linearly dependent because $v_1 = x^2, v_2 = 2x^2$

Example

Let V be a vector space.
Let $S = \{0\}$
We see that 2 holds for $v_1 = 0$ (e.g. $1v_1 = 0$)
So S is linearly dependent.

$$v_1 = \sum_{i \neq 1} v_i = \sum_{i \in \emptyset} \square = 0$$

by convention, so $Span(\emptyset) = \{0\}$

Linear Independence

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Linear Independence

A subset S of a vector space V is linearly independent if it is not linearly dependent.

Example

In $\mathbb{R}^3$, $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is linearly independent

Proof:
We need to show that the list $v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (0,0,1)$ is not linearly dependent.
Suppose $a_1, a_2, a_3 \in \mathbb{R}$ and let $a_1v_1 + a_2v_2 + a_3v_3 = 0$
$a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1) = (a_1, a_2, a_3) = (0,0,0) = 0 \; iff \; a_1 = a_2 = a_3 = 0.$
So S is linearly independent.

Example

In $\mathbb{Z}_5^3$, is $S = \{v_1 = (1,2,3), v_2 = (2,3,4), v_3 = (3,4,0)\}$ linearly dependent?
Ans: Let $a_1, a_2, a_3 \in \mathbb{Z}_5$ and that
$a_1(1,2,3) + a_2(2,3,4) + a_3(3,4,0) = (0,0,0)$

$a_1 + 2a_2 + 3a_3 \equiv 0 \; (mod\ 5)$
$2a_1 + 3a_2 + 4a_3 \equiv 0 \; (mod\ 5)$
$3a_1 + 4a_2 \equiv 0 \; (mod\ 5)$

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 0 \end{vmatrix} \; subtract\ multiples\ of\ 1st\ line\ from\ 2nd\ and\ 3rd\ lines$$

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 4 & 3 & 0 \\ 0 & 3 & 1 & 0 \end{vmatrix} \; multiply\ 2nd\ line\ by\ 4^{-1} = 4\ and\ 3rd\ line\ by\ 3^{-1} = 2$$

$$\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{vmatrix} subtract\ 2nd\ line\ from\ 3rd\ line, and\ twice\ 2nd\ line\ from\ first$$

$$\begin{vmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Solution:
$a_3 \in \mathbb{Z}_3$ is arbitrary (a free parameter)
$a_2 = -2a_3 = 3a_3$
$a_1 = -4a_3 = a_3$

So there is a solution with $a_3 \neq 0$, so yes, S is linearly dependent.

Example

Let v be a vector space over $\mathbb{R}$
Suppose that $\{v_1, v_2\}$ is linearly independent.
Show that the set $\{2v_1 + 3v_2, 4v_1 - 5v_2\}$ is linearly independent.

Proof:
Let $a_1, a_2 \in \mathbb{R}$ and that $a_1(2v_1 + 3v_2) + a_2(4v_1 - 5v_2) = 0$
$(2a_1 + 4a_2)v_1 + (3a_1 - 5a_2)v_2 = 0$
Because v_1, v_2 are linearly independent,
$2a_1 + 4a_2 = 0$
$3a_1 - 5a_2 = 0$

$2a_1 + 4a_2 = 0 \; (retained)$
$\left(-\frac{3}{2}4 - 5\right)a_2 = 0$
$So\ a_2 = 0, and\ therefore\ a_1 = 0.$ So $\{2v_1 + 3v_2, 4v_1 - 5v_2\}$ is linearly independent.

Gaussian and Jordan Eliminations

January-24-11
11:31 AM

Example: Gaussian Elimination

Solve

$$\begin{cases} 2a_1 + 2a_2 + 4a_3 = 2 \\ a_1 - a_2 + 7a_3 = 5 \\ a_1 + 8a_3 = 0 \end{cases}$$

$$\begin{cases} 2a_1 + 3a_2 + 4a_3 = 2 \\ -\frac{5}{2}a_2 + 5a_3 = 4 \\ -\frac{3}{2}a_2 + 6a_3 = -1 \end{cases}$$

$$\begin{cases} 2a_1 + 3a_2 + 4a_3 = 2 \\ -\frac{5}{2}a_2 + 5a_3 = 4 \\ 3a_3 = -\frac{17}{5} \end{cases}$$

End of Gaussian Elimination, write out the general solution:

$$a_3 = -\frac{17}{15}$$

$$a_2 = \frac{4 - 5a_3}{-\frac{5}{2}} = \frac{8 - 10\left(-\frac{17}{15}\right)}{-5} = -\frac{58}{15}$$

$$a_3 = \frac{2 - 3a_2 + 4a_3}{2} = \frac{2 - 3\left(-\frac{58}{15}\right) - 4\left(-\frac{17}{15}\right)}{2}$$

Jordan Elimination Steps

Used to reduce the system further

1. Multiply the lines to set the 1st non-zero coefficients equal to 1
2. Eliminate the variables from the lines above each 1

Continuing from the system above:

$$\begin{cases} a_1 + \frac{3}{2}a_2 + 2a_3 = 1 \\ a_2 - 2a_3 = -\frac{8}{5} \\ a_3 = -\frac{17}{15} \end{cases}$$

$$\begin{cases} a_1 + 5a_3 = \frac{17}{5} \\ a_2 - 3a_3 = -\frac{8}{5} \\ a_3 = -\frac{17}{15} \end{cases}$$

$$\begin{cases} a_1 = \frac{136}{15} \\ a_2 = -5 \\ a_3 = -\frac{17}{15} \end{cases}$$

Why no work? :(

Augmented Matrix

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_m = b_n \end{cases}$$

Represented by

$$\left|\begin{matrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{matrix}\right| \left|\begin{matrix} b_1 \\ b_2 \\ \dots \\ b_m \end{matrix}\right|$$

Set Theory Cont.*

January-24-11
3:34 PM

Let X and Y be sets.

Injective

A function $f: X \to Y$ is injective (one-to-one) if
$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ or alternatively
$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Smaller Cardinality

A set X is said to be of smaller cardinality than set Y if there is an injective map $f: X \to Y$

Surjective

A function $f: X \to Y$ is surjective (or onto) if for all $y \in Y$ there exists $x \in X$ so that $f(x) = y$

Proposition

These statements are equivalent:
For two sets X, Y

1. There is an injective function $f: X \to Y$
2. There is a surjective function $g: Y \to X$

Equal Cardinality

Two set X, Y are of equal cardinality if there exists $f: X \to Y$ which is injective and surjective (bijective)

Theorem (Bernstein)

Let X and Y be sets. If there exists an injective $f: X \to Y$ and an injective $g: X \to Y$ there exists a bijective $h: X \to Y$
Rephrase: If $|X| \leq |Y|, |Y| \leq |X|, then\ |X| = |Y|$

Immediate clear is that if X is finite with n distinct and Y has fewer elements than X then no $f: X \to Y$ can be injective.

Example of cardinality differences:
[0, 4] has a smaller cardinality than [0, 1]
$f: [0, 4] \to [0, 1], x \to \frac{1}{4}x$

Similarly, [0, 1] has smaller cardinality than [0, 4]

Proof of Proposition

Suppose we have a surjective $g: Y \to X$

For each $x \in X$, consider $S_x = \{y \in Y: g(y) = x\} \subseteq Y$
As g is surjective, each S_x is non-empty. Moreover, $x_1 \neq x_2$ implies S_{x_1} and S_{x_2} are disjoint.
The family $\{S_x: x \in X\}$ form a partition of Y
By the axiom of choice, there is a function (choice)
$f: X \to \bigcup_{x \in X} S_x = Y\ so\ that\ f(x) \in S_x$
Obviously, f is injective

Basis

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11:33 AM

Basis

Let V be a vector space over F. A subset $B \subseteq V$ is called a basis for V if it satisfies:

1. B is linearly independent

Intuitively, B is "small", that no element of B is a linear combination of the others.

2. B spans V, i.e. $span(B) = V$

Finite Dimensional

If V has a finite set B which forms a basis, then we say V is finite dimensional.

Theorem

Suppose that V has a finite basis B with n elements. Then all other bases must have n elements. We call n the dimension of V.

Example

Consider $\mathbb{R}^3$. Subsets satisfying the 1st properties are, e.g.
$\emptyset, \{(1,0,0)\}, \{(1,0,0),(1,1,0)\}, \{(1,0,0),(1,1,0),(0,0,2)\}$

Of these examples
$span(\emptyset) = \{(0,0,0)\}$
$span\{(1,0,0)\} = (x,0,0) = the\ x-axis$
$span\{(1,0,0),(1,1,0)\} = (x+y,y,0) = the\ xy\ plane$
$span\{(1,0,0),(1,1,0),(0,0,2)\} = (x+y,y,2z) = \mathbb{R}^3$
So the last is a basis.

Example

$In\ P(\mathbb{R})$
$B = \{1,x,x^2,x^3,\dots,x^n,\dots\} = \{x_n : x \in \mathbb{N}, or\ n = 0\}$ $x^0 = 1$by convention
is a basis.

Proof:

To check for linear independence:

Let a finite number of terms be extracted from B (all terms are distinct)

WLOT that the list is $1, x, x^2, \dots x^n$
Will show that the list is not linearly dependent. Let $a_0, a_1, \dots, a_n$ be scalars and that
$a_0 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$
By definition of equality between polynomials, $a_0 = a_1 = \cdots = a_n = 0$

Hence, every finite list of distinct terms from B is linearly independent. So B is linearly independent.

Next check if $span(B) \supseteq \mathcal{P}(\mathbb{R})$
Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ for some $a_i \in \mathbb{R}, n \in \mathbb{N}$
Therefore, clearly $p(x) \in Span\{1, x, x^2, \dots, x^n\} \supseteq Span\ (B)$
Hence $\mathcal{P}(\mathbb{R}) \subseteq span(B)$. Equality follows. So B is a basis.

Example

$V = \{A \in M_{3\times4}(\mathbb{R}) : column\ sums\ of\ A\ are\ zero\}$

$$e.g. \begin{vmatrix} 1 & 4 & \pi & 0 \\ 2 & 5 & e & 0 \\ -3 & -1 & -\pi - e & 0 \end{vmatrix}$$

The dimensionality is the number of free scalars. In this case
$\dim V = 8$

Replacement

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Theorem 1.8
Let S be a linearly independent subset of a vector space V and let x be an element of V that is not in S. Then $S \cup \{x\}$ is linearly dependent iff $x \in span(S)$

Theorem 1.9
If a vector space V is generated by a finite set S_0 then a subset of S_0 is a basis for V. Hence V has a finite basis.

Replacement Theorem 1.10
Let V be a vector space having a basis β containing exactly n elements. Let $S = \{y_1, \dots, y_m\}$ be a linearly independent subset of V containing exactly m elements, where m ≤ n. Then there exists a subset S_1 of β containing exactly n-m elements such that $S \cup S_1$ generates V.

Corollary 1
Let V be a vector space having a basis β containing exactly n elements. Then any linearly independent subset of V containing exactly n elements is a basis for V.

Corollary 2
Let V be a vector space having a basis β containing exactly n elements. Then any subset of V containing more than n elements is linearly dependent. Consequently, any linearly independent subset of V contains at most n elements.

Corollary 3
Let V be a vector space having a basis β containing exactly n elements. Then every basis for V contains exactly n elements.

Definition
A vector space V is called finite-dimensional if it has a basis consisting of a finite number of elements; the unique number of elements in each basis for V is called the dimension of V and is denoted $\dim(V)$. If a vector space is not finite dimensional, then it is called infinite-dimensional.

Corollary 4
Let V be a vector space having dimension n and let S be a subset of V that generates V and contains at most n elements. Then S is a basis for V and hence contains exactly contains exactly n elements.

Corollary 5
Let β be a basis for a finite-dimensional vector space V and let S be a linearly independent subset of V. There exists a subset S_1 of β such that $S \cup S_1$ is a basis for V. Thus every linearly independent subset of V can be extended to a basis for V.

Proof of Theorem 1.8
Suppose $S \cup \{x\}$ is linearly dependent.
Then

$$0 = a_0 x + \sum_{i=1}^{n} a_i x_i$$

With not all $a_i = 0$ and since S is linearly independent, $a_0 \neq 0$ so

$$x = \sum_{i=1}^{n} -\frac{a_i}{a_0} x_i$$

So $x \in span(S)$
Suppose $x \in span(S)$, then

$$x = \sum_{i=1}^{n} a_i x_i$$

so $S \cup \{x\}$ is linearly dependent. ■

Proof of Theorem 1.9
If $S_0 = \emptyset$ or $S_0 = \{0\}$ then $V = \emptyset$ and $\emptyset$ is a basis for V.
Otherwise pick $x_1 \in S_0$. $\{x_1\}$ is linearly independent.
Now with a linearly independent set of $n-1$ vectors $x_i \in S_0$ if $S_0 \subseteq span(\{x_1, \dots, x_{n-1}\})$ then done since the set is linearly independent and generates V so it is a basis. Otherwise find $x_n \in S_0, x_n \notin span(\{x_1, \dots, x_{n-1}\})$ By theorem 1.8 $\{x_1, \dots, x_n\}$ is linearly independent. Continue until terminating after finitely many x_i since S_0 is finite.

