AMATH 350 notes: Differential Equations in Business and Economics

Johnew Zhang

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1 Basic Concept & Terminology

Differential Equation A differential equation is an equation relating a function to its own derivatives.

Solving a DE means finding the functions which satisfy it.

For example,

$$\frac{dy}{dx} = 2y + 1$$

A solution is

$$y = 5e^{2x} - \frac{1}{2}$$

We can verify this easily: $\frac{dy}{dx} = 10e^{2x}$ while $2y + 1 = 2(5e^{2x} - \frac{1}{2}) + 1 = 10e^{2x}$

DE's usually have multiple solutions.

General Solution The general solution is an expression which represents all (or nothing all) of the solution.

Singular Solution Occasionally there are singular solutions, which don't fit general pattern.

For example, consider the DE $y' = y$. The general solution is $y = Ce^x$.

Consider the equation,

$$\frac{dy}{dx} = -4xy^2$$

has general solution $y = \frac{1}{2x^2+c}$. However, $y = 0$ is also a solution.

Hence, a singular solution is a solution that does not match the general patterns.

Partial differential equations In multivariable calculus we encounter partial differential equations (PDEs), which involve partial derivatives. However, some other are very hard to solve. Consider if $\frac{dy}{dx} = x$, then $y = \frac{1}{2}x^2 + C$. For example, $U_t = U_{xx}$ where $u = u(x, t)$. But if $u_x = x$, then $u(x, t) = \frac{1}{2}x^2 + g(t)$.

Order The order of a DE is the order of the highest-order derivative. For example, $y''' + 3y' - y = 0$ is a third-order ODE. $(y')^3 - xy = x^2$ is first-order ODE. $u_t = u_x + sinx$ is a first-order PDE.

boundary conditions In application we will usually be seeking one particular solution. To determine this we need (for first-order ODE) the value of the function at one point (at initial condition). For an nth-order ODE, we need n inputs to solve the equation. If they are all given at the same value of x we call them initial conditions (e.g. $y(0), y'(0), y''(0)$, etc.). If they are given at different values we call them **boundary conditions**. For example, $y'' = -y, y(0) = 0, y(1) = 0$. Solution: $y = Asin(n\pi x)$.

Linear An important distinction: a linear ODE has the structure $\cdots + f_2(x)y'' + f_1(x)y' + f_0(x)y = f(x)$.

2 First-Order ODEs

Most general expression: $F(x, y, y') = 0$. We will consider only equations which can be solved for y': $\frac{dy}{dx} = f(x, y)$

Theorem. *Existence& uniqueness of solutions: The initial value problem* $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ *has a unique solution, defined on some interval around* x_0*, if* $f(x, y)$ *and* $f_y(x, y)$ *are continuous within some rectangle containing* (x_0, y_0).

For example, $y' = x^2y + e^{xy}$ has a unique solution for any initial condition. However, $y' = (xy)^{2/3}$ may not have unique solutions for initial condition with $y = 0$.

Separable equations Suppose $\frac{dy}{dx} = f(x, y)$. If $f(x, y)$ can be factored as $g(x)h(y)$, then the equation is separable, and can be solved thus:

$$\frac{dy}{dx} = g(x)h(y)$$

Divide by $h(y) : \frac{1}{h(y)}\frac{dy}{dx} = g(x)$

Integrate on x: $\int \frac{1}{h(y)}\frac{dy}{dx}dx = \int g(x)dx$

Now $\frac{dy}{dx}dx$ is just dy, so

$$\int \frac{1}{h(y)}dy = \int g(x)dx$$

(as a shortcut : $\frac{dy}{dx} = g(x)h(y) \implies \frac{dy}{h(y)} = g(x)dx \implies \int \frac{dy}{h(y)} = \int g(x)dx$)

Example Solve the initial value problem $\frac{dy}{dx} = e^{x+y}, y(0) = 0$.

$$\frac{dy}{dx} = e^x e^y \implies \frac{dy}{e^y} = e^x dx \implies \int e^{-y}dy = \int e^x dx \implies -e^{-y} = e^x + c_1$$

Solve for y:

$$e^{-y} = -e^x - c_1 \implies -y = \ln(-e^x - c_1) \implies y = -\ln(-e^x - c_1) \implies y = \ln[\frac{1}{c - e^x}](c = -c_1)$$

Apply the initial condition: we will get $c = 2$ and hence $y = \ln[\frac{1}{2-e^x}]$

The only dander here is overlooking singular solutions. For example, find the general solution to

$$\frac{dy}{dx} = -4xy^2$$

it is easy to get the solution $\begin{cases} y = \frac{1}{2x^2+c} & y \neq 0 \\ y = 0 & y = 0 \end{cases}$(watch out the singular solutions)

Well here has an example that does not satisfy the theorem. Consider the differential equation

$$\frac{dy}{dx} = (xy)^{2/3}$$

Notice that $f(x, y) = (xy)^{2/3}$ is continuous but $f_y(x, y) = \frac{2}{3}x^{2/3}y^{-1/3}$ is not when $y = 0$.

Solve:

$$\int \frac{dy}{y^{2/3}} = \int x^{2/3}dx \implies y = (\frac{1}{5}x^{5/3} + c)^3 (\text{ if } y \neq 0)$$

3 Mathematical Models of Population Growth

Strictly speaking, differential equations should apply only to continuous processes, but quantities which change sequentially can be approximated by continuous functions (populations, prices, etc.)

Suppose we wish to predict future values of a population, $P(t)$. We will assume that $P(t)$ is differentiable. To find $P(t)$, we will make assumptions about $\frac{dP}{dt}$.

3.1 The Malthusian Model (T. Robert Malthus)

The most basic assumptions under ideal circumstances (unlimited food, space, etc.), the rate of change of population should be proportionate to the population itself. That is, $\frac{dP}{dt} = rP$ for some $r \in \mathbb{R}$. To complete the model we need an initial condition: $P(0) = P_0$. Solve: $\int \frac{dP}{P} = \int rdt \implies \ln P = rt + C_1 \implies P = Ce^{rt}$.From the initial condition, we can solve $C = P_0$. To determine the rate r, we need some point in the future for the population. For example If $P(10) = 2.4P_0$, then $r = \ln 2.4/10$.

3.2 The Logistic Model

Of course, Malthus realized that resources were not unlimited. to improve on the model, he suggested the concept of a maximum sustainable population (a "carrying capacity") which we'll call K. With this, we should have

$$\frac{dP}{dt} = 0 \text{ when } P = K; \ \frac{dP}{dt} < 0 \text{ if } P > K; \ \frac{dP}{dt} \approx rP \text{ when } P \ll K$$

Malthus suggested the Logistic Equation:

$$\frac{dP}{dt} = rP(1 - \frac{P}{K})$$

Solution: this is separable:

$$\frac{dP}{P(1-\frac{P}{K})} = rdt \implies \int \frac{KdP}{P(K-P)} = \int rdt \implies \int \frac{1}{P} + \frac{1}{K-P} dP = \int rdt$$

Hence we get $\ln P - \ln|K-P| = rt + C_1 \implies \ln|\frac{P}{K-P}| = rt + C_1 \implies \frac{P}{K-P} = C_2 e^{rt}$. Eventually, we get

$$P = \frac{KC_2e^{rt}}{1+C_2e^{rt}} = \frac{K}{C_3e^{-rt}+1}(C_3 = \frac{1}{C_2})$$