Proof of Theorem 1.10
Proof by induction on m.

If m = 0, then $S = \emptyset$ and n - m = n so take $S_0 = \beta, S \cup S_1 = \emptyset \cup \beta = \beta$ is a basis for V

Now suppose the statement holds true for $m - 1$.
Let $S_0 = \{y_1, \dots, y_{m-1}\}$. $\exists \beta_0 \subset \beta$ with $|\beta_0| = n - (m-1)$ such that $span(S_0 \cup \beta_0) = V$ by induction supposition.
So

$$y_m = \sum_{x_i \in S_0} a_i x_i + \sum_{z_j \in \beta_0} b_j z_j$$

But S is linearly independent so at least one $b_j \neq 0$, say b_1
Then

$$z_1 = \frac{y_m}{b_1} + \sum_{x_i \in S_0} -\frac{a_i}{b_1} x_i + \sum_{\substack{z_j \in B_0 \\ j \neq 1}} -\frac{b_j}{b_1} z_j$$

So $z_1 \in span(\{y_1, \dots, y_m, z_2, \dots z_{n-m+1}\})$
Clearly $y_1, \dots, y_{m-1}, z_2, \dots, z_{n-m+1} \in span(\{y_1, \dots, y_m, z_2, \dots, z_{n-m+1}\})$
$S_1 = \beta_0 \setminus \{z_1\}$
So $S_0 \cup \beta_0 \subseteq span(S \cup S_1)$
$span(S_0 \cup \beta_0) = V$ so $span(S \cup S_1) = V$
So there is a subset of β such that $span(S \cup S_1) = V$ ∀m, by the induction principle. ■

Corollary 1
Let S be a linearly independent subset of V with exactly n elements.
Then $\exists S_1$ such that $span(S \cup S_1) = V$ and $|S_1| = n - n = 0 \Rightarrow S_1 = \emptyset$
so $span(S \cup S_1) = span(S) = V$ so S is a basis for V.

Corollary 2
Let S be a subset of V with more than n elements. Suppose that S is linearly independent, then there is an $S_0 \subset S$ with n elements. By Corollary 1, S_0 is a basis so $span(S_0) = V$. Let $x \in S, x \in S_0$, then $S_0 \cup \{x\}$ is linearly dependent, contradicting the supposition that S is linearly independent. Therefore, S is linearly dependent. ■

Corollary 3
Let S be a basis for V. We know $|S| \leq n$ since $|\beta| = n$. Suppose $|S| < n$, then by Corollary 2 β would not be linearly independent, a contradiction, so $|S| = n$. ■

Corollary 4
By Theorem 1.9, $\exists S_1 \subseteq S$ such that S_1 is a basis for V. $|S_1| = n, |S_1| \leq |S| \leq n$ so $|S| = n$ so $S_0 = S$ and S is a basis for V. ■

Corollary 5
$|S| = m \leq n, |\beta| = n$ so by Theorem 1.10, $\exists S_1 \subseteq \beta, |S_1| = n - m$ such that $S \cup S_1$ generates V.
Since $|S \cup S_1| = n$, by Corollary 4 $S \cup S_1$ generates V.

General Bases

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11:31 AM

Proposition

Let V be a vector space. Let $L \subset V$ be linearly independent. Then the following two statements are equivalent.

1. $v \in V, v \notin L$ and $L \cup \{v\}$ is linearly independent.
2. $v \notin span(L)$

Proposition

Let V be a vector space. Let $L \subset V$ be linearly independent, $G \subset V$ be generating, $L \subset G$. Suppose that v is such that $v \notin L, L \cup \{v\}$ is still linearly independent.

Then there exists a $u \in G$ so that $u \notin L$ and $L \cup \{u\}$ is (still) independent.

Remark

If V is a finite vector space.
If F is infinite, like $\mathbb{C}$, then $V = \{0\}$
If F is finite, then $|V| = |F|^n$ for some $n \in \mathbb{N}^0$

Proof of Proposition 1

Suppose v satisfied 1. To argue for 2, assume to the contrary that v ∈ Span(L). Then

$$v = \sum_{i=1}^{n} \lambda_i v_i$$

for some distinct v_i's in L and $\lambda_i \in F$
As $v \notin L, v_1, \dots, v_n, v$ *are all distinct*, we have a set of distinct vectors such that one is a linear combination of the rest, so the set $L \cup \{v\}$ is linearly dependent, a contradiction.

Conversely, suppose that 2 holds, we need to show 1
As $Span(L) \supset L$, it is clear that $v \notin L$. To show that $L \cup \{v\}$ is linearly independent, suppose that

$$\sum_{(i=0)}^{n} \lambda_i v_i = 0$$

where $v_i, \dots, v_n$ are distinct elements from $L \cup \{v\}$

Case 1:

Suppose that none of the v_i are v. Then by linear independence of L, all $\lambda_i = 0$

Case 2:

One of $v_1, \dots, v_n$ is equal to v. WLOG say that $v_n = v$
Suppose that $\lambda_n = 0$ Then

$$\sum_{i=1}^{n-1} \lambda_i v_i = 0$$

By the linear independence of L, we set $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$
Thus $\lambda_1, \dots \lambda_n$ are 0

Suppose that $\lambda_n \neq 0$ Then from

$$\sum_{i=1}^{n} \lambda_i v_i = 0$$

$$v = v_n = \sum_{i=1}^{n-1} -\frac{\lambda_i}{\lambda_n} v_i$$

So $v \in Span(L)$

Proof of Proposition 2

$v \notin Span(L)$
It is a linear combination of things in G
So , (WLOG, n is the least number which satisfies the linear combination)

$$v = \sum_{i=1}^{n} \lambda_i u_i = \sum_{i=1}^{k} \lambda_i u_i + \sum_{i=k+1}^{n} \lambda_i u_i$$

where $u_1, \dots, u_n$ are distinct vectors in G
WLOG, $u_1, \dots u_k \in L, u_{k+1}, \dots, u_n \notin L$
At least one u_i $(k+1 \leq i \leq n)$ is present with $\lambda \neq 0$. Take $u = u_{k+1}$
This means $u \neq span(L)$ since the above is the smallest representation and if $u \in span(L)$ then u could be written as part of $\sum_{i=1}^{k} \lambda_i u_i$

Suppose $L \cup \{u\}$ were linearly dependent. Then

$$0 = \lambda u + \sum_{i=1}^{m} \lambda_i v_i \ \ for\ v_i \in L, \lambda_i not\ all\ 0$$

L is linearly independent so $\lambda \neq 0$ so

$$u = \sum_{i=1}^{m} -\frac{\lambda_i}{\lambda} v_i$$

So $u \in span(L)$, a contradiction. So $L \cup \{u\}$ is linearly independent. ■

Example

Basis of any size.

Let S be any set $(\neq \emptyset)$. We now construct a vector space V over F having a basis B with $|B| = |S|$

Consider the subspace $\mathcal{F}_0$ of $\mathcal{F}(S, F)$ consisting of functions $f: S \to F$ with $f(s) = 0$ for all but finitely many s. For each fixed element $s \in S$, let $\chi_s: S \to F$ be $\chi_s(t) = \begin{cases} 1\ for\ t = s \\ 0\ for\ t \neq s \end{cases}$

χ is the characteristic function.

Clearly $\chi_s \in \mathcal{F}_0$
Let $B = \{\chi_s : s \in S\} \subset \mathcal{F}_0$
Be is a basis for $\mathcal{F}_0$ because

1. Let $f \in \mathcal{F}_0$ be given. Then ∃ finitely many $s_1, s_2, \dots, s_n \in S$ with $f(s) = 0\ if\ s \notin \{s_1, \dots, s_n\}$
   Let $\lambda_i = f(s_i)$ for $i \in \{1, \dots, n\}$
   Then $f = \lambda_1 \chi_{s_1} + \lambda_2 \chi_{s_2} + \cdots + \lambda_n \chi_{s_n}$
   Therefore $f \in Span(B)$
2. Let $\chi_{s_1}, \chi_{s_2}, \dots, \chi_{s_n}$ be a finite list of distinct vectors in B and that $\lambda_1, \lambda_2, \dots, \lambda_n$ are scalars from F with

   $$\sum_{i=1}^{n} \lambda_i \chi_{s_i} = 0$$

   Since χ_{s_i} are distinct, clearly s_i are distinct. Fix any $i_0 \in \{1, \dots, n\}$

$$\left(\sum_{i=1}^{n} \lambda_i \chi_{s_i}\right)(s_{i_0}) = 0\,(s_{i_0}) = 0$$

$$= \sum_{i=1}^{n} \lambda_i \chi_{s_i}(s_{i_0}) = \lambda_{i_0} 1 = 0$$

So $\lambda_i = 0\ \forall i$

So B is linearly independent.

$\chi: S \to B$

$\chi(s) = \chi_s$

is clearly bijective. So B is of the same cardinality as S. ■

* Maximal Principle

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11:31 AM

Maximal Principle

Let X be a set. Let $\mathcal{C}$ be a collection of subsets of X. A sub-collection of $\mathcal{C}$, say $\mathcal{T} \subseteq \mathcal{C}$ is called a tower (or chain) if for any two elements $T_1, T_2 \in \mathcal{T}$, either $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$.

Suppose that $\mathcal{C}$ has the property that every tower $\mathcal{T}$, there exists $C \in \mathcal{C}$ such that $C \supseteq T$ for all $T \in \mathcal{T}$. (C is called an upper bound for $\mathcal{T}$)

Then $\mathcal{C}$ has a maximal element M $\in \mathcal{C}$ i.e. no $C \in \mathcal{C}$ contains M strictly.

Application

Let V be any vector space over F. Let $\mathcal{C}$ be the set of all linearly independent subsets of V.
If $\mathcal{T}$ is a tower in $\mathcal{C}$ it is not difficult to check that

$$\bigcup_{T \in \mathcal{T}} T$$

is also linearly independent. So it is in $\mathcal{C}$ and it is an upper bound for $\mathcal{T}$. So by the maximal principle, there is a maximal $M \in \mathcal{C}$.
M will be a basis for V.

Example

Let $\mathcal{C}$ be the set of all finite open intervals of $\mathbb{R}$
$\mathcal{T} = \{(0, n) : n \in \mathbb{N}\}$ is a tower/chain
This tower has no upper bound in $\mathcal{C}$

No member of $\mathcal{C}$ is maximal because for every $C \in \mathcal{C}, c = (a, b)$ finite a, b the element $(a, b+1)$ is strictly larger.

Example

Let X be any non-empty set.
Let $\mathcal{C} = \{C : C \subset X\}$
Then $M = X \backslash \{x\}\ for\ some\ x \in X$ is a maximal element for $\mathcal{C}$

Examples

$X = \mathbb{R}, M = \mathbb{R} \backslash \{1\}$ is M maximal, yes.
$N = \mathbb{R} \backslash \{2\}$ is also maximal
$C = \mathbb{R} \backslash \{1, 2\}$ is not maximal
Look at $\mathcal{T} = \{[-n, n] : n \in \mathbb{N}\}$ is a tower with no upper bound

So every tower having an upper bound ⇒ there is a maximal element
There being a maximal element ⇏ every tower has an upper bound

Linear Mappings

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11:32 AM

Linear Mapping

Let U and V be vector spaces over F. A mapping (function) $L: U \to V$ is linear if:

1. L preserves summation
   $L(u_1 + u_2) = L(u_1) + L(u_2)$
2. L preserves scalar multiplication
   $L(\lambda u) = \lambda L(u) \text{ for } \lambda \in F$

Proposition

For any linear $L: U \to V$

1. $L(0) = 0$
2. $L(-u) = -L(u)$
3. $L\left(\sum_{i=1}^{n} \lambda_i u_i\right) = \sum_{i=1}^{n} \lambda_i L(u_i)$
   L preserves linear combinations

Kernel (Nullspace)

Let $L: U \to V$ be linear
$Ker(L) = Nullspace(L) \coloneqq \{u \in U | L(u) = 0\}$

Range (Image)

Let $L: U \to V$ be linear
$Range(L) = Im(L) \coloneqq \{L(u) | u \in U\}$

Proposition

$Ker(L)$ is a subspace of U
$Range(L)$ is a subspace of V

Example

$L: \mathbb{R}^3 \to \mathbb{R}^3, L(x, y, z) = (x, 0, z)\ \forall (x, y, z) \in \mathbb{R}^3$.
Then L is linear.

Proof:

1. Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$
   Then $L\big((x_1, y_1, z_1) + (x_2, y_2, z_2)\big) = L(x_1 + x_2, y_1 + y_2, z_1 + z_2) = (x_1 + x_2, 0, z_1 + z_2)$
   $L(x_1, y_1, z_1) + L(x_2, y_2, z_2) = (x_1, 0, z_1) + (x_2, 0, z_2) = (x_1 + x_2, 0, z_1 + z_2)$
2. Let $\lambda \in \mathbb{R}, (x, y, z) \in \mathbb{R}^3$
   $L\big(\lambda(x, y, z)\big) = L(\lambda x, \lambda y, \lambda z) = (\lambda x, 0, \lambda z) = \lambda(x, 0, z) = \lambda L(x, y, z)$

Example

$L: \mathbb{R}^3 \to M_{3\times3}(\mathbb{R}), L(x, y, z) = \begin{vmatrix} x & y & z \\ z & y & 2x \\ 0 & x + y & z \end{vmatrix}$

This is a linear mapping

Example

$L: P_2(\mathbb{R}) \to P_3(\mathbb{R}), L\big(p(x)\big) = xp(x)$ L is linear

Example of a non-linear map

$f: \mathbb{R}^2 \to \mathbb{R}^2, f(x, y) = (x + 1, y)$
Then f is not linear

$f(x_1, y_1) + f(x_2, y_2) = (x_1 + 1, y_1) + (x_2 + 1, y_2) = (x_1 + x_2 + 2, y_1 + y_2)$
$f\big((x_1, y_1) + (x_2, y_2)\big) = f(x_1 + x_2, y_1 + y_2) = (x_1 + x_2 + 1, y_1 + y_2)$
So f does not preserve summation. Similarly, it does not preserve multiplication.

Proof of Proposition

1. Because L preserves addition, $L(0 + 0) = L(0) + L(0) \Rightarrow L(0) = L(0) + L(0)$ so $L(0) = 0 \in V$
2. $L(-u) = L\big((-1)u\big) = (-1)L(u) = -L(u)$
3. Follows directly from preservation of addition and scalar multiplication

Dimension Theorem

February-09-11
11:32 AM

Nullity and Rank

Let $L: U \to V$ be linear. Suppose that U is finite dimensional.
The nullspace (kernel) of L, $N(L) = \{u \in U \mid L(u) = 0\}$, is a subspace of U.
Then N(L) is finite dimensional. $Nullity\ (L) = \dim N(L)$

The dimension of the range space, $R(L) = \{L(u) | u \in U\}$ is called the Rank of L, denoted $rank(L)$

Dimension Theorem (Rank and Nullity Theorem)

For linear $L: U \to V$, finite dimensional U,
$\dim(U) = rank(L) + nullity\ (L)$

Example

$L: \mathbb{R}^3 \to \mathbb{R}^4$ is given by $L(x, y, z) = (x + y, y + z, 0, 0)$ has range
$R(L) = \{(a, b, 0, 0) |\ a, b \in \mathbb{R}\}$ and $rank(L) = 2$
It has $N(L) = \{(x, y, z) \in \mathbb{R}^3 \mid (x + y, y + z, 0, 0) = (0, 0, 0, 0)\}$

$$= \left\{(x, y, z) \in \mathbb{R}^3 \mid \begin{matrix} x + y = 0 \\ y + z = 0 \end{matrix}\right\}$$

Nullity (L) = $\dim N(L) = 1$

Proof of Rank and Nullity Theorem

Pick a basis for N(L), say $\{u_1, u_2, \ldots, u_k\}$
Now, $nullity(L) = k$
Extend the linearly independent set $\{u_1, u_2, \ldots, u_k\}$ to a basis for U
say that $\{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ is a basis for U
So $\dim(U) = n$

Claim: $\beta = \{L(u_{k+1}), L(u_{k+2}), \ldots, L(u_n)\}$ is a basis for Range(L). Thus $rank(L) = n - k$

1. Show that β spans $Range(L)$
   Let $v \in Range(L)$ be given. Then $\exists u \in U\ s.t.\ L(u) = v$
   Since $\{u_1, \ldots, u_n\}$ spans U, there exist scalars $\lambda_1, \ldots, \lambda_n$ so that
   $$u = \sum_{i=1}^{n} \lambda_i u_i$$
   Now,
   $$v = L(u) = L\left(\sum_{i=1}^{n} \lambda_i u_i\right) = \sum_{i=1}^{n} \lambda_i L(u_i)$$, since L is linear
   $But\ L(u_i) = 0\ \forall i \in \{1, \ldots, k\}\ so$
   $$v = \sum_{i=k+1}^{n} \lambda_i L(u_i)$$
   So $v \in span\ \beta$
   So $span\ \beta = Range(L)$
2. Show that β is linearly independent.
   Suppose $\lambda_{k+1} L(u_{k+1}) + \cdots + \lambda_n L(u_n) = 0 \Rightarrow L(\lambda_{k+1} u_{k+1} + \cdots + \lambda_n u_n) = 0$
   So $\lambda_{k+1} u_{k+1} + \cdots + \lambda_n u_n \in N(L)$
   As $u_1, \ldots, u_n\ spans\ N(L)$
   $\lambda_{k+1} u_{k+1} + \cdots + \lambda_n u_n = \lambda_1 u_1 + \cdots + \lambda_k u_k$ for some scalars λ_i
   So $\lambda_1 u_1 + \cdots + \lambda_k u_k - \lambda_{k+1} u_{k+1} - \cdots - \lambda_n u_n = 0$
   So $\lambda_1 = \cdots = \lambda_k = \lambda_{k+1} = \cdots = \lambda_n = 0$
   So β is linearly independent.

Example

Let $L: \mathbb{R}^3 \to M_{6\times 6}(\mathbb{R})$ be

$$L(x, y, z) = \begin{bmatrix} x & y & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank(L) = 3, N(L) = 0

Rank/Nullity

February-11-11
11:28 AM

Proposition

Every Linear $L: U \to V$ is completely determined by the restriction, $L|_B$, to a basis B for U

Simple consequences of the dimension (rank/nullity) theorem.

Observations.

1. If $L: U \to V$ is linear, then L is injective iff Ker(L) = {0}

   Proof: Suppose that $L(u_1) = L(u_2)$ for given $u_1, u_2 \in U$
   Now $L(u_1) - L(u_2) = 0$. As L is linear, $L(u_1 - u_2) = 0$ so $u_1 - u_2 \in Ker(L)$. Since Ker(L) = {0}, we set $u_1 - u_2 = 0 \Rightarrow u_1 = u_2$

   For the converse: Suppose L is injective
   Because L is linear, L(0) = 0, so 0 in Ker(L)
   To get Ker(L) = {0}, we need to show that for any given u in Ker(L), we have u = 0
   Let u in Ker(L) be given. Then L(u) =0. Since L is linear, L(0) = 0. So L(u) = L(0)
   As L in injective, u = 0 follows.

   Restate: Linear L is injective iff dim Ker(L) = 0 iff nullity(L) = 0

2. Linear $L: U \to V$ is surjective iff L(U) = V. If V is finite dimensional, then L is surjective iff dim L(U) = dim V, iff rank(L) = dim V

   By the dimensional theorem, we get the Corollary:
   If $L: U \to V$ is linear and both U and V are of the same dimension, then the following two statements are equivalent:
   1. L is injective
   2. L is surjective

   Basic idea: dim U = rank(L) + nullity(L) = dim V
   Injective <=> nullity(L) = 0 <=> rank(L) = dim V <=> surjective

   In particular, if U is finite dimensional and L is a linear operator on U, then L is injective iff it is surjective.

Example of Proposition:

Suppose that$L: \mathbb{R}^2 \to P_2(\mathbb{R})$ is linear, and that B = {(1, 0), (0, 1)}
If we know L(1, 0) and L(0, 1) (that is, we know $L|_B$), we should be able to tell L(x, y) for general $(x, y) \in \mathbb{R}^2$
Reason: $(x, y) = x(1, 0) + y(0, 1), so\ L(x, y) = L(x(1, 0) + y(0, 1) = xL(1, 0) + yL(0, 1)$

Proof of Proposition:

Let $B = \{b_i |\ i \in I\}$ be a basis for U.
Given any vector u ∈ U, we can write $u = \sum_{i=1}^{n} \lambda_i b_i$ for finitely many $b_i \in B$
Now,

$$L(u) = \sum_{i=1}^{n} \lambda_i L(b_i)$$

Example

WE could define a linear map $L: \mathbb{R}^2 \to \mathbb{R}^2$ by specifying L(1, 0) and L(0, 1), say L(1, 0) = (1, 1) and L(0, 1) = (-1, -1), Implicitly, we know L fully
Explicitly : $L(x, y) = x\, L(1, 0) + y\, L(0, 1) = x(1, 1) + y(-1, -1) = (x - y, x - y)$
Rank(L) = 1, Nullity(L) = 1
Range(L) = span(L(1, 0), L(0, 1)) = span{(1, 1), (-1, -1)} = span{(1, 1)}, a basis is (1, 1)
N(L) is $\{(x, y) \in \mathbb{R}^2 \mid x - y = 0\}$, a basis is (1, 1)

Example

$D\colon P_{10}(\mathbb{R}) \to P_{10}(\mathbb{R}). D(p(x)) = p'(x)$
$D(1) = 0, D(x) = 1, D(x^2) = 2x, \dots, D(x^{10}) = 10x^9$
Note $\{1, x, x\text{^}2, \dots, x^{10}\}$ is a basis for P_{10}
$R(D) = P_9(\mathbb{R})$
$N(D) = P_0(\mathbb{R}) = span\{1\}$
$Rank(D) = 10, nullity(D) = 1, dim\, P_{10}(\mathbb{R}) = 11$

Coordinatization

February-14-11
11:33 AM

Coordinatizing a Space

Let U be a finite dimensional space.
Fix a basis $\beta = \{u_1, u_2, \dots, u_n\}$ and order it as presented.
Every vector $u \in U$ can be uniquely written:

$$u = \sum_{i=1}^{n} a_i u_i \, , a_i \in F$$

$$(a_1, \dots, a_n) \neq (b_1, \dots, b_n) \Rightarrow \sum_{i=1}^{n} a_i u_i \neq \sum_{i=1}^{n} b_i u_i$$

Coordinates

We call $(a_1, a_2, \dots, a_n)$ the coordinates of u with respect to (relative to) β. Notation:

$$|u|_\beta = \begin{vmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{vmatrix}, or \ (a_1, \dots, a_n)$$

Proposition

Let U be a space with ordered basis β.
The correspondence
$u \in U \rightarrow |u|_\beta \in F^n$
is a bijective linear map. Thus U is isomorphic to F^n

It is easy to check that $|u_1 + u_2|_\beta = |u_1|_\beta + |u_2|_\beta$
$|\lambda u|_\beta = \lambda |u|_\beta$

Representation of Linear Maps

A linear map $L: U \rightarrow V$ can by represented by a matrix.

Let U, V be finite dimensional. Let α, β be ordered bases for U and V, respectively.
$\alpha = \{u_1, \dots, u_n\}, \beta = \{v_1, \dots, v_m\}$
Now $L: U \rightarrow V$, linear, is determined by knowing
$L(u_1), L(u_2), \dots, L(u_n)$. Each $L(u_i)$ is determined by knowing $|L(u_i)|_\beta$ - (column formation)

The matrix
$||L(u_1)|_\beta \quad |L(u_2)|_\beta \quad \dots \quad |L(u_n)|_\beta|$
Size $m \times n$ is called the matrix representation of L with respect to α, β

Proposition

Let $L_1, L_2: U \rightarrow V$ be linear. $x \in F$
Let α for U and β for V be fixed finite ordered bases.
Then $L_1 + L_2: U \rightarrow V, (L_1 + L_2)(u) = L_1(u) + L_2(u)$
$\lambda L_1: U \rightarrow V, (\lambda L_1)(u) = \lambda(L_1(u))$ are linear (exercise)

$|L_1 + L_2|_\alpha^\beta = |L_1|_\alpha^\beta + |L_2|_\alpha^\beta, \qquad |\lambda L_1|_\alpha^\beta = \lambda |L_1|_\alpha^\beta$
Thus $|\square|_\alpha^\beta$: all linear maps from U to V → $M_{m \times n}(F)$
is linear.