To apply the initial condition that $P(0) = P_0 \implies C_2 = \frac{P}{K-P_0}$, so $C_3 = \frac{K-P_0}{P_0}$, and $P = \frac{K}{(\frac{K-P_0}{P_0})e^{-rt}+1}$. The inflection points occur when $P = K/2$. How do we know this? Consider:

$$\frac{dP}{dt} = rP(1 - \frac{P}{K})$$

Differentiate with respect to t:

$$\frac{d^2P}{dt^2} = r\frac{dP}{dt}(1-\frac{P}{K}) - rP(\frac{1}{K}\frac{dP}{dt}) = r\frac{dP}{dt}[1-\frac{2P}{K}]$$

so

$$\frac{d^2P}{dt^2} = r^2P[1-\frac{P}{K}][1-\frac{2P}{K}]$$

and so $\frac{d^2P}{dt^2} = 0$ when $P = 0, K,$ or $K/2$

4 The Principle of Dimensional Homogeneity

A useful check on our calculations is the realization that in any equation, the units of each term must be the same. e.g., In the equation $\frac{dP}{dt} = rP$. $\frac{dP}{dt}$ has "dimensions" of $\frac{\text{population}}{\text{time}}$ Therefore rP must also have dimensions of $\frac{\text{population}}{\text{time}}$ so r must have dimensions of $\frac{1}{\text{time}}$ (r is a frequency!) (we say $[r] = T^{-1}$)

Now consider the solution: $P(t) = P_0e^{rt}$. We see that e^{rt} must be dimensionless. Why? Consider that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

This cannot make sense if x has dimension!

Now consider the logistic equation:

$$\frac{dP}{dt} = rP(1 - \frac{P}{K})$$

where $\frac{\text{population}}{\text{time}} = (\frac{1}{\text{time}})(\text{population})(\text{dimensionless})$ and its solution:

$$P(t) = \frac{KP_0}{(K - P_0)e^{-rt} + P_0}$$

Comments: Angle is dimensionless because $\theta = \frac{S}{r}$

5 First-Order Linear ODEs

Consider

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

As motivation, consider the example

$$x\frac{dy}{dx} + y = e^x$$

This is just $\frac{d}{dx}(xy) = e^x$ so $xy = \int e^x dx = e^x + C$ so $y = \frac{1}{x}e^x + \frac{C}{x}$

To solve $a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$, we

1. write it in standard form:

$$\frac{dy}{dx} + p(x)y = q(x)$$

2. Multiply through by an unknown function $\mu(x)$ (an "integrating factor")

$$\mu(x)\frac{dy}{dx} + \mu(x)p(x)y = \mu(x)q(x)$$

3. Now assume that the LHS is $\frac{d}{dx}(\mu(x)y(x))$

   This requires that

$$\mu y' + \mu p y = \mu y' + \mu' y$$

   so $\mu(x)p(x)y(x) = \mu'(x)y(x)$. That is $\frac{d\mu}{dx} = \mu p$, so $\int \frac{d\mu}{\mu} = \int p(x)dx$ and so $\ln \mu = \int p(x)dx + C$ and so $\mu(x) = e^{\int p(x)dx + C}$

4. The DE will now be

$$\frac{d}{dx}(\mu y) = \mu q$$

   and so $\mu(x)y(x) = \int \mu(x)q(x)dx$ and $y(x) = \frac{1}{\mu(x)} \int \mu(x)q(x)dx$

Example 1

Solve $\frac{dy}{dx} + xy = x$.

Solution: This is linear in standard form so we identify $p(x) = x$ and calculate $\mu(x) = e^{\int x dx} = e^{\frac{1}{2}x^2}$ Factor this in

$$e^{\frac{1}{2}x^2}\frac{dy}{dx} + xe^{\frac{1}{2}x^2}y = xe^{\frac{1}{2}x^2}$$

This is $\frac{d}{dx}(e^{\frac{1}{2}x^2}y) = xe^{\frac{1}{2}x^2}$

so $e^{\frac{1}{2}x^2}y = \int xe^{\frac{1}{2}x^2}dx = e^{\frac{1}{2}x^2} + C$

$\implies y = 1 + Ce^{-\frac{1}{2}x^2}$

Example 2

$xy' - 2y = x^3 \cos x$ (assume $x > 0$)

Solution: In standard form we have

$$y' - \frac{2}{x}y = x^2 \cos x$$

so $\mu(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln x^{-2}} = e^{\ln(x^{-2})} = \frac{1}{x^2}$

This gives us

$$\frac{1}{x^2}y' - \frac{2}{x^3}y = \cos x$$

That is $\frac{d}{dx}(\frac{1}{x^2}y) = \cos x \implies \frac{y}{x^2} = \sin x + C$

Example 3

Solve the IVP $\frac{dx}{dt} + x = \sqrt{1 + \cos^2 t}$, $x(1) = 4$.

Solution: Here $\mu(t) = e^{\int 1 dt} = e^t$ so we have

$$e^t\frac{dx}{dt} + e^t x = e^t\sqrt{1 + \cos^2 t}$$

That is $\frac{d}{dt}(e^t x) = e^t\sqrt{1 + \cos^2 t}$

$\implies e^t x = \int e^t\sqrt{1 + \cos^2 t}dt$

We may write this as

$$e^t x = \int_1^t e^\tau\sqrt{1 + \cos^2 \tau}d\tau + C$$

$$\implies x(t) = e^{-t}\int_1^t e^\tau\sqrt{1 + \cos^2 \tau}d\tau + Ce^{-t}$$

Use the initial condition $x(1) = 4$.

$$4 = e^{-1} \cdot 0 + Ce^{-1} \implies c = 4e$$

so $x(t) = e^{-t}\int_1^t e^\tau\sqrt{1 + \cos^2 \tau}d\tau + 4e^{1-t}$

5.1 Substitution

If a first-order ODE is neither linear nor separable, it might be possible to convert it into a separable or linear equation by introducing an appropriate change of variable.

5.1.1 Homogeneous Equations

If the equation has the form $\frac{dy}{dx} = f(\frac{y}{x})$, we can replace y with $u = \frac{y}{x}$. (Think: $u(x) = \frac{y(x)}{x}$)

Since $y = ux$, we have

$$\frac{dy}{dx} = x\frac{du}{dx} + u$$

Example: Solve $\frac{dy}{dx} = \frac{x^2-y^2}{3xy}$

Note that this is not linear or separable but

$$\frac{dy}{dx} = \frac{1}{3}(\frac{x}{y} - \frac{y}{x})$$

Let $u = \frac{y}{x}$, so $\frac{dy}{dx} = x\frac{du}{dx} + u$

$$x\frac{du}{dx} + u = \frac{1}{3}(\frac{1}{u} - u)$$
$$\implies \frac{xdu}{dx} = \frac{1}{3u} - \frac{4}{3}u$$
$$\implies \frac{3du}{\frac{1}{4} - 4u} = \frac{dx}{x}$$
$$\implies 3\int \frac{u}{1-4u^2}du = \int \frac{dx}{x} \text{ Let } z = 1-4u^2, dz = -8udu$$
$$\implies -\frac{3}{8}\int \frac{dz}{z} = \int \frac{dx}{x}$$
$$\implies -3\ln|z| = 8\ln|x| + c_1$$
$$\implies z^{-3} = c_2x^8 \implies \frac{1}{(1-4u^2)^3} = c_2x^8$$
$$\implies c_2x^8(1-4(\frac{y}{x})^2)^3 = 1$$

5.1.2 The Form $y' = G(ax + by)$

If $y' = G(ax + by)$, then let $u = ax + by$.