Example

Let $U = P_2(\mathbb{R})$. Let $\beta = \{x^2, x, 1\}$ (ordered)
Let $u = 4 + 2x + 5x^2 = 5(x^2) + 2(x) + 4(1)$
So

$$u = \begin{vmatrix} 5 \\ 2 \\ 4 \end{vmatrix} \ or \ (5, 3, 4)$$

P_2 is isomorphic to $\mathbb{R}^3$

Example

Let $D: P_2 \rightarrow P_2$ over $\mathbb{R}, D(f) = f'$
Let $\alpha = \{1, x, x^2\}$ for the domain and $\beta = \{x, 1, x^2\}$ for the codomain

$$|D|_\alpha^\beta = \left| |D(1)|_\beta, |D(x)|_\beta, |D_{x^2}|_\beta \right| = \left| |0|_\beta, |1|_\beta, |2x|_\beta \right| = \begin{vmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

* Cardinality

February-14-11
3:30 PM

Countable

A set X is countable iff $|X| = |\mathbb{N}|$
A set X is at most countable if $|X| \le |\mathbb{N}|$

Facts

1. $|X| = |X|, |X| \le |X|, |X| \ge |X|$
2. If $|X| \le |Y|$ and $|Y| \le |Z| \Rightarrow |X| \le |Z|$
3. $|X| \le |Y|$ iff $|Y| \ge |X|$
4. $|X| \le |Y|$ and $|Y| \le |X| \Rightarrow |X| = |Y|$
5. $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$
6. $|A| = |X|, |B| = |Y| \Rightarrow |A \times B| = |X \times Y|$
7. $A \subset B \subset C \text{ and } |A| = |C| \Rightarrow |A| = |B| = |C|$
8. $|0, 1| = |(0, 1) \times (0, 1)|$
9. For any infinite set X, removing a finite subset will not change the cardinality
10. $|(0, 1)| = |[0, 1|$
11. $|(0, 1) \times (0, 1)| = ||0, 1| \times |0, 1||$
12. $|\mathbb{R}| = |0, 1|$
13. $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$
14. $|\mathbb{R}^n| = |\mathbb{R}|$

Proof of Fact 5

Define the mapping $\varphi: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$
$\varphi(1) = (1, 1)$
$\varphi(2) = (1, 2)$
$\varphi(3) = (2, 1)$
$\varphi(4) = (1, 3)$
$\varphi(5) = (2, 2)$
...
This function is bijective, so $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

Proof of Fact 6

$\exists$bijection $f: A \to X, g: B \to Y$
Consider $\varphi: A \times B \to X \times Y$, $(a, b) \to (f(a), g(b))$
Then φ is bijective

Example

$|\mathbb{N} \times \mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

Proof of Fact 7

Consider the map
$\varphi: (0, 1) \to |0, 1| \times |0, 1| \setminus \{(0, 0), (1, 1)\}$
$\varphi(x = 0. a_1 a_2 a_3 \dots) = (0. a_1 a_3 a_5 \dots, 0. a_2 a_4 a_6 \dots)$
In the event that x can be written in two ways, use the representation which is not terminated by repeating 9's.

This is injective. And surjective
$(0, 1) \times 0.5 \subset (0,1) \times (0,1) \subset |0, 1| \times |0, 1| \setminus \{(0, 0), (1, 1)\}$
$|0, 1| = |(0, 1) \times 0.5| = ||0, 1| \times |0, 1| \setminus \{(0, 0), (1, 1)\}$
So $|(0, 1)| = |(0, 1) \times (0, 1)|$

Matrices

February-16-11
11:32 AM

Matrix Representation

Let $L: U \to V$ be linear
Let $\alpha = \{u_1, \dots, u_n\}, \beta = \{v_1, \dots, v_m\}$ be ordered bases for U and V, respectively

$$|L|_\alpha^\beta = \left| |L(u_1)|_\beta, |L(u_2)|_\beta, \dots, |L(u_n)|_\beta \right| = |a_{ji}|_{(m\times n)}$$

Matrix - Tuple Multiplication

Let $A = |a_{ji}|,\ X = \begin{vmatrix} a_1 \\ \vdots \\ a_n \end{vmatrix}$

$$AX = \begin{bmatrix} \sum_{i=1}^{n} a_{1i}a_i \\ \sum_{i=1}^{n} a_{2i}a_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}a_i \end{bmatrix}$$

With that, we have the formula:
$|L(u)|_\beta = |L|_\alpha^\beta |u|_\alpha$

Matrix Representation

Let $L: U \to V$ be linear.
Let $\alpha = \{u_1, \dots, u_n\}$ and $\beta = \{v_1, \dots, v_m\}$ be ordered bases for U and V respectively.
Each vector $u \in U$ has the representation

$$|u|_\alpha = \begin{vmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{vmatrix} i.e.\, u = \sum_{i=1}^{n} a_i u_i\,;$$

$and\ L(u)$ in the codomain V, has

$$|L(u)|_\beta = \begin{vmatrix} b_1 \\ \vdots \\ b_m \end{vmatrix}, i.e. \sum_{j=1}^{m} b_j v_j$$

$$|L|_\alpha^\beta = \left| |L(u_1)|_\beta, |L(u_2)|_\beta, \dots, |L(u_n)|_\beta \right| = |a_{ji}|$$

Hence

$$L(u_i) = \sum_{j=1}^{m} a_{ji} v_i$$

How should $|L(u)|_\beta, |u|_\alpha, and\ |L|_\alpha^\beta$ relate?

$$L(u) = L\left(\sum_{i=1}^{n} a_i u_i\right) = \sum_{i=1}^{n} a_i L(u_i) = \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{m} a_{ji} v_j\right)$$

Note change of scope:

a_i comes from the vector $|u|_\alpha$

a_{ji} comes from the matrix $|L|_\alpha^\beta$

$$L(u) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ji} a_i v_j = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ji} a_i\right) v_j = \sum_{j=1}^{m} b_j v_j$$

$$\therefore b_j = \sum_{i=1}^{n} a_{ji} a_i\,, \qquad j = 1, 2, \dots, m$$

b_j comes from the vector $|L(u)|_\beta$

Get:

$$\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{vmatrix} = |a_{ji}| \begin{vmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{vmatrix} \Rightarrow |L(u)|_\beta = |L|_\alpha^\beta |u|_\alpha$$

Example

Let $L: \mathbb{R}^3 \to \mathbb{R}^2$. Let $\alpha = \{(1,0,0), (0,1,0), (0,0,1)\}$ be the standard ordered basis for $\mathbb{R}^3$ and $\beta = \{(1,0), (0,1)\}$, the standard ordered basis for $\mathbb{R}^2$
Let le be having

$$|L|_\alpha^\beta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}_{2\times 3}$$

Find $L(x, y, z)$
Step 1:

$$|L(x,y,z)|_\beta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} |(x,y,z)|_\alpha = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{vmatrix}$$

$$\therefore L(x,y,z) = (x + 2y + 3z)(1,0) + (4x + 5y + 6z)(0,1) = (x + 2y + 3z, 4x + 5y + 6z)$$

Example

If $T: \mathbb{R}^2 \to \mathbb{R}^3$ is given by $T(x, y) = (x + 2y, 3x + 4y, 5x + 6y)$
Using the standard bases α, β

$$|T|_\alpha^\beta = \begin{vmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{vmatrix}_{(3\times 2)}$$

Example

Let $L: P_2 \to P_2$ over $\mathbb{R}$
Let $\alpha = \beta = \{1, x, x^2\}$

$$If\ |L|_\alpha^\beta = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix}$$

Find $L(a_0 + a_1 x + a_2 x^2)$
Solution:

$$|L(a_0 + a_1 x + a_2 x^2)|_\beta = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} \begin{vmatrix} a_0 \\ a_1 \\ a_2 \end{vmatrix} = \begin{vmatrix} a_0 + 2a_1 + a_2 \\ a_0 + a_1 + a_2 \\ 2a_2 \end{vmatrix}$$

$$\therefore L(a_0 + a_1 x + a_2 x^2) = (a_0 + 2a_1 + a_2) + (a_2 + a_1 + a_2)x + 2a_2 x^2$$

Composition of Linear Maps

February-18-11
11:37 AM

Linearity of Composition

If $L_1: U \to V$ and $L_2: V \to W$ are linear.
Then there are compositions
$L_2 \circ L_1: U \to W$ is linear.

$$|L_2|_\beta^\gamma |L_1|_\alpha^\beta = |L_2 \circ L_1|_\alpha^\gamma$$

Matrix Multiplication

Let $A_{(i \times j)}, B_{(j \times k)}$ be matrices.

$$AB = \left| \sum_{(j=1)} a_{ij} b_{jk} \right|_{(i,k)}$$

Note

For A times B to make sense, the number of columns in A must equal the number of rows in B.

Proof of Linearity of Composition

$$(L_2 \circ L_1)(\lambda u_1 + u_2) = L_2(L_1(\lambda u_1 + u_2)) = L_2(\lambda L_1(u_1) + L_1(u_1))$$
$$= \lambda L_2(L_1(u_1)) + L_2(L_1(u_2)) = \lambda (L_2 \circ L_1)(u_1) + (L_2 \circ L_1)(u_2)$$

Finite Bases

Let α, β, γ be ordered bases for U, V, W, respectively, assuming that U, V, W are finite dimensional.

Then as $|L_1|_\alpha^\beta$ determines L_1, $|L_2|_\beta^\gamma$ determines L_2. They also determine $L_2 \circ L_1$ and subsequently $|L_2 \circ L_1|_\alpha^\gamma$
This motivates the definition of matrix multiplication.
$|L_2|_\beta^\gamma |L_1|_\alpha^\beta = |L_2 \circ L_1|_\alpha^\gamma$

Example

Let $L_1: \mathbb{R}^2 \to \mathbb{R}^3, L_1(x, y) = (x + 2y, 3x, 4y)\ and$
$L_2: \mathbb{R}^3 \to \mathbb{R}^2, L_2(x, y, z) = (x + y - z, x + y + z)$
$Let\ \alpha = \{(1, 0), (0, 1)\}$ for the domain of L_1
$\beta = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ for the domain of L_2
$\gamma = \{(0, 1), (1, 0)\}$ for the range of L_2

$$|L_1|_\alpha^\beta = \begin{vmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 4 \end{vmatrix}, \qquad |L_2|_\beta^\gamma = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$|L_2|_\beta^\gamma |L_1|_\alpha^\beta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 4 \end{vmatrix} = \begin{vmatrix} 1 \times 1 + 1 \times 3 + 1 \times 0 & 1 \times 2 + 1 \times 0 + 1 \times 4 \\ 1 \times 1 + 1 \times 3 - 1 \times 0 & 1 \times 2 + 1 \times 0 - 1 \times 4 \end{vmatrix}$$
$$= \begin{vmatrix} 4 & 6 \\ 4 & -2 \end{vmatrix}$$

$$L_2 \circ L_1 = L_2(L_1(x, y)) = L_2(x + 2y, 3x, 4y) = (4x - 2y, 4x + 6y)$$

$$|(4x - 2y, 4x + 6y)|_\alpha^\gamma = \begin{vmatrix} 4 & 6 \\ 4 & -2 \end{vmatrix}$$
Which agree. Excellent.

Properties of Matrix Operations

March-02-11
1:38 AM

Under addition and scalar multiplication $M_{n\times n}(F)$ is a vector space.
There is a third operation, "matrix multiplication."

The following additional properties hold:

Properties of Matrix Multiplication:

- Multiplicative Identity
  The identity matrix served as the identity element
  $$I, or\ I_n = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{vmatrix}$$
  $i.e. AI = A = IA\ \forall A \in M_{n\times n}(F)$
- Associativity of Multiplication
  $(AB)C = A(BC)\ \ \forall A, B, C \in M_{n\times n}(F)$
  Note: $AB \neq BA$ in general
- Distributivity:
  $A(B + C) = AB + AC$
  $(A + B)C = AC + BC$
  $(\lambda A)B = \lambda(AB) = A(\lambda B),$
  $\forall A, B, C \in M_{n\times n}(F), \lambda \in F$

Linear Algebra

A vector space (or a linear space) under a binary operation called multiplication which satisfies the listed properties above is called a linear algebra.