e.g. Solve $\frac{dy}{dx} = sin(x - y)$

Let $u = x - y$, so $\frac{du}{dx} = 1 - \frac{dy}{dx}$. The DE becomes: $1 - \frac{du}{dx} = \sin(u)$, $\frac{du}{dx} = 1 - \sin(u)$ separable. This implies $\int \frac{du}{1-\sin u} = \int dx$. Therefore, we can get $\int (sec^2u + sec(u)tan(u)du = \int dx$. Hence $tan(u) + sec(u) = x + c$. At the end we get an expression

$$tan(x - y) + sec(x - y) = x + c$$

Here is another example: $\frac{dy}{dx} = \sqrt{x+y} - 1$
Let $u = x + y$. Then $\frac{du}{dx} = 1 + \frac{dy}{dx}$. The DE becomes

$$\frac{du}{dx} - 1 = \sqrt{u} - 1 \implies \frac{du}{dx} = \sqrt{u} \implies \int u^{-1/2} du = \int dx$$

$$\implies 2\sqrt{u} = x + c_1 \implies u = (\frac{x}{2} + c_2)^2$$

$$\implies x + y = (\frac{x}{2} + c_2)^2 \implies y = \frac{1}{4}(x+c)^2 - x$$

5.1.3 Bernoulli Equation

A Bernoulli equation has the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

We can solve these by letting $v = y^{1-n} = y^{-(n-1)}$
Example: Solve $\frac{dy}{dx} - 5y = -\frac{5}{2}xy^3$
Let $y = y^{-2}$, then $\frac{dy}{dx} = -2y^{-3}\frac{dy}{dx}$
This is easiest to use if we first divide the original equation by y^3.

$$\frac{1}{y^3}\frac{dy}{dx} - \frac{5}{y^2} = -\frac{5}{2}x \implies -\frac{1}{2}\frac{dv}{dx} - 5v = -\frac{5}{2}x$$

i.e. $\frac{dv}{dx} + 10v = 5x$.
We need $u(x) = e^{\int 10dx} = e^{10x}$

$$e^{10x}\frac{dv}{dx} + 10e^{10x}v = 5xe^{10x} \implies \frac{d}{dx}(e^{10x}v) = 5xe^{10x}$$

$$\implies e^{10x}v = \frac{1}{2}xe^{10x} - \frac{1}{20}e^{10x} + c \implies \frac{1}{y^2} = \frac{1}{2}x - \frac{1}{20} + ce^{-10x} \text{ or } y = 0$$

5.2 Graphing Families of Solutions

We can gain information about solutions directly from the DE. Example: suppose $\frac{dy}{dx} = y^2 - 4$

- We can see immediately that $y = \pm 2$ are equilibrium solutions , and $y' \neq 0$ anywhere else.

- Also, $y' > 0$ when $|y| > 2$; $u' < 0$ when $|y| < 2$; $y' = 0$, when $|y| \approx 2$
- We could use these observation to plot direction field.
- of course, we can solve this DE. We find

$$y = 2(\frac{e^{-4x} + c}{e^{-4x} - c}) \text{ (or } y = -2)$$

If we have both the DE and its general solution, we can sketch the solutions quite quickly. Example, Consider the equation $\frac{dy}{dx} = y - x^2$. The solution is $y = Ce^x + x^2 + 2x + 2$

- From the solution we can see that one "exceptional" solution is $y = x2 + 2x + 2 = (x+1)^2 + 1$ (all solutions have an exponential term, except for this one!)
- From the DE we can see that $\frac{dy}{dx} = 0$, when $y = x^2$ (this is a horizontal isocline, or "curve of zero slope")
- All solutions move towards the exceptional solution as $\to -\infty$ and are repelled by it as $x \to \infty$.

Example: $\frac{dy}{dx} + \frac{1}{x}y = 3x$
$u(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$
$\implies x\frac{dy}{dx} + y = 3x^2$
$\frac{d}{dx}(xy) = 3x^2 \implies xy = x^3 + c \implies y = x^2 + \frac{c}{x}$

- Exceptional solution $y = x^2$
- Other solutions $\implies x^2$ as $x \to \pm\infty$ but $\implies \pm\infty$ as $x \to 0$
- $y' = 0$ when $y = 3x^2$ (horizontal isocline)

Example: Recall the equation $\frac{dy}{dx} = (xy)^{2/3}$. This has solution $y = (\frac{1}{5}x^{5/3} + C)^3$ or $y = 0$. Exceptional solutions: $y = \frac{1}{125}x^5, y = 0$

6 Higher-order Linear ODEs

We now turn to equations of the form

$$a_n(x)\frac{d^ny}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x) \tag{1}$$

Definition. *The equation is called homogeneous if $F(x)$ is the zero function; otherwise, it is inhomogeneous.*

6.1 Existence & Uniqueness Theorem

The IVP consisting of equation above and the n initial condition $y(x_0) = p_0, y'(x_0) = p_1, \cdots, y^{(n-1)}(x_0) = p_{n-1}$ has a unique solution or an interval I containing x_0. If

1. $a_n, a_{n-1}, \cdots, a_0$ and F are continuous on I
2. $a_n(x) \neq 0$ anywhere on I

6.1.1 Operator

Writing equations like (1) is tiresome so we have a more concise notation.

Definition. *An operator is a mapping from functions to functions.*

For example, we can speak of the differential operator - D which produces the derivative of the input.

$$Df(x) = f'(x)$$

$$D(x^2 + 7) = 2x$$

The identity operator is I: $If(x) = f(x)$. We can combine these to produce new operators. For example Letting $D^2 = D \cdot D$, we could write

$$(D^2 + 5I)f(x) = D^2 f + 5If = D(Df) + 5f = Df'(x) + 5f(x) = f''(x) + 5f(x)$$

If we give the operator $D^2 + 5I$ a new name say $\Theta(D)$, then we can write

$$y'' + 5y = \cos x$$

as

$$\Theta(D)[y] = \cos x \text{ or just } \Theta[y] = \cos x$$

Similarly, by letting

$$\Theta = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)I$$

we can write equation (1) as

$$\Theta[y] = F(x)$$

6.2 The Principle of Superposition (1st-order version)

Let $y_1(x)$ be a solution to the equation $y' + p(x)y = F_1(x)$ and let $y_2(x)$ be a solution to the equation $y' + p(x)y = F_2(x)$.

Then $y_1 + y_2$ is a solution to the equation

$$y' + p(x)y = F_1(x) + F_2(x)$$

.

Proof. If y_1 and y_2 are as described then

$$(y_1 + y_2)' + p(x)(y_1 + y_2) = y_1' + p(x)y_1 + y_2' + p(x)y_2 = F_1(x) + F_2(x)$$

□

6.2.1 Special Cases

1. If y_n is a solution to $y' + p(x)y = 0$ and y_p is a solution to $y' + p(x)y = f(x)$, then $y_n + y_p$ is also a solution to $y' + p(x)y = f(x)$

2. If y_1 and y_2 are both solutions to $y' + p(x)y = 0$, then so is $y_1 + y_2$. Furthermore, any linear combination of y_1 and y_2 will also be a solution.

   Proof. If $y = c_1y_1 + c_2y_2$, then

   $$\begin{aligned} y' + p(x)y &= c_1y_1' + c_2y_2' + p(x)y_1c_1 + c_2p(x)y_2 \\ &= c_1(y_1' + p(x)y_1) + c_2(y_2' + p(x)y_2) = 0 \end{aligned}$$

   □

   These results generalize naturally to higher orders.

6.3 Linear Independence of Functions

Definition. *The function $f_1(x), f_2(x), \cdots, f_n(x)$ are linearly dependent on an interval I if there exist constants $c_1, c_2, \cdots, c_n$ (not all zero) such that*

$$c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0$$

for all $x \in I$.

For example, $e^x, e^{-x}, \sinh x$ ($\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$) is a linearly dependent set.

Otherwise, linearly independent. For example, $1, x, x^2$ is a linearly independent set, since $c_1 + c_2x + c_3x^2 = 0, \forall x \implies c_1 = c_2 = c_3 = 0$.Another example, $\cos^{-1} x, \sin^{-1} x, 1$ are linearly dependent, since $\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$

Definition. *The Wronskian of the n functions $f_1, f_2, \cdots, f_n$ is the determinant of the $n \times n$ matrix*

$$\begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}$$

We will denote this by $W(x)$ or $W(f_1, f_2, \cdots, f_n)$

Example: If $f_1 = \sin x$ and $f_2 = e^{2x}$ then $W(x) = det(\begin{pmatrix} \sin x & e^{2x} \\ \cos x & 2e^{2x} \end{pmatrix}) = 2e^{2x}\sin x - e^{2x}\cos x$

Theorem. *If f_1 and f_2 are linearly dependent on an interval I, then $W(f_1, f_2) = 0$ for some $x_0 \in I$*

Proof. Suppose W is nonzero for all $x \in I$, and suppose that $c_1 f_1 + c_2 f_2 = 0$. Differentiating gives $c_1 f_1' + c_2 f_2' = 0$, so we may write $\begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since $W \neq 0$ this matrix is invertible, so $c_1 = c_2 = 0$, meaning that f_1 and f_2 are linearly independent. The converse is not generally true. However, it does hold in the context of linear ODEs!. □

Theorem. *Suppose y_1 & y_2 are solutions to the equations $y'' + p(x)y' + q(x)y = 0$ and $p(x)$ and $q(x)$ are continuous on an interval I. If $W(y_1, y_2) = 0$ for some $x_0 \in I$, then y_1 and y_2 are linearly dependent.*

Proof. Assume that $c_1 y_1 + c_2 y_2 = 0$ for all $x \in I$ (we will construct a non-zero c_1, c_2) Then $c_1 y_1' + c_2 y_2' = 0$ so $\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since $W = 0$ for some $x_0 \in I$, there exists a non-zero solution $[c_1, c_2]^T$ when $x = x_0$.

Now let $u(x) = c_1 y_1 + c_2 y_2$. Then $u'(x) = c_1 y_1' + c_2 y_2'$ and so $u(x_0) = 0$ and $u'(x) = 0$. This means that $u(x)$ is a solution to the IVP $u'' + p(x)y' + q(x)y = 0$ with $y(x_0) = 0$ and $y'(x_0) = 0$. By inspection, the function $y = 0$ is a solution, and by the Existence and Uniqueness Theorem the solution is unique so $u(x)$ is the zero function. Thus we've found nonzero c_1, c_2 such that $c_1 y_1 + c_2 y_2 = 0$ for all $x \in I$, so y_1 and y_2 are linearly independent. □

(so y_1&y_2 are linearly dependent if and only if $W(x) = 0$ for a some $x_0 \in I$)

Theorem. *If y_1 and y_2 are solutions to a linear DE, then $W(t)$ is either zero everywhere or nowhere.*

Proof. Abel's Formula: $W(x) = W(x_0)e^{-\int_{x_0}^{x} p(\epsilon)d\epsilon}$.

Proof. If $y'' + p(x)y_1' + q(x)y_1 = 0$ and $y_0'' + p(x)y_2' + q(x)y_2 = 0$, multiply the first equation by y_2 and multiply the second by y_1 and subtract them.

$$(y_1 y_2'' - y_1'' y_2) + p(x)(y_1 y_2' - y_1' y_2) = 0$$

Hence

$$W'(x) + p(x)W(x) = 0$$

so we have $W'(x) + p(x)W(x) = 0$. Hence $W(x) = c_1 e^{-\int p(x)dx} = c_1 e^{-\int_{x_0}^{x} p(\epsilon)d\epsilon}$

Setting $x = x_0$, gives $c_1 = W(x_0)$. □

□

6.4 Homogeneous Linear DEs Finding Solutions

1. The Existence and Uniqueness theorem tells us that a second-order linear ODE needs to be accompanied by two ICs. The general solution need s two arbitrary constant.

2. The principle of superposition tells us that if y_1 and y_2 are solutions to a linear homogeneous DE, then so is $c_1y_1 + c_2y_2$. This suggests the following:

   Theorem. *Suppose $p(x)$ and $q(x)$ are continuous on I. If y_1 and y_2 are linearly independent solutions to the DE $y'' + p(x)y + q(x)y = 0$, then the general solution is $y = c_1y_1 + c_2y_2$.*

Proof. Let $\Phi(x)$ be any solution (we will show that $\exists c_1, c_2$ such that $\Phi = c_1y_1 + c_2y_2$) Consider the IVP consisting of the DE and the ICs $y(x_0) = \Phi(x_0)$ and $y'(x_0) = \Phi'(x_0)$ If we match the proposed general soliton $y = c_1y_1 + c_2y_2$ to these ICs, we

$$c_1y_1(x_0) + c_2y_2(x_0) = \Phi(x_0)$$

$$c_1y'(x_0) + c_2y_2'(x_0) = \Phi'(x_0)$$

That is

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \Phi(x_0) \\ \Phi'(x_0) \end{pmatrix}$$

Since $W(y_1, y_2) \neq 0$, we can apply the inverse matrix to find $c_1 \& c_2$. Hence $\Phi(x)$ is a linear combination of $y_1 \& y_2$. □

6.5 Solving Homogeneous Constant-Coefficient Linear ODEs

We have difficulty solving $y'' + p(x)y' + q(x)y = 0$, unless $p(x)\&q(x)$ are constants.

Example: Consider $y'' = y$.We have $y_1 = e^x$ and $y_2 = e^{-1}$, and so the general solution is $y = c_1e^x + c_2e^{-x}$

Example: Consider $y'' = -y$. We have $y_1 = \sin x$ and $y_2 = \cos x$ and so $y = c_1 \cos x + c_2 \sin x$.

Clever Observation

$\cos x, \sin x$ are related to e^x. Euler's formal: $e^{ix} = \cos x + i \sin x$. In fact, $e^{ix}($ and $e^{-ix})$ are also solutions to $y'' = -y$, so the general solution can be written as

$$y = c_1e^{ix} + c_2e^{-ix}$$

Perhaps every equation $y'' + py' + qy = 0$ has exponential solutions? Let's assume that $y = e^{mx}$. then $y' = me^{mx}$ and $y'' = m^2e^{mx}$

Substituting these into the DE gives,

$$m^2e^{mx} + pme^{mx} + qe^{mx} = 0$$

$$\implies m^2 + pm + q = 0$$

We call this the characteristic equation of the DE. If $m = \frac{-p\pm\sqrt{p^2-4q}}{2}$ then e^{mx} is a solution

Case I: Distinct Real Roots. If we have solution m_1, m_2, then e^{m_1x} and e^{m_2x} are linearly independent solutions, so $y = c_1e^{m_1x} + c_2e^{m_2x}$ is the general solution.

Eg. Suppose $y'' - y' - 2y = 0$. The characteristic equation is $m^2 - m - 2 = 0$ so $m = -1, 2$.

$$\implies y = c_1e^{-x} + c_2e^{2x}$$

Case II: Complex Conjugate Roots.Here we can still say that the general solution is $y = c_1e^{m_1x} + c_2e^{m_2x}$ but $m_1, m_2 = \alpha \pm i\beta$

However, we can assume that y is real (we know that real solutions exist) so let's rewrite this :

$$\begin{aligned} y &= c_1e^{(\alpha+i\beta)x} + c_2e^{(\alpha-i\beta)x} \\ &= e^{\alpha x}[c_1e^{i\beta x} + c_2e^{-i\beta x}] \\ &= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)] \\ &= e^{\alpha x}[(c_1+c_2)\cos\beta x + (c_1-c_2)i\sin\beta x] \end{aligned}$$

since this is real, $(c_1 + c_2) \in \mathbb{R}$ and $(c_1 - c_2)i \in \mathbb{R}$. Let's rename them!

$$y = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$$

Eg/ Suppose $y'' + 2y' + 5y = 0$. The characteristics equation is

$$m^2 + 2m + 5 = 0 \implies (m+1)^2 + 4 = 0 \implies m = -1 \pm 2i$$

$$\implies y = e^{-x}[c_1\cos 2x + c_2\sin 2x]$$

Case III: Repeated Real Roots. In this case our guess (of e^{mx}) has yielded only one solution. We need a second one.