$M_{n\times n}(F)$ is a linear algebra

Support for $(AB)C = A(BC)$
There is a bijective map from $\mathcal{L}(F^n, F^n)$, or all linear maps from F^n to F^n (a subspace of $\mathcal{F}(F^n, F^n)$)
$[\square]_\alpha: L \to [L]_\alpha$, where α is a fixed, ordered basis for $\mathcal{L}(F^n, F^n)$
It preserves the linear algebra operations:

$[L_1 + L_2]_\alpha = [L_1]_\alpha + [L_2]_\alpha$
$[\lambda L]_\alpha = \lambda [L]_\alpha$
$[L_1 L_2]_\alpha = [L_1 \circ L_2]_\alpha = [L_1]_\alpha [L_2]_\alpha$

In short, the matrix representation $[]_\alpha$ from $\mathcal{L}(F^n, F^n)$ to $M_{(n\times n)}(F)$ is a linear algebra isomorphism.

Composition is an associative operation on $\mathcal{L}(F^n, F^n)$:
$(L_1 \circ L_2) \circ L_3 = L_1 \circ (L_2 \circ L_3) \Leftrightarrow \big((L_1 \circ L_2) \circ L_3\big)(v) = \big(L_1 \circ (L_2 \circ L_3)\big)(v)\ \forall v \in F^n$
$\Leftrightarrow (L_1 \circ L_2)(L_3(v)) = L_1\big((L_2 \circ L_3)(v)\big) \Leftrightarrow L_1(L_2(L_3(v))) = L_1(L_2(L_3(v)))$
The latter is obviously true so due to the isomorphism matrix multiplication must be associative.

Example

Let
$$A = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$
Then
$$A^{20} = \begin{vmatrix} \cos 20\theta & -\sin 20\theta \\ \sin 20\theta & \cos 20\theta \end{vmatrix}$$

Example

Let $D: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the differentiation operator.
Let the domain and codomain be given the (ordered basis) $\alpha = \{1, x, x^2\}$
Then $[D]_\alpha = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{vmatrix}$
because:
$$D(1) = 0, [0]_\alpha = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}, \qquad D(x) = 1, [1]_\alpha = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}, \qquad D(x^2) = 2x, [2x]_\alpha = \begin{vmatrix} 0 \\ 2 \\ 0 \end{vmatrix}$$
Find $[2I + 4D + 5D^5]_\alpha$

<u>Solution 1:</u>
$(2I + 4D + 5D^5)(a + bx + cx^2) = 2(a + bx + cx^2) + 4(b + 2cx) + 5(0)$
$= (2a + 4b) + (2b + 8c)x + 2cx^2$
$[2I + 4D + 5D^5] = \begin{vmatrix} 2 & 4 & 0 \\ 0 & 2 & 8 \\ 0 & 0 & 2 \end{vmatrix}$

<u>Solution 2:</u>
$[]_\alpha$ is a linear algebra isomorphism
$$[2I + 4D + 5D^5]_\alpha = 2[I]_\alpha + 4[D]_\alpha + 5[D]_\alpha^5 = 2\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + 4\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{vmatrix} + 5\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{vmatrix}$$
$$= \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 4 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 0 \\ 0 & 2 & 8 \\ 0 & 0 & 2 \end{vmatrix}$$

Example

Give an example of a 3×3 real matrix satisfying $A^3 = 0$ but $A^2 \neq 0$
Is there a linear operator $L: \mathbb{R}^3 \to \mathbb{R}^2$ so that $L^3 = 0, L^2 \neq 0$
$L(x, y, z) = (y, z, 0), L^2(x, y, z) = (z, 0, 0) \neq 0, L^3 = (0, 0, 0) = 0$
So
$\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$ satisfies the statement.

Sum of Vector Spaces *

March-02-11
2:05 AM

Sum of Vector Spaces

Let V be a vector space. Let W_1 and W_2 be two subspaces of V.
The sum of W_1 and W_2 is defined by:

$$W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$$

Fact: $W_1 + W_2$ is a subspace.

Direct Sum

The sum $W_1 + W_2$ is direct if $W_1 \cap W_2 = \{0\}$. In that case, we write $W_1 \oplus W_2$

Theorem

Suppose that $V = W_1 \oplus W_2$
If α is a basis for W_1 and β is a basis for W_2, then $\alpha \cup \beta$ is a basis for V.
Conversely, if $W_1 \& W_2$ are subspaces of V and $\alpha \cup \beta$ (disjoint union, XOR) is a basis for $W_1 + W_2$, then $\alpha \cup \beta$ is a basis for V

Example
$V = \mathbb{R}^3, W_1 = x - y\ plane, W_2 = y - z\ plane$
Then $W_1 + W_2 = \mathbb{R}^3$

Example
$V = \mathcal{F}([-1,1], \mathbb{R})$
$W_1 = \text{Subspace of all even functions}$
$W_2 = \text{Subspace of all odd functions}$
$W_1 + W_2 = V$

Proof of Theorem

First, $\alpha\ and\ \beta$ are disjoint. Will show that $\alpha \cup \beta$ spans V.
Let $v \in V$ be given. Then $v = w_1 + w_2$ for some $w_1 \in W_1, w_2 \in W_2$, because $V = W_1 + W_2$

Now,

$$w_1 = \sum_{i \in I_1 \subset I} \lambda_i \alpha_1, w_2 = \sum_{j \in J_1 \subset J} \mu_j \beta_j, \qquad I_1, J_2\ finite$$

$\alpha = \{\alpha_i \mid i \in I\}, \beta = \{\beta_j \mid j \in J\}$

$$v = \sum_{i \in I_1} \lambda_i \alpha_i + \sum_{j \in J_1} \mu_j \beta_j, \qquad \alpha_i, \beta_j \in \alpha \cup \beta$$

To show that $\alpha \cup \beta$ is linearly independent, let $\gamma_1, \dots, \gamma_n$ be a finite list of distinct vectors from $\alpha \cup \beta$and that $\eta_1\gamma_1 + \eta_2\gamma_2 + \cdots + \eta_n\gamma_n = 0$
Each γ_i is in either α or β in exactly one way. Re-label those in α as α_i and those in β as β_j;
We set

$$\sum \lambda_i \alpha_i + \sum \mu_j \beta_j = 0 \Rightarrow \sum \lambda_i \alpha_i = -\sum \mu_j \beta_j$$

And since the left side is in W_1 and the right side is in W_2, the only element common to both subspaces is 0. And since W_1 and W_2 are linearly independent, $\lambda_i, \mu_j = 0$ so $\eta_i = 0\ \forall i$

Row Reducing

March-02-11
12:06 PM

Row Reduced Echelon Form

Let A be a $n \times m$ matrix over F. It is in Row Reduced Echelon Form if it has the following features:

1. If there are zero rows, these are at the bottom
2. For each non-zero row, the first (leading, scanned left to right) non-zero entry is 1. We call such positions the leading 1's positions.
3. Leading 1s with higher row numbers should have higher column numbers.
4. All entries above and below the leading 1s are zero

Proposition

Every A can be changed to a Row Reduced Echelon Form using three kinds of row operations in a finite number of steps:

1. Interchange two rows
2. Multiply a row by a non-zero scalar
3. Adding a multiple of a row to a different row

Interpretations of RREF

Could consider the matrix, A, short hand for a system of linear equations. Hence the RREF of A records a system of equations equivalent to that of A.

Could be interpreted as a linear equation of column vectors.

Statement

Every $m \times n$ matrix A has a unique RREF.

The Matrix A and its RREF, in general, do not represent the same linear map.

E.g.

$$\begin{vmatrix} 0 & 1 & * & * \\ 1 & * & * & * \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Not in RREF, second 1 has higher row number but lower column number.

$$\begin{vmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Satisfies 1,2,3

$$\begin{vmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Is in Row-Reduced Echelon Form

Example

Use row operations to reduce

$$A = \begin{vmatrix} 4 & 0 & 8 \\ -9 & 0 & 5 \\ 0 & 0 & 4 \end{vmatrix}$$

to reduced row echelon form:

$$Step\ 1: \frac{1}{4} \times R_1 \to R_1\ we\ get \begin{vmatrix} 1 & 0 & 2 \\ -9 & 0 & 5 \\ 0 & 0 & 4 \end{vmatrix}$$

$$Step\ 2: 9 \times R_1 + R_2 \to R_2, we\ get \begin{vmatrix} 1 & 0 & 2 \\ 0 & 0 & 23 \\ 0 & 0 & 4 \end{vmatrix}$$

$$Step\ 3: \frac{1}{23} \times R_2 \to R_2\ we\ get \begin{vmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{vmatrix}$$

$$Step\ 4: \begin{matrix} -2 \times R_2 + R_1 \to R_1 \\ -4 \times R_2 + R_3 \to R_3 \end{matrix}\ we\ get \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

Example

The matrix

$$A = \begin{vmatrix} 0 & -5 & -15 & 4 & 7 \\ 1 & -2 & -4 & 3 & 6 \\ 2 & 0 & 4 & 2 & 1 \\ 3 & 4 & 18 & 1 & 4 \end{vmatrix}$$

has reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -\frac{25}{4} \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & \frac{27}{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Maple Command:

```
[> linalg[rref] (A);
```

If this is interpreted as a linear system of equations, the general solution of

$$\begin{cases} 0x_1 + (-5)x_2 + (-15)x_3 + 4x_4 + 7x_5 = 0 \\ 1x_1 + (-2)x_2 + (-4)x_3 + 3x_4 + 6x_5 = 0 \\ 2x_1 + 0x_2 + 4x_3 + 2x_4 + 1x_5 = 0 \\ 3x_1 + 4x_2 + 18x_3 + 1x_4 + 4x_5 = 0 \end{cases}$$

is:

Let x_3 and x_5 be free (non-pivot variables)

$$\begin{cases} x_1 = -2x_3 + \frac{23}{4}x_5 \\ x_2 = -3x_3 - 4x_5 \\ x_4 = -\frac{27}{4}x_5 \end{cases}$$

Alternate interpretation:

$$x_1 \begin{vmatrix} 0 \\ 1 \\ 2 \\ 3 \end{vmatrix} + x_2 \begin{vmatrix} -5 \\ -2 \\ 0 \\ 4 \end{vmatrix} + x_3 \begin{vmatrix} -15 \\ -4 \\ 4 \\ 18 \end{vmatrix} + x_4 \begin{vmatrix} 4 \\ 3 \\ 2 \\ 1 \end{vmatrix} + x_5 \begin{vmatrix} 7 \\ 6 \\ 1 \\ 4 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

It concerns the linear dependence or independence of the five column vectors of A in $\mathbb{R}^4$

We wee that the five columns form a dependent set (there are free variables in giving the scalars)

In REF, 3rd column = 2*first column + 3 * second column

That is, a particular solution $(x_1, x_2, x_3, x_4, x_5)$ is $(2, 3, -1, 0, 0)$ which are not all zero.

A basis for span

$$\left\{ \begin{vmatrix} 0 \\ 1 \\ 2 \\ 3 \end{vmatrix}, \begin{vmatrix} -5 \\ -2 \\ 0 \\ 4 \end{vmatrix}, \ldots, \begin{vmatrix} 7 \\ 6 \\ 1 \\ 4 \end{vmatrix} \right\}$$

is

$$\left\{ \begin{vmatrix} 0 \\ 1 \\ 2 \\ 3 \end{vmatrix}, \begin{vmatrix} -5 \\ -2 \\ 0 \\ 4 \end{vmatrix}, \begin{vmatrix} 4 \\ 3 \\ 2 \\ 1 \end{vmatrix} \right\}$$

Rationale for RREF Uniqueness

Different RREF will lead to different solutions to the system of equations $AX = 0$

Example

Clearly all possible RREF must be the same size.

$\begin{vmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$

In first case, dimension of solution space is 1, in second space dimension of solution space is 2

So the number of zero rows at the bottom must be the same in all solutions.

$\begin{vmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$

$x_2 = -3x_3, x_2 = -5x_3$

So the solutions to the first two matrices are not the same.

x_3 arbitrary in first case, 0 in last case. So different solutions.

The Matrix A and its RREF, in general, do not represent the same linear map.

Example

$A = \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix}$ represents $L_A = \left(\begin{vmatrix} x \\ y \end{vmatrix}\right) = A \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 2x \\ 0 \end{vmatrix}$

its RREF is $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = F, L_R \left(\begin{vmatrix} x \\ y \end{vmatrix}\right) = R \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} x \\ 0 \end{vmatrix}$

Elementary Matrices

March-07-11
11:31 AM

Elementary Matrices

There are three types of elementary row operations. When we apply a single elementary row operation to I_n, the resulting matrix is called an elementary matrix.