D'Alembert's Method of discovery: Having 2 identical roots should not be much different from having 2 nearly identical roots. So, suppose we have two roots, m and $m + \epsilon$. Then the general solution is $y = c_1e^{mx} + c_2e^{(m+\epsilon)x}$

As $\epsilon \to 0$ it appears that the solutions all merge. However, if we have $c_1 = -\frac{1}{\epsilon}$ and $c_2 = \frac{1}{\epsilon}$ then $y = \frac{1}{\epsilon}e^{(m+\epsilon)x} - \frac{1}{\epsilon}e^{mx} = e^mx[\frac{e^{\epsilon x}-1}{\epsilon}]$

Now $\lim_{\epsilon\to 0}\frac{e^{\epsilon x}-1}{\epsilon} = \lim_{\epsilon\to 0}\frac{xe^{\epsilon x}}{1} = x$

Therefore a second (linearly independent) solution is xe^{mx}, and the general solution is

$$y = c_1 e^{mx} + c_2 x e^{mx}$$

For example, for $y'' + 6y' + 9y = 0$, we have $m^2 + 6m + 9 = 0$ i.e. $(m+3)^2 = 0$, so $m = -3$ and so $y = C_1 e^{-3x} + c_2 x e^{-3x}$

6.5.1 Generalization

For nth-order equations

$$y^{(n)} + b_{n-1} y^{(n-1)} + \cdots + b_1 y' + b_0 y = 0$$

, the characteristic equation is

$$m^n + b_{n-1} m^{n-1} + \cdots + b_1 m + b_0 = 0$$

. For every distinct real root m, e^{mx} is a solution. For every complex pair $\alpha \pm i\beta$, $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$ are solutions. If we have repeated roots, we multiply by x, repeatedly if necessary.

Examples

1. $y''' - 4y'' + 7y' - 6y = 0$. Characteristic equation $m^3 - 4m^2 + 7m - 6 = (m-2)(m^2 - 2m + 3) = 0$. Hence $m = 1 \pm \sqrt{2}i$. Hence we have $y = c_1 e^{2x} + c_2 e^x \cos \sqrt{2}x + c_3 e^x \sin \sqrt{2}x$
2. $y''' + 3y'' + 3y' + y = 0$. Characteristics equation is $m^3 + 3m^2 + 3m + 1 = (m+1)^3 = 0$, so $m = -1, -1, -1$. The general solution is $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$
3. $y^{(4)} + 2y'' + y = 0$ Characteristics equation is $m^4 + 2m^2 + 1 = 0$. i.e. $(m^2 + 1)^2 = 0$, so $m = i, i, -i, -i$

   The general solution is $y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x$.

6.5.2 Solving Inhomogenuous Linear Equation

From our discussion of the theory of linear DEs. If we can solve the corresponding homogeneous solution (call the solution y_h) and we can find one solution to the full DE (call it y_p), then the general solution will be $y = y_h + y_p$.

How do we find y_p? One option: **The Method of Undetermined Coefficient**

Example 1

Consider the DE $y'' + y' - 6y = e^{4x}$. What kind of function could satisfy this? Maybe a multiple of e^{4x}? We guess that $y_p = Ae^{4x}$. Then $y_p' = 4Ae^{4x}$, and $y_p'' = 16Ae^{4x}$ so the DE becomes

$$16Ae^{4x} + 4Ae^{4x} - 6Ae^{4x} = e^{4x} \implies 14A = 1$$

so $A = \frac{1}{14} \implies \frac{1}{14}e^{4x}$ is a solution.

Now for the homogeneous problem we have

$$y'' + y' - 6y = 0$$

$\implies m^2 + m - 6 = 0$
$(m+3)(m-2) = 0 \implies m = -3, 2$ so $y_h = c_1e^{-3x} + c_2e^{2x}$

Example 2

Consider $y'' + y' - 6y = 6x^2$? We try

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

Substitution into the DE gives

$$2A + (2Ax + B) - 6(Ax^2 + Bx + C) = 6x^2$$

$$\implies -6Ax^2 + (2A - 6B)x + (2A + B - 6C) = 6x^2$$

$$\implies A = -1, B = -\frac{1}{3}, C = -\frac{7}{18}$$

so $y_p = -x^2 - \frac{1}{3}x - \frac{7}{18}$ is a solution. Example: $y'' + y' - 4y = \cos 2x$

We try

$$y_p = A\cos 2x + B\sin 2x$$

$$y_p' = -2A\sin 2x + 2B\cos 2x$$

$$y_p'' = -4A\cos 2x - 4B\sin 2x$$

We find

$$-4A\cos 2x - 4B\sin 2x$$

$$-2A\sin 2x + 2B\cos 2x$$

$$-4A\cos 2x - 4B\sin 2x = \cos 2x$$

$$\implies A = -\frac{2}{17}, B = \frac{1}{34}$$

$\implies y_p = -\frac{2}{17}\cos 2x + \frac{1}{34}\sin 2x$ is a solution.

Summary

Inhomogeneous Term	Trial function
e^{kx}	Ae^{kx}
x^n	$A_n x_n + \cdots + A_0$
$\sin kx$ or $\cos kx$	$A\cos kx + B\sin kx$

1. If $F(x)$ is a sum of functions, we try the sum of the corresponding trial functions.

   For example, for $y'' + y' + y = e^{3x} + x$, we will try $y_p = Ae^{3x} + Bx + C$.

2. If $F(x)$ is a product of functions, we guess the product of the corresponding trial f)unctions but (for example $y'' + y' + y = x^2e^x$ we'd try $y_p = (Ax^2 + Bx + C)e^x$) one constant will be redundant, and must be omitted. Another example will be $y'' + y' + y = x\cos x$, we'd try $y_p = (Ax + B)(C\cos x + D\sin x)$

3. There is one other special case. Consider $y'' - y' - 2y = e^{2x}$. Try $y_p = Ae^{2x}$, $y_p' = 2Ae^{2x}$, $y_p'' = 4Ae^{2x}$. The DE becomes $4Ae^{2x} - 2Ae^{2x} - 2Ae^{2x} = e^{2x}$, (i.e. $0 = 1$)

   What happened? e^{2x} is one of the solutions of the homogeneous problem, so it can't solve the inhomogeneous one!. What else might work? We try Axe^{2x}. If $y_p = Axe^{2x}$, then $y_p' = Ae^{2x} + 2Axe^{2x}$ and $y_p'' = 4Ae^{2x} + 4Axe^{2x}$.

   Plug into DE.

   $$4Ae^{2x} + 4Axe^{2x} - Ae^{2x} - 2Axe^{2x} - 2Axe^{2x} = e^{2x}$$

   so $A = \frac{1}{3}$ and $y_p = \frac{1}{3}xe^{2x}$ is a solution.

   This works in general:

   (a) If any term in the "usual" trial function is a solution to the homogeneous problem, then multiply it by x. repeat if necessary. If this duplicates another term in your trial function, then multiply that term by x as well.
   E.g. For $y'' + 2y' + y = e^{-x}$, we have $y_h = c_1e^{-x} + c_2xe^{-x}$ so we try $y_p = Ax^2e^{-x}$
   Comment: It is advisable to always find y_n first.
   Comment 2: For other forcing terms we will probably not be able to guess the correct form. For example $y'' + y' + y = \frac{1}{x}$

6.6 Variation of Parameters

6.6.1 First Order

Consider $y' + xy = \frac{1}{x}$. Find solution to homogeneous: $y' + xy = 0 \implies y_h = Ce^{-\frac{1}{2}x^2}$ to find y_p we try: $y_p = u(x)e^{-\frac{1}{2}x^2}$

$$y_p' = u'e^{-\frac{1}{2}x^2} - xue^{-\frac{1}{2}x^2}$$

$$u' = \frac{1}{x}e^{\frac{1}{2}x^2} \implies u(x) = \int \frac{1}{x}e^{\frac{1}{2}x^2} \implies u(x) = \int_{x_0}^{x} \frac{1}{t}e^{\frac{1}{2}t^2}dt + C$$

$y_p = e^{-\frac{1}{2}x^2}\int_{x_0}^{x}\frac{1}{t}e^{\frac{1}{2}t^2}dt \implies y = y_p + y_h$

Note: we could keep the "$+C$" in u to get the general solution.

6.6.2 Second Order

Now suppose $y'' + py' + qy = F$. Suppose we can find $y_n = c_1y_1 + c_2y_2$, so we try $y_p = u_1y_1 + u_2y_2$ and $y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$. If we differentiate again, will get very messy so we impose a second condition: $u_1'y_1 + u_2'y_2 = 0$. Hence $y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$. Therefore $u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' + pu_1y_1' + pu_2y_2' + qu_1y_1 + qu_2y_2 = F$. That is $u_1'y_1' + u_2'y_2' = F$. Therefore we can solve u_1' and u_2'.

Example: $y'' + qy = 9\sec^2(3t)$ (for $t \in (0, \frac{\pi}{6})$). Firstly solve $y'' + 9y = 0 \implies y_h = c_1\cos 3t + c_2\sin 3t$. Then we try $y = u_1\cos 3t + u_2\sin 3t \implies u_1'\cos 3t + u_2'\sin 3t = 0$ and $-u_1'\sin 3t + u_2'\sin 3t = 3\sec^2 3t$. Therefore, $u_2' = 3\sec 3t \implies u_2 = \ln|\sec 3t + \tan 3t| + c_2$ and $u_1 = -\sec 3t + c_1$

6.7 The "Reduction of Order technique"

The same trick (variation of parameters) can be used to find full solutions to homogeneous problems, when we know one solution. When used in this context, it is called the reduction of order technique.

Example: Consider

$$x^2y'' + 2xy' - 2y = 0 \text{ (for } x > 0)$$

we might be able to spot one solution: $y_1 = x$. To find the second solution, we write

$$y_2 = u(x)y_1 = xu$$

Then $y_2' = xu' + u$

$$y_2'' = xu'' + 2u'$$

Plug this into the DE:

$$x^2(xu'' + 2u') + 2x(xu' + u) - 2xu = 0$$

$$\implies x^3u'' + 4x^2u' = 0 \implies xu'' + 4u' = 0 \text{ (A first-order DE for u')}$$

For convenience, let $v = u'$, so $x\frac{dv}{dx} + 4v = 0$

$$\implies \int \frac{dv}{v} = \int -4\frac{dx}{x} \implies \ln v = -4\ln x \implies v = x^{-4} \implies u = \frac{-1}{3}x^{-3}$$

Therefore a second solution is $y_2 = x(-\frac{1}{3}x^{-3})$ or just $y_2 = \frac{1}{x^2}$ The general solution to this DE is $y = c_1x + \frac{c_2}{x^2}$

Example: Consider

$$y^{(4)} + 4y''' + 6'' + 4y' + y = 0$$

The characteristic equation is $m^4+4m^3+6m^2+4m+1 = 0$ (i.e. $(m+1)^4 = 0$) so $y_1 = e^{-x}$ is one solution so suppose $y = ue^{-x}$ Then $y' = u'e^{-x} - ue^{-x}$ and $y'' = u''e^{-x} - 2u'e^{-x} + ue^{-x}$. Next, $y''' = u'''e^{-x} - 3u''e^{-x} + 3u'e^{-x} - ue^{-x}$. Lastly $y'''' = u''''e^{-x} - 4u'''e^{-x} + 6u''e^{-x} - 4u'e^{-x} + ue^{-x}$. Then DE becomes

$$u''''e^{-x} = 0$$

This implies $u'''' = 0, u''' = c_1, u''' = c_1x + c_2, u'' = \frac{c_1x^2}{2} + c_2x + c_3, \cdots$.

So the general solution is $y = (c_1x^3 + c_2x^2 + c_3x + c_4)e^{-x}$ This even works for inhomogeneous equations!

Example:

$$x^2y'' + xy' - y = 72x^5$$

Notice that a solution to the homogeneous problem is $y_1 = x$. To find the general solution, we have two options:

1. Use reduction of order to find y_n and then variance of parameters.
2. Use reduction of order directly!

We will try both:

1. Consider $x^2y'' + xy' - y = 0$

   Suppose $y_2 = ux$ Then $y_2' = u'x + u$ and $y_2'' = u''x + 2u'$. This implies $x^2(u''x + 2u') + x(u'x + u) - ux = 0$. Hence $xu'' + 3u' = 0$. Let $v = u'$ so $xv' + 3v = 0$. Hence

   $$\frac{dv}{v} = \int \frac{-3dx}{x} \implies \ln v = -3\ln x \implies y_2 = \frac{1}{x}$$

   Therefore $y_h = c_1x + \frac{c_2}{x}$ Now, for $x^2y'' + xy' - y = 75x^5$, we try $y = u - 1x + \frac{u_2}{x}$. We need to solve

   $$u_1'y_1 + u_2'y_2 = 0$$

   $$u_1'y_1' + u_2'y_2' = F(x)(\leftarrow \text{This requires the DE to be in standard form!})$$

   That is

   $$u_1'x + \frac{u_2'}{x} = 0 \ (1)$$

   $$u_1' - \frac{u_2'}{x^2} = 72x^3 \ (2)$$

   Then

   $$(1)/x + (2) \implies u_1' = 36x^3 \implies u_1 = 9x^4 + c_1$$

   $$1/x - 2 \implies u_2' = 36x^5 \implies u_2 = -6x^6 + c_2$$

   Hence

   $$y = c_1x + \frac{c_2}{x} + 9x^5 - 6x^5 = c_1x + \frac{c_2}{x} + 3x^5$$

2. Use reduction of order directly:

   Assume that $y = u(x)x$ so $y' = u'x + u$ and $y'' = u''x + 2u'$. Substitute into DE,

$$x^2(u''x + 2u') + x(u'x + u) - ux = 72x^5$$

$$\implies x^3u'' + 3x^2u' = 72x^5 \implies xu'' + 3u' = 72x^3$$

   Let $v = u' : x\frac{dv}{dx} + 3v = 72x^3$. That is

$$\frac{dv}{dx} + \frac{3}{x}v = 72x^2$$

   Integrating factor $u(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$. Hence $x^3\frac{dv}{dx} + 3x^2v = 72x^5$.

$$x^3v = 12x^6 + C \implies v = 12x^3 + \frac{C}{x^3} \implies u = 3x^4 - \frac{C}{x^2} + C_1$$

$$\implies y = C_1x + \frac{C_2}{x} + 3x^5$$

7 Boundary Value Problems (BVP)

A boundary value problem involves a DE with conditions specified at different points. E.G. $y'' + y = 0, y(0) = 0, y(1) = 1$. There is no E/U Theorem for such problems.

Usually there are no solutions, and when they do exist there are often infinitely many of them.