Proposition

Let A be any $m \times n$ matrix.
When we apply an elementary row operation on A, the outcome is equivalent to multiplying A on the left side by an elementary matrix.

Corollary

Every $m \times n$ matrix A can be changed to its RREF by repeatedly multiplying on the left by a finite sequence of elementary matrices.

Examples of Elementary Matrices

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Not elementary:

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

Example

$Let\ A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}$

$I_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

and that the operation is $2R_2 + R_1 \to R_1$

$A \to \begin{vmatrix} 2a_{21} + a_{11} & 2a_{22} + a_{12} & 2a_{23} + a_{12} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}$

$I_2 \to \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$

$$\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} = \begin{vmatrix} 2a_{21} + a_{11} & 2a_{22} + a_{12} & 2a_{23} + a_{12} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}$$

Example

$Let\ A = \begin{vmatrix} 0 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix} (F = \mathbb{Z}_5)$

Then $A \to^{R_1 \leftrightarrows R_2} \begin{vmatrix} 1 & 2 & 4 \\ 0 & 3 & 1 \end{vmatrix} \to^{2R_2 \to R_1} \begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \end{vmatrix} \to^{-2R_2 + R_1} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix}$

$\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} \left(\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \left(\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} A \right) \right) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix}$

$= \begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} A = \begin{vmatrix} -4 & 1 \\ 2 & 2 \end{vmatrix} A$

Matrices & Maps

March-09-11
11:36 AM

Let $L: U \to V$ be a bijective linear map. If W is a subspace of U, then $L(W)$ is a subspace of V. If α is a basis for W, then $L(\alpha)$ is a basis for $L(W)$
In particular, if $\dim W$ = k, then $\dim L(W) = k$

If L is bijective and linear: $U \to V$
then $L^{-1}: V \to U$ is also linear.
$L \circ L^{-1} = identity\ map\ on\ U$
$L^{-1} \circ L = identity\ map\ on\ V$

If α, β are bases for U, V respectively, then
$[L]_\alpha^\beta [L^{-1}]_\beta^\alpha = [L \circ L^{-1}]_\alpha = I_n$
$[L^{-1}]_\beta^\alpha [L]_\alpha^\beta = [L^{-1} \circ L]_\beta = I_n$

Invertible Map / Matrix

A map which is called bijective is called invertible.

An $n \times n$ matrix is invertible if there exists $n \times n$ B so that $AB = BA = I_n$. If such B exists, it is unique and is denoted by A^{-1}
In particular, if $A = [L]_\alpha$ (bijective operator L), then A is invertible and $A^{-1} = [L^{-1}]_\alpha$

Proposition

The three elementary row operations are invertible linear maps.

Statement:

Composition of linear maps is invertible.

Rank of a Matrix

Let $A \in M_{m\times n}(F)$. The rank of A, $rank(A)$, is the rank of $L_A: F^n \to F^m$

Proposition

Range of $L_A = span\{L_A(e_1), L_A(e_2), \dots, L_A(e_n)\}$ where $\{e_1, \dots, e_n\}$ is the standard basis for F^n. $range(L_a) = span\ \{c_1, c_2, \dots, c_n\}$ where c_i is the i^{th} column of A.
$rank(A) = \#\ of\ linearly\ independent\ columns\ that\ form\ a\ basis$
$\quad = \#\ of\ leading\ 1's\ in\ RREF\ of\ A$

Nullity of a Matrix

Nullity of A = $Nullity(L_A) = \dim N(L_A)$ = $\dim\{X \in F^n : AX = 0\}$

Let $B = RREF\ of\ A$
$\dim\{X \in F^n : AX = 0\} = \dim\{X \in F^n: BX = 0\} = \#\ of\ free\ variables$
$= n - \#\ leading\ 1s = n - rank(A)$

Example

$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \end{bmatrix}$
$RREF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} \end{bmatrix}$
$rank(A) = 2, nullity(A) = 1$

Matrix Multiplication

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11:32 AM

Matrix Multiplication in Blocks

$A|B|C| = |AB|AC|$

$\begin{vmatrix} C \\ - \\ D \end{vmatrix} B = \begin{vmatrix} CB \\ - \\ DB \end{vmatrix}$

$\begin{vmatrix} A_1 & | & A_2 \\ - & + & - \\ A_3 & | & A_4 \end{vmatrix} \begin{vmatrix} B_1 & | & B_2 \\ - & + & - \\ B_3 & | & B_4 \end{vmatrix} = \begin{vmatrix} A_1B_1 + A_2B_3 & | & A_1B_2 + A_2B_4 \\ - & + & - \\ A_3B_1 + A_4B_3 & | & A_3B_2 + A_4B_4 \end{vmatrix}$

Matrix Inversion

In general, for $n \times n$ A, to find A^{-1} if it exists we row reduce $|A|I_n|$ $(Solving\ AB = I_n)$ to RREF on the A side only.

Case 1

If RREF of A is I_n then we have $|I_n|A^{-1}|$

Case 2

If RREF of A is not I_n, then A is not invertible.

Solving Equations

To solve the equation $AX = B$ where

$A = \begin{vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{vmatrix}, B = \begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{vmatrix}$

we could find the RREF of $|A|B|$
and then determine the solutions,

Suppose we want to solve two parallel equations.
$AX = B_1, AX = B_2$ (separately, parallel means not related, different X)

It can be done by finding $RREF$ of $|A|B_1|$ and of $[A|B_2]$
The job can be done in one round: Find RREF of $|A|B_1|B_2|$ and then read the solutions.

Example

Let $A = \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}$. Find A^{-1} if A has an inverse.
Solution:
We seek B (2×2) such that $AB = I$
Let $B = |X_1|X_2|$. The equation is $A|X_1|X_2| = \begin{vmatrix} 1 \\ 0 \end{vmatrix}\begin{vmatrix} 0 \\ 1 \end{vmatrix}$

$AX_1 = \begin{vmatrix} 1 \\ 0 \end{vmatrix}, AX_2 = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$

Consider

$\begin{vmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 3 & | & 0 & 1 \end{vmatrix}$ and use row ops. to bring it to RREF (on A partition)

$\begin{vmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 1 & | & 0 & \frac{1}{3} \end{vmatrix} \left(\frac{1}{3}R_2 \to R_2\right)$

$\begin{vmatrix} 1 & 0 & | & 1 & -\frac{2}{3} \\ 0 & 1 & | & 0 & \frac{1}{3} \end{vmatrix} (-2R_2 + R_1 \to R_1)$

$X_1 = \begin{vmatrix} 1 \\ 0 \end{vmatrix}, X_2 = \begin{vmatrix} -\frac{2}{3} \\ \frac{1}{3} \end{vmatrix}, B = \begin{vmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{vmatrix}$

Example

Solve $\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} X = \begin{vmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \end{vmatrix}$

$\begin{vmatrix} 1 & 2 & 3 & 5 & 7 \\ 0 & 3 & 4 & 6 & 8 \end{vmatrix} \to \begin{vmatrix} 1 & 2 & | & 3 & 5 & 7 \\ 0 & 1 & | & \frac{4}{3} & 2 & \frac{8}{3} \end{vmatrix} \to \begin{vmatrix} 1 & 0 & | & \frac{1}{3} & 1 & \frac{5}{3} \\ 0 & 1 & | & \frac{4}{3} & 2 & \frac{8}{3} \end{vmatrix}$

$X = \begin{vmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{4}{3} & 2 & \frac{8}{3} \end{vmatrix}$

Example

Express $\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}$ as a product of elementary matrices.
Solution:

$\underset{E_2}{\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix}} \underset{E_1}{\begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{vmatrix}} \underset{A}{\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}} = I_2$

$A = E_1^{-1}E_2^{-1}I_2 = \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$

Column Operations

March-14-11
11:32 AM

Proposition

If $T_1 : U \to V$ and $T_2 : V \to W$ are linear and T_2 is an isomorphism on finite dimensional spaces U, V, and W.

$Range(T_2T_1) = (T_2T_1)(U)$ by definition of range

$= T_2(T_1(U))$

$= T_2(range(T_1))$

When T_2 is an isomorphism, the subspace $range(T_1)$ of V is mapped to a subspace of W of the same dimension.

Therefore, $rank(T_2 \circ T_1) = Rank\ T_1$

Converting that statement to $n \times n$ matrices A and B, we get

$rank(AB) = rank(B)$ when A is invertible (i.e. equivalently rank(A) = n)

In parallel, we get $rank(AB) = rank(A)$ if B is invertible.

Corollary

For any matrix A, an elementary row operation performed on A does not change the rank.

$rank(EA) = rank(A)$

Since E is invertible.

In particular, $rank(A) = rank(RREF(A))$

Theorem

Elementary column operations does not change the rank of a matrix.

$rank(AE) = rank(A)$ since E is invertible.

Theorem

By using both elementary row and column operations, we can reduce a matrix to the form

$$\begin{vmatrix} I_r & | & 0 \\ - & + & - \\ 0 & | & 0 \end{vmatrix}$$

where r is the rank of the original matrix.

Corollary

Let A be any matrix ($m \times n$). Then there exist invertible P & Q such that

$$PAQ = \begin{vmatrix} I_r & | & 0 \\ - & + & - \\ 0 & | & 0 \end{vmatrix}$$

Observations

Observe that rows of A are the same as the columns of A^t. Therefore, action on rows of A becomes action on the columns of A^t.

Every theorem on row operations has a corresponding theorem on column operations.

Example

Every matrix can be reduced to a unique RREF using elementary row operations.

In parallel, we have:

> Every matrix can be reduced to a unique reduced column echelon form using elementary column operations.

Notice that transpose has the property

$(AB)^t = B^tA^t$

The statement : an elementary row operation performed on A has the effect of multiplying A on the left by an elementary matrix translates into multiplying A on the right by an elementary matrix.

Demonstration

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} \to (C_1 \rightleftarrows C_2) \to \begin{vmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \end{vmatrix}$$

Example

Let A be 2×3 and that under the use of row operations we bring it to

$$\begin{vmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \end{vmatrix} (RREF)$$

Using further column operations, we can bring it down to CREF

$$\to (C_2 \leftrightarrows C_1) \to \begin{vmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \end{vmatrix} \to (-3C_1 \to C_3) \to \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

* Dot product on R^n

March-14-11
3:33 PM

Dot Product on R^n

Let $\vec{x} = (x_1, \dots, x_n), \vec{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$

$$\vec{x} \cdot \vec{y} = (x_1, \dots, x_n) \cdot (y_1, \dots, y_n) \coloneqq \sum_{i=1}^{n} x_i y_i$$

It is seen within matrix multiplication, and also in equations like $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$
$(a_1, \dots, a_n) \cdot (x_1, \dots, x_n) = 0$

Norm of a Vector in $\mathbb{R}^n$

In $\mathbb{R}^n$, $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

If $\vec{x} \neq 0$, then $\|\vec{x}\| > 0$
If $\vec{x} = 0$ *then* $\|\vec{x}\| = 0$
$\|\lambda \vec{x}\| = |\lambda| \|\vec{x}\| \; \forall x \in \mathbb{R}^n$
$\left\| \frac{x}{\|x\|} \right\| = \left| \frac{1}{\|x\|} \right| \|\vec{x}\| = \frac{1}{\|x\|} \|\vec{x}\| = 1$

Normal Vector

A vector whose norm is 1

Normalisation

We call the division of $\vec{x} \neq 0$ by $\|\vec{x}\| > 0$ the normalisation of $\vec{x}$

Distance

Distance between $\vec{x}, \vec{y}$:
$d(\vec{x}, \vec{y}) = \|\vec{y} - \vec{x}\| = \|\vec{x} - \vec{y}\|$

Theorem

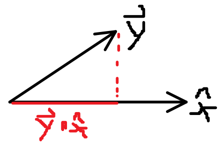
$Proj_x : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map

Geometric Interpretation in $\mathbb{R}^2$

$(x_1, x_2) \cdot (y_1, y_2) = 0$ means the vectors $\vec{x}, \vec{y}$ are perpendicular.
Same story for x^3

Dot Product

Interpretation of non-zero dot product:



Orthogonal projection of a vector $\vec{y} \in \mathbb{R}^n$ on a normal vector x is
$Proj_x(\vec{y}) = (\vec{y} \cdot x)x$

$Range(Proj_x) = span\{x\}$
$Nullspace(Proj_x) = \{\vec{y} \in \mathbb{R}^n : Proj_x(\vec{y}) = 0\} = \{\vec{y} \in \mathbb{R}^n : (\vec{y} \cdot x) = 0\} = \{\vec{y} \in \mathbb{R}^n : \vec{y} \perp x\}$
$\mathbb{R}^n = Nullspace(Proj_x) \oplus Range(Proj_x)$

Let $Proj_x = T, \; T^2 = T$

Projection

$Let\ V = W_1 \oplus W_2$
Then for $v \in V, v = w_1 + w_2$
Define $Proj_{W_2}(v) = w_2$ and $Proj_{W_1}(v) = w_1$

Abstract Definition of Projection

A linear operator L such that $L^2 = L$

Determinant

March-16-11
11:31 AM

The Determinant Function

Let A be a 1×1 matrix. The determinant of A, $\det(A)$ is equal to the entry of A.