Examples:

1. $y'' + \pi^2y = 0, y(0) = 0, y(1) = 1$. The general solution is $y = C_1\cos\pi x + C_2\sin\pi x$ If $y(0) = 0$, then $C_1 = 0$. If $y(1) = 1$, then $C_1 = -1$. Hence this problem has no solutions.

2. $y'' + \pi^2y = 0, y(0) = 0, y(1) = 0$.

$$y = C_1\cos\pi x + C_2\sin\pi x$$

   again. $C_1 = 0, C_1 = 0$. Therefore C_2 is free. Every multiple of $\sin\pi x$ is a solution!

   The characteristic equation is $m^2 + k = 0$.

   Case 1: If $K < 0$, we can call $k = -\lambda^2$. We have $m = \pm\lambda$ and $y = c_1e^{\lambda x} + c_2e^{-\lambda x}$. If $y(0) = 0$, then $c_1 + c_2 = 0$. If $y(1) = 0$ then

$$c_1e^{\lambda} + c_2e^{-\lambda} = 0 \implies c_1e^{2\lambda} + c_2 = 0$$

$$\implies c_1e^{2\lambda} - c_1 = 0 \implies c_1(e^{2\lambda} - 1) = 0 \implies c_1 = 0 \text{ since } \lambda \neq 0$$

Hence $c_2 = 0$, and $y = 0$ is the only solution for $k < 0$.

Case 2: If $k = 0$, then $y = c_1 x + c_2$. Again

$$y(0) = 0 \implies c_2 = 0$$

$$y(1) = 0 \implies c_1 + c_2 = 0$$

so $y = 0$ is the only solution.

Case 3: If $K > 0$, let $K = \lambda^2$.

$$y'' + \lambda^2 y = 0$$

General solution is $y = c_1 \cos \lambda x + c_2 \sin \lambda x$. Therefore, $y(0) = 0 \implies c_1 = 0$, $y(1) = 0 \implies c_2 \sin \lambda = 0$. Therefore either $c_2 = 0$ (so $y = 0$ again). or $\lambda = n\pi, n = \pm 1, \pm 2, \cdots$ (i.e. $k = n^2\pi^2$) in which case the solutions are

$$y = C \sin n\pi x$$

Terminology: The numbers

$$n^2\pi^2 = \pi^2, 4\pi^2, 9\pi^2. \cdots$$

are called the eigenvalues of the BVP. The functions $\sin n\pi x = \sin \pi x \sin 2\pi x, \cdots$ are called the eigenfunctions. There BVPs are referred to as eigenvalue problems.

Example: Find the eigenvalues & eigenfunctions of the BVP

$$y'' + 2y' + ky = 0, y(0) = y(1) = 0$$

Solution: the characteristics equations is $m^2 + 2m + k = 0$ so $m = -1 \pm \sqrt{1-k}$.

- If $K < 1$ the solutions are exponentials and cannot satisfy the BCs (unless $y = 0$).
- If $K = 1$, the solutions are $y = c_1 e^{-x} + c_2 x e^{-x}$

  $y(0) = 0 \implies c_1 = 0, y(1) = 0 \implies C_2 e^{-1} = 0 \implies C_2 = 0$, so $y = 0$ again.
- If $K > 1$, then

  $$y = e^{-x}[C_1 \cos \sqrt{k-1} x + C_2 \sin \sqrt{k-1} x]$$

  $y(0) = 0 \implies C_1 = 0$, $y(1) = 0 \implies C_2 e^{-1} \sin \sqrt{k-1} = 0$. Hence $C_2 = 0$ or $\sqrt{k=1} = n\pi$ (i.e. $k = n^2\pi^2 + 1, n \in \mathbb{Z}$. (These are the eigenvalues)

The eigenfunctions are $y = e^{-x} \sin n\pi x$

7.1 System of ODEs

In some applications, we encounter "coupled" equations; the rates of change of two quantities may each depend on both quantities. Example: Suppose we have two populationsL one predator and one prey. Let $x9t)$ be the population of prey. Let $y(t)$ be the population of the predators. We may argue that

$$\frac{dx}{dt} = r_1 x - \alpha_1 xy$$

where r_1 is natural growth rate of prey, without predation and α_1 is the death rate of prey caused by interaction with predators. and

$$\frac{dy}{dt} = -r_2 y + \alpha_2 xy$$

where r_2 is the natural death rate of predators in absence of prey and α_2 is the growth rate for predators due to interaction with prey.

Unfortunately, we can't solve this system, because it's non-linear. We'll stick to linear systems, such as

$$\begin{aligned} x'(t) &= a_1 x + b_1 + c_1 z + f_1(t) \\ y'(t) &= a_2 x + b_2 y + c_2 z + f_2(t) \\ z'(t) &= a_3 x + b_3 y + c_3 z + f_3(t) \end{aligned}$$

Linear systems can be expressed as vector DEs. Let

$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Then

$$\vec{x}'(t) = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix}$$

or just $\vec{x}' = A\vec{x} = A\vec{x} + \vec{f}$

Our analysis of these problems will be shaped by 2 theorems.

Theorem. *Any higher-order linear ODE, or system of higher-order Des, can be converted into a system of first-order linear ODEs*

Proof. (for a pair of second-order equations) Suppose

$$x'' + p_1 x' + q_1 x = f_1(t)$$

$$\text{and } y'' + p_2 y' + q_2 y = f_2(t)$$

If we let $\vec{x}(t) = \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}$, then

$$\vec{x}' = \begin{pmatrix} x' \\ x'' \\ y' \\ y'' \end{pmatrix} = \begin{pmatrix} x' \\ -p_1 x' - q_1 x + f_1(t) \\ y' \\ -p_2 y' - q_2 y + f_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -q_1 & -p_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -q_2 & -p_2 \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ f_1(t) \\ 0 \\ f_2(t) \end{pmatrix}$$

Implication: we really only need to study first-order vector equations. □

Theorem. *(partial converse - not in notes) Every system of first-order linear constant-coefficient ODEs can be converted into a higher-order linear constant-coefficient ODE.*

Proof. (2D-Case) Suppose $x' = ax + by + f(t)$ (1) and $y' = cx + dy + g(t)$ (2) . Solve (1) for $y : y = \frac{x'}{b} - \frac{a}{b}x - \frac{f(t)}{b} \implies y' = \frac{x''}{b} - \frac{a}{b}x' - \frac{f'(t)}{b}$. Hence (2) becomes

$$x'' - (a + d)x' + (ad - bc)x = f' - df + bg$$

Implication: The theory of linear constant-coefficient ODEs will carry over to vector DEs. □

In particular, we have

Existence & Uniqueness: The IVP $\vec{x}' = A\vec{x} + \vec{f}(t)$, $\vec{x}(0) = \vec{x}$ has a unique solution on an interval I provided that $\vec{f}(t)$ is continuous on I. (This holds even if $A = A(t)$, as long as $A(t)$ is continuous). **The Principle of Superposition** applies. If $\vec{x}_1$ and $\vec{x}_2$ are solutions to $\vec{x}' = A\vec{x}$ then so is $c_1\vec{x}_1 + c_2\vec{x}_2$ (etc.).