Let A be a 2×2 matrix $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$. Then
$det(a) = a_{11}a_{22} - a_{12}a_{21} = a_{11} \det|a_{22}| - a_{12} \det|a_{21}|$

Let $A = |a_{ij}|$ be 3×3. We define
$\det A$
$= a_{11} \det \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \det \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$
$+ a_{13} \det \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Recursively, we define for $n \times n$ matrix A

$$\det A = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det|A_{1j}|$$

Where A_{1j} is the sub matrix of A obtained when we remove row 1 and column j

Area Magnitude

Area is considered positive when the points are defined in a widdershins fashion about the shape. When the points are described clockwise, the area can be considered negative.

Multiplying the area by -1 means a change in orientation.

Fact

A 2×2 matrix A is invertible iff $\det A \neq 0$. In general, for any $n \times n$ A, A is invertible iff $\det A \neq 0$

Theorem

For any $n \times n$ A over F, A is invertible iff $\det A \neq 0$.

Proposition

Let A be $n \times n$. Holding all rows but the $1st$ row fixed, $\det A$ is a linear map of the first row R_1. It is a function from F^n to F

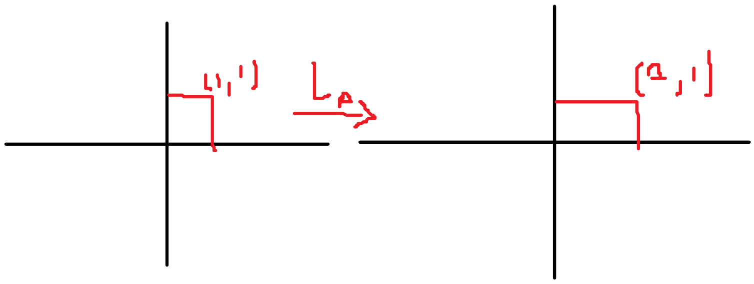
Interpretation of Determinant

Interpretation for 2×2 matrix A and $\det A$

e.g. Let $A = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}$. Then $\det A = (2)(1) - (0)(0) = 2$

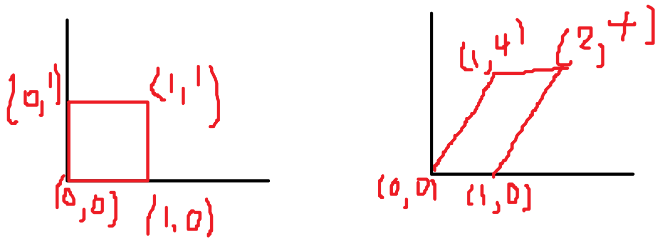
Consider $L_A: \mathbb{R}^2 \to \mathbb{R}^2$. The map is $L_A(x,y) = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 2x \\ y \end{vmatrix} = (2x, y)$

The figure:



The area under the region is doubled by the transform.

Let $A = \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix}$. Then $\det A = 4$. $L_A(x,y) = (x+y, 4y)$



So the Area was multiplied by a factor of 4.

Determinant Properties

March-18-11
11:32 AM

Properties of Determinants

In the textbook, properties of determinant are built up in this sequence:
Theorems
(4.3) det A is linear as a function of each row when other rows are fixed.
Corollary: If A has a zero row, then det A = 0

$$(4.4)\ det\ A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \text{ for any fixed } i$$

(Co-Factor expansion along row i)
A lead to (4.4) is the Lemma: If B is $n \times n, n \geq 2$ has row I equal to e_k(standard basis for F^k) then
$\det B = (-1)^{i+k} \det B_{ik}$

Corollary: If A has two identical rows, then $\det A = 0$

(4.5) IF B is obtained from A by interchanging two rows, then $\det B = -\det A$
(4.6) If B is obtained from A by $\lambda R_i + R_j \to R_j$ $(i \neq j)$action, then $\det B = \det A$
Corollary: If $rank\ (A), n \times n\ A$, is below n, then $\det A = 0$

Corollary

If a matrix is upper triangular A, $A_{ij} = 0\ for\ i > j$ then

$$\det A = \prod_{i=1}^{n} A_{ii} = \text{product of all diagonal entries}$$

Illustration of Theorem 4.3

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_1 + kc_1 & b_2 + kc_2 & b_3 + kc_3 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Claim:

$$\det A = \det \begin{vmatrix} a_{11} & a_{12} & a_{12} \\ b_1 & b_2 & b_3 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + k \det \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ c_1 & c_2 & c_3 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$LHS = a_{11} \det \begin{vmatrix} b_2 + kc_2 & b_3 + kc_3 \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \det |\dots| + a_{13} \det \begin{vmatrix} b_1 + kc_1 & b_2 + kc_2 \\ a_{31} & a_{32} \end{vmatrix}$$

By induction
LHS

$$= a_{11} \det \left(\begin{vmatrix} b_2 & b_3 \\ a_{32} & a_{33} \end{vmatrix} + k \begin{vmatrix} c_2 & c_3 \\ a_{32} & a_{33} \end{vmatrix} \right) - a_{12} \det |\dots|$$

$$+ a_{13} \det \left(\begin{vmatrix} b_1 & b_2 \\ a_{31} & a_{32} \end{vmatrix} + k \begin{vmatrix} c_1 & c_2 \\ a_{31} & a_{32} \end{vmatrix} \right) = RHS$$

Illustration of Lemma for Theorem 4.4

$$B = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det B = a_{11} \det \begin{vmatrix} 0 & 1 \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \det \begin{vmatrix} 0 & 1 \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \det \begin{vmatrix} 0 & 0 \\ a_{31} & a_{32} \end{vmatrix}$$

The new determinants are either 0 or same form but smaller so use induction.

Proof of Corollary

Use brute force to check it is true for 2×2 matrices.
For larger n, pick a row which is not part of the 2 identical rows. The determinant calculated using that row will be 0 because there are 2 identical rows in every sub-matrix, by induction.

Illustration of Theorem 4.6

Let B be obtained from A using $\lambda R_i + R_j \to R_j$

$$\det B = \det \begin{bmatrix} R_1 \\ \vdots \\ R_{j-1} \\ \lambda R_i + R_j \\ R_{j+1} \\ \vdots \\ R_n \end{bmatrix} = \lambda \det \begin{bmatrix} R_1 \\ \vdots \\ R_{j-1} \\ R_i \\ R_{j+1} \\ \vdots \\ R_n \end{bmatrix} + \det \begin{bmatrix} R_1 \\ \vdots \\ R_{j-1} \\ R_j \\ R_{j+1} \\ \vdots \\ R_n \end{bmatrix} = \det \begin{bmatrix} R_1 \\ \vdots \\ R_{j-1} \\ R_j \\ R_{j+1} \\ \vdots \\ R_n \end{bmatrix}$$

Since the first matrix has two identical rows and thus has determinant 0.

Example

$$\text{Evaluate } \det \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 6 & 7 & 8 \end{vmatrix}$$

$$= 1 \det \begin{vmatrix} 5 & 0 \\ 7 & 8 \end{vmatrix} - 2 \det \begin{vmatrix} 0 & 0 \\ 6 & 8 \end{vmatrix} + 3 \det \begin{vmatrix} 0 & 5 \\ 6 & 7 \end{vmatrix} = 1 \times 40 - 2 \times 0 + 3 \times -20 = -50$$

or

$$= (-1)^{2+2} \times 5 \times \det \begin{vmatrix} 1 & 3 \\ 6 & 8 \end{vmatrix} = 5 \times -10 = -50$$

Example

Find $\det \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 1 & 1 & 1 \end{vmatrix}$ over $\mathbb{Z}_7$

Lin comb of rows, then multiply a row by $2 = \frac{1}{4}$

$$= \det \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 3 \\ 0 & 6 & 5 \end{vmatrix} = 4 \times \det \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 6 & 5 \end{vmatrix} = 4 \times \det \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 4 \end{vmatrix}$$

$$= 4 \times (-1)^{3+3} \times 4 \times \det \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 4 \times 4 \times 1 = 2$$

Example

$$\text{Evaluate } \det \begin{vmatrix} 1 & 1 & 1 \\ 1 & x & x^2 \\ 1 & y & y^2 \end{vmatrix}$$

It is some multinomial involving x and y of degree at most 3.
By inspection, factors should be
$(x-1)(y-1)(x-y)$

$\det \begin{vmatrix} 1 & 1 & 1 \\ 1 & x & x^2 \\ 1 & y & y^2 \end{vmatrix} = a(x-1)(y-1)(x-y)$ for some constant a

If over $\mathbb{R}$, pick $x = 0, y = 2$

$$a(-1)(1)(-2) = 2a = (-1)^{1+2} \det \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = -1 \times 2 = -2$$

$a = -1$

More Det. Properties

March-23-11
11:34 AM

Theorem

$\det(AB) = \det A \det B$

Similar Matrices

Two $n \times n$ matrices A & B are similar if there exists invertible P so that
$A = P^{-1}BP$

Result

If A and B are similar then $\det A = \det B$

Example

Let $T: V \to V$ be linear, $\dim V = n$. Let α be a basis, and led β be another. Then $|T|_\alpha$ and $|T|_\beta$ are similar.

Determinant of Operator

Let $T: V \to V$ be a linear operation on n-dimensional V. Then $\det T \coloneqq \det(|T|_\alpha)$ for any ordered basis α

Theorem

$\det(T_1 \circ T_2) = \det T_1 \det T_2$

Determinant Properties Cont.

$\det(A) = \det(A^T)$

Proof of Theorem

First see that it is true for elementary matrix $A = E$

Case 1:

E is from I_n $\lambda R_i \to R_i$
$\det(E) = \lambda \det(I_n) = \lambda$
$\det(EB) = \lambda \det(B) = \det(E)\det(B)$

Case 2:

Suppose E is from I_n by the action $R_i \leftrightarrows R_j$
Then $\det E = -\det(I_n) = -1$
$\det(EB) = -\det(B) = \det(E)\det(B)$

Case 3:

Suppose E is from I_n by the action $\lambda R_i + R_j \to R_j$
Then $\det(E) = \det(I_n) = 1$
$\det(EB) = \det(B) = \det(E)\det(B)$

Next, if A is equal to $E_1 E_2 \dots E_k$, then $\det(AB) = \det(A) + \det(B)$
$\det(AB) = \det(E_1)\det(E_2, \dots, E_k) = \cdots = \det(E_1)\det(E_2)\dots\det(E_k)\det(B)$
$= \det(E_1 E_2 \dots E_k)\det(B) = \det(\mathrm{A})\det(\mathrm{B})$

Finally, if A is not invertible then AB is not invertible. Since A is not invertible, the RREF has a 0 row at the bottom so $\det A$ is 0, as for AB so $\det AB = 0$ so $\det(A) = \det(A)\det(B) = 0 \times \det(B) = 0$
∎

Proof of Result

$\exists P, A = P^{-1}BP, \det(A) = \det(P^{-1}BP) = \det(P^{-1})\det(B)\det(P) = \det(P^{-1})\det(P)\det(B)$
$= \det(P^{-1}P)\det(B) = \det(I_n)\det(B) = \det(B)$

Proof of Example

Recall the rule $|L_1 \circ L_2|_{\gamma_1}^{\gamma_3} = |L_1|_{\gamma_2}^{\gamma_3}|L_2|_{\gamma_1}^{\gamma_2}$
$V \xrightarrow{T} V$
$\alpha \xrightarrow{|T|_\alpha} \alpha$
$\downarrow \quad \uparrow$
$V \xrightarrow{T} V$
$\beta \xrightarrow{|T|_\beta} \beta$
So:
$T = 1 \circ T \circ 1$
$|T|_\alpha = |1 \circ T \circ 1|_\alpha = |1|_\beta^\alpha |T|_\beta |1|_\alpha^\beta$
Testing: $|1|_\beta^\alpha |1|_\alpha^\beta = |1|_\alpha = I_n$

Example

Let $T: V \to V$
$\alpha = \{v_1, v_2\}, \beta = \{v_2, v_1\}$ *be bases*
Let $|T|_\alpha = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
What is $|T|_\beta$?
Ans: Given $\begin{cases} T(v_1) = av_1 + cv_2 \\ T(v_2) = bv_1 + dv_2 \end{cases}$
Hence $T(v_2) = bv_1 + dv_2 = dv_2 + bv_1$
$T(v_1) = av_1 + cv_2 = cv_2 + av_1$
So
$|T|_\beta = \begin{vmatrix} d & c \\ b & a \end{vmatrix}$

Corollary

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and $\begin{vmatrix} d & c \\ b & a \end{vmatrix}$ are similar.

Proof of Theorem

$\det(T_1 \circ T_2) = \det|T_1 \circ T_2|_\alpha = \det|T_1|_\alpha \det|T_2|_\alpha$ ∎

Proof of $\det(A) = \det(A^T)$

For $\lambda R_i \to R_i$ and $R_1 \leftrightarrows R_j$, $E^T = E$
For $\lambda R_i + R_j \to R_j$
Each E, E^T are upper or lower triangular so $\det(E) = \det(E^T) = 1$
Since this is true for elementary matrices, it should be true for all invertible matrices.
$\det A^T = \det(E_1 E_2 \dots E_n)^T = \det(E_n^T \dots E_2^T E_1^T) = \det(E_n^T)\dots\det(\mathrm{E}_2^{\mathrm{T}})\det(\mathrm{E}_1^{\mathrm{T}})$
$= \det(\mathrm{E_n})\dots\det(\mathrm{E_2})\det(\mathrm{E_1}) = \det(E_1)\det(E_2)\dots\det(\mathrm{E_n}) = \det(\mathrm{E_1 E_2 \dots E_n}) = \det(\mathrm{A^T})$
And for non-invertible A, A^T is non-invertible so $\det A = \det A^T = 0$
Suppose $A^T B = 1$, *then* $(A^T B)^T = 1^T \Rightarrow B^T A = 1$ so A^T not invertible $\Leftrightarrow A$ not invertible

Similar Maps

March-28-11
11:30 AM

Proposition

If A and B are similar, then $p(A)$ is similar to $p(B)$.
Where p is a polynomial expression

$$p = \sum_{i=0}^{n} a_i x^i$$

Similar Maps

Let L_1 and $L_2 : V \to V$ be linear operators. we say that L_1 is similar to L_2 if there exists an isomorphism $P: V \to V$ so that $L_1 = P^{-1} \circ L_2 \circ P$

Proposition

If V is finite dimensional, then operators $L_1, L_2: V \to V$ are similar iff $|L_1|_\alpha$ and $|L_2|_\alpha$ are similar.

Characteristic Polynomial

$\det|A - \lambda I_n|$ is the characteristic polynomial of $(n \times n)$ A

Characteristic roots (Eigenvalues)

The roots of the characteristic polynomial of A are called the characteristic roots of A.

Proof of Proposition

$$Let\ p(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

i) A^2 is similar to B^2.
Let $A = P^{-1}BP$. Then $A^2 = P^{-1}BPP^{-1}BP = P^{-1}BIBP = P^{-1}B^2P$
ii) Similarly, A^k is similar to B^k for each $k \geq 3$
iii) $p(A) = P^{-1}p(B)P$

$$p(A) = \sum_{i=0}^{n} a_i A^i = \sum_{i=0}^{n} a_i P^{-1} B^i P$$

Example

Let $L_1: \mathbb{R}^2 \to \mathbb{R}^2$ by the rotation $\circlearrowleft by\ 20°$. Let $P: \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the y-axis.
[i.e. $P(x, y) = (-x, y)$]
Let $L_2 = P^{-1}L_1P$. Then L_1 and L_2 are similar.
L_2 is the rotation $\circlearrowright$ by 20°

Try

Is rotation counter clockwise by 20° similar to rotation counter clockwise by 30°
May be on exam

Proof of Proposition

$(\Rightarrow)$ Suppose that there is an isomorphism $T: V \to V$ so that
$L_1 = T^{-1}L_2T$
Let α be any fixed basis. Then
$|L_1|_\alpha = |T|_\alpha^{-1}|L_2|_\alpha|T|_\alpha$. Take $P = |T|_\alpha$
$(\Leftarrow)$ Converse left as exercise

Example

Consider the two similar rotations mentioned earlier. Pick $\alpha = standard\ basis.$ We get

$|L_1|_\alpha = \begin{vmatrix} \cos 20 & -\sin 20 \\ \sin 20 & \cos 20 \end{vmatrix}$ is similar to

$|L_2|_\alpha = \begin{vmatrix} \cos 20 & \sin 20 \\ -\sin 20 & \cos 20 \end{vmatrix}$ under $P = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$

Example of characteristic polynomials

$A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

Then its characteristic polynomial is

$\det|A - \lambda I| = \det\left(\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} - \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix}\right) = \det\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - (2)(3) = \lambda^2 - 5\lambda - 2$

* Axiom of Choice

March-28-11
3:38 PM

If X is a finite set with n elements then X can be partitioned into two (disjoint) parts of same cardinality iff n is even.

Proposition

If X is an infinite set, then it can be partitioned into two parts of the same cardinality.

Function Extension

Say $G: A_2 \to B_2$ extends $F: A_1 \to B_1$
if $A_2 \supseteq A_1$ and $B_2 \supseteq B_1$ and $G(A_1) = B_1$

Proof of Proposition

Consider the class $\mathcal{C}$ of all bijective functions from a set $A \subset X$ onto $B \subset X, A \cap B = \emptyset$
$\mathcal{C}$ is non-empty.
Define in $\mathcal{C}$, $f \leq g$ when g extends f.

$\mathcal{C}$ is partially ordered by ≤
We seek maximal f.
Let C be a chain in $\mathcal{C}$

$Let\ A = \bigcup_{f \in C} dom\ f$ and $B = \bigcup_{f \in C} range\ f$
$f: A \to B$ by if $a \in A$ then $a \in dom\ f_i$ for some $f_i \in C$ let $f(a) = f_i(a)$.
If $a \in dom\ f_j$ for some $f_j \in C$ then WLOG say that $f_i \leq f_j$ so $f_i(a) = f_j(a)$.
Hence f is well defined

$dom\ f = A, range\ f = B$. It is easy to observe that A and B are disjoint and f is a bijection from A to B
So $f \in \mathcal{C}$

The maximal principle asserts that maximal f_0 exists.
The union of the domain A of f_0 and its range B is either the whole X or is $X \backslash \{x_0\}$
We are done if $A \cup B = X$
Else, $A \cup B \cup \{x_0\} = X$
Select a sequence of distinct elements $(a_n)_{n=1}^{\infty}$ from A.
Adjust f_0 to g:

$g: A \cup \{x_0\} \to B$
$g(x_0) = f_0(a_1)$
$g(a_n) = f_0(a_{n+1})$
$g(a) = f_0(a), for\ a \notin \{a_n\} \cup \{x_0\}$

Hence $A \cup \{x_0\}$ and B is a partition of X, and the presence of bijective g means $A \cup \{x_0\}$ and B are of the same cardinality.

Eigenvalues/vectors

March-30-11
11:34 AM

Eigenvalues and Eigenvectors

Let V be a vector space over F. Let $L: V \to V$ be a linear operator. A scalar λ is an **eigenvalue** of L if there exists $v \neq 0$ so that $L(v) = \lambda v$.

If $v \neq 0$ and $L(v) = \lambda v$ for some $\lambda \in F$, then v is called an **eigenvector** of L.

Proposition

Eigenvalues of $L_A: F^n \to F^n$ $(n \times n\ A)$ are given by the characteristic roots of A.

Hence, L_A has at most n distinct eigenvalues.

Remark

Let $L: V \to V$ be an operator on finite dimensional V. Then λ is an eigenvalue of L iff it is a characteristic root of $[L]_\alpha$ for any fixed basis α for V.

Example

Let $L: \mathbb{R}^2 \to \mathbb{R}^2$ be $Proj_{\vec{x}}$

Then each non-zero vector on the line spanned by $\vec{x}$ is an eigenvector of L, and $\lambda = 1$ is an eigenvalue.

Each $v \neq 0$, perpendicular to $\vec{x}$ is also an eigenvector of L, and $\lambda = 0$ is an eigenvalue of L.

Proof of Proposition

Let λ be an eigenvalue of L_A. Then, by definition, there exists $X \neq 0 \in F^n$ so that $L_A(X) = \lambda X$. That is, $AX = \lambda X$

$AX - \lambda X = 0 \Rightarrow (A - \lambda I_n)X = 0$

This is equivalent to that $A - \lambda_n$ is not invertible.

Therefore, $\det(A - \lambda I_n) = 0$

Therefore, λ is a characteristic root.

The converse is also true and can be observed through the proof done backwards.

Example

Let V be the space of all infinitely differentiable functions on the real line into the real line. (A subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$)

Let $D: V \to V$ be the differentiation.

Each function $e^{\lambda x}$ is an eigenvector of D. Hence λ is an eigenvalue of D for every $\lambda \in \mathbb{R}$.

Computational comments

April-01-11
11:32 AM

Given a finite list of vectors $v_1, \ldots v_k$ in F^n, how to extract a subset which is a basis for $span\,\{v_1, \ldots, v_k\}$ and extend that to a basis for the full F^n

Method

Form the matrix
$|v_1|v_2| \ldots |v_k|\ e_1|\ e_2| \ldots |e_n|$ and find its RREF, then read an answer out.

Example

Suppose that $k = 4, n = 6$ and that RREF of A is

$$\begin{bmatrix} 0 & 1 & 0 & * & * & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 1 & * & * & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The then answer is $\{v_2, v_3\}$ is a basis for $span\{v_1, v_2, v_3, v_4\}$. An extension to a basis for F^6 is $\{v_2, v_3, e_2, e_3, e_5, e_6\}$

If mission is to find a basis for $span\{v_1, v_2, \ldots, v_k\}$ in F^k then we could form

$$A = \begin{bmatrix} v_1 \\ - \\ v_2 \\ - \\ \vdots \\ - \\ v_k \end{bmatrix}$$

and find its RREF. At the end we produce a basis. For instance

$$k = 4, n = 6, RREF\ of\ A\ is \begin{vmatrix} 1 & *_1 & *_2 & 0 & *_3 & *_4 \\ 0 & 0 & 0 & 1 & *_5 & *_6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

Then a basis for $span\,\{v_1, v_2, v_3, v_4\}\ is\ \{\ (1,*_1,*_2,0,*_3,*_4), (0,0,0,1,*_5,*_6)\}$, not $\{v_1, v_2\}$

Comments

April-04-11
11:33 AM

The following are undefined:

$L: U \to V$ a linear map, $\dim(L)$.
$Vectors\ v_1, v_2, \dots, v_n$. $\dim\{v_1, \dots, v_n\}$
Matrix A, $\dim A$

$v_1, v_2, \dots v_n$. They form a basis for V. Avoid saying $v_1, v_2, \dots, v_n$ is a basis.
Correct: $\{v_1, v_2, \dots, v_n\}$ is a basis

$\dim M_{3\times 4}(F)$ is defined, though $\dim A$ is undefined for $A \in M_{3\times 4}(F)$

$L: V \to V$ a linear operator, V finite dimensional, $\det L$ is defined by $\det([L]_\alpha)$
When V is infinite dimensional, $\det L$ is undefined.
e.g. If D is the differentiation operator, then det D is defined when the space it acts on is finite dimensional, like $P_n(\mathbb{R})$. It is undefined on $P(\mathbb{R})$

The characteristic polynomial of A is defined by $\det(A - \lambda I_n)$.
It cannot be computed using the RREF of A.

*Might be on exam
If A is similar to B, then $\det A = \det B$
$trace\ A = traceB$
$rank\ A = rank\ B, nullity\ A = nullity\ B$
Characteristic polynomial of A = B?

$A \sim B \Rightarrow A^2 \sim B^2$
$A \sim B \Rightarrow p(A) \sim p(B)$
$A \sim B\ \&\ C \sim D \Rightarrow AC \sim BD$?

$\lambda\ is\ an\ eigenvalue\ of\ A$
$(\exists\ X \neq 0\ so\ that\ AX = \lambda X)$
then λ^2 is an eigenvalue of A^2
As $A^2X = A(AX) = A(\lambda X) = \lambda A(x) = \lambda(\lambda x) = \lambda^2 x$
Similarly $\lambda\ is\ a\ root\ of\ \det(A - \lambda I_n) \Rightarrow \lambda^2\ is\ root\ of\ \det(A^2 - \lambda I_n)$