- An n-dimensional homogeneous system will require n linearly independent solutions, and hence n arbitrary constants.
- The solution to an inhomogeneous system can be found as $\vec{x} = \vec{x}_h + \vec{x}_p$.
- The Wronskian must be modified.

$$W(\vec{f}_1, \cdots, \vec{f}_n) = det[\vec{f}_1, \cdots, \vec{f}_n]$$

Example: Suppose the functions, $\vec{x}_1 = e^{2t}\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$, $\vec{x}_2 = e^{2t}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\vec{x}_3 = e^{3t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are solutions to a linear vector DE. Are they independent? Check $W(t) = -e^{3t}(1-5e^{4t}) \neq 0 \implies yes.$

7.2 Homogeneous Systems with Constant Coefficients Method of Solution

Consider the DE $\vec{x}' = A\vec{x}$ (i.e. the system $x'(t) = a_{11}x + a_{21}y$ and $y'(t) = a_{12}x + a_{22}y$)

Recalled that the ODE $y' = ay$ has solution $y = Ce^{ax}$ we guess that $\vec{x}' = A\vec{x}$ has solution $\vec{x} = \vec{v}e^{mt}$, for some $m \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^2$. This gives $\vec{x}' = m\vec{v}e^{mt}$ so plugging this into the DE gives

$$m\vec{v}e^{mt} = A\vec{v}e^{mt}$$

so $A\vec{v} = m\vec{v}$. That is, $\vec{x} = \vec{v}e^{mt}$ is a solution to $\vec{x}' = A\vec{x}$, if m is an eigenvalue of A and $\vec{v}$ is a corresponding eigenvector.

Recall: $A\vec{v} = \lambda\vec{v}$ can be rewritten as $(A - \lambda I)\vec{v} = \vec{0}$.

- This will have a nonzero solution if and only $(A - \lambda I)^{-1}$ does not exist (otherwise $\vec{v} = (A - \lambda I)^{-1}\vec{0} = \vec{0}$).
- A matrix is invertible if and only if its determinant is non-zero.
- Therefore we need to solve $det(A - \lambda I) = 0$ for λ. We call this the characteristic equation of the DE.
- Once we have found the eigenvalues λ, we can find corresponding eigenvectors by solving $(A - \lambda I)\vec{v} = \vec{0}$ for $\vec{v}$ (there will be one free variable here).
- An $n \times n$ matrix will have n eigenvectors if multiplicity is counted.
- The eigenvectors corresponding to distinct eigenvalues will be linearly independent.
- Complex eigenvalues always occur in conjugate pairs.
- If an eigenvalue has multiplicity m, it will possess anywhere from 1 to m linearly independent eigenvectors.

7.3 The 2D Case

Case 1: Distinct Real eigenvalues

Example: Solve $\vec{x}' = A\vec{x}, A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ (that is $x'(t) = 2x + 3y$ and $y'(t) = 2x + y$)

Find the eigenvectors

For $\lambda = 4$: $(A - \lambda I)\vec{v} = \vec{0}$ becomes $\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $2v_1 - 3v_2 = 0$, i.e. $v_2 = \frac{2}{3}v_1$. Setting $v_1 = 3$, we get $\vec{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Therefore one solution to the DE is $\vec{x}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$.

For $\lambda = -1$: $(A - \lambda I)\vec{v} = \vec{0} \implies \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $v_1 + v_2 = 0$, and we may use $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Therefore a second solution is $\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$. Hence the general solution is $\vec{x} = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$. Of course we may have initial conditions. Suppose $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \implies c_1 = \frac{2}{5}, c_2 = \frac{-1}{5}$. After all, we can plug c_1, c_2 in and get the general solution.

Case 2: Complex Eigenvalues:

Example: Solve $\vec{x}' = \begin{pmatrix} -2 & 1 \\ -3 & -4 \end{pmatrix} \vec{x}$. Find λ: $A - \lambda I = \begin{pmatrix} -2-\lambda & 1 \\ -3 & -4-\lambda \end{pmatrix}$ Hence $det(A - \lambda I) = (\lambda + 3)^2 + 2$. Hence $\lambda = -3 \pm \sqrt{2}i$. Find $\vec{v}$: $\vec{x}_1 = \begin{pmatrix} 1 \\ -1 + \sqrt{i} \end{pmatrix} e^{(-3+\sqrt{2}i)t}$. We break this into its real and imaginary parts: $\vec{x}_1 = e^{-3t}[\cos\sqrt{2}t + i \sin\sqrt{2}t][\begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}]$.
Therefore $\vec{x}_1 = e^{-3t}[\begin{pmatrix} \cos\sqrt{2}t \\ -\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{pmatrix} + i \begin{pmatrix} \sin\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t - \sin\sqrt{2}t \end{pmatrix}]$

From this we can conclude immediately that the general solution is

$$\vec{x} = e^{-3t}[c_1 \begin{pmatrix} \cos\sqrt{2}t \\ -\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{pmatrix} + c_2 \begin{pmatrix} \sin\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t - \sin\sqrt{2}t \end{pmatrix}]$$

Case 3: Repeated Eigenvalues:

Consider $\vec{x}' = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix} \vec{x}$.

$$A - \lambda I = \begin{pmatrix} -3-\lambda & 4 \\ -1 & 1-\lambda \end{pmatrix}$$

so $det(A - \lambda I) = (\lambda + 1)^2$. Hence $\lambda = -1$ (repeated). Eigenvectors? $(A - \lambda I)\vec{v} = \vec{0} \implies \begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Therefore our eigenvector is $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Hence one solution is $\vec{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$. For the second solution, try multiplying by t? Trying $\vec{x}_2 = te^{\lambda t}\vec{v}$, gives $\vec{x}_2' = e^{\lambda t}\vec{v} + \lambda te^{\lambda t}\vec{v}$. so $\vec{x}' = A\vec{x} \implies e^{\lambda t}\vec{v} + \lambda te^{\lambda t}\vec{v} = Ate^{\lambda t}\vec{v}$. Hence $\vec{v} = \vec{0}$ (but $\vec{v} \neq \vec{0}$!)

We need more constants! It turns out that the right "guess" is $\vec{x}_2 = te^{\lambda t}\vec{v} + e^{\lambda t}\vec{w}$. This gives $\vec{x}_2' = e^{\lambda t}\vec{v} + \lambda te^{\lambda t}\vec{v} + \lambda e^{\lambda t}\vec{w}$ so $\vec{x}' = A\vec{x} \implies e^{\lambda t}\vec{v} + \lambda te^{\lambda t}\vec{v} + \lambda e^{\lambda t}\vec{w} = Ate^{\lambda t}\vec{v} + Ae^{\lambda t}\vec{w}$. Therefore, $A\vec{w} = \lambda\vec{w} + \vec{v}$. This implies $(A - \lambda I)\vec{w} = \vec{v}$. Note: observe that if we apply

$(A-\lambda I)$ to this we get $(A-\lambda I)^2\vec{w} = (A-\lambda I)\vec{v} = \vec{0}$. $\vec{w}$ is called a generalized eigenvector of A. Back to our example: we need to solve

$$\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Therefore $\vec{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. The general solution is $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1 e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2[te^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} -1 \\ 0 \end{pmatrix}]$

7.4 Generalizations for Higher-Order Systems

- If A is $n \times n$ it has n eigenvalues
- Each distant real eigenvalue will give us one solution
- Each pair of complex eigenvalues will give us two solutions.
- If an eigenvalue is repeated k times, it may still have k linearly independent eigenvectors, in which case we obtain k independent solutions
- If an eigenvalue has multiplicity k but has $< k$ independent eigenvectors, we will need generalized eigenvectors.

For example, if λ has multiplicity 2 and 1 eigenvector $\vec{v}$, then one solution is $\vec{u}e^{\lambda t}$ and another is $\vec{u}te^{\lambda t} + \vec{v}e^{\lambda t}$, where $(A-\lambda I)\vec{v} = \vec{u}$.

Another example is that if λ has multiplicity 3 and 1 eigenvector $\vec{u}$, then the solutions are $\vec{u}e^{\lambda t}$, $\vec{u}te^{\lambda t} + \vec{v}e^{\lambda t}$ where $(A-\lambda I)\vec{v} = \vec{u}$, $\frac{1}{2}\vec{u}t^2e^{\lambda t} + \vec{v}te^{\lambda t} + \vec{w}e^{\lambda t}$ where $(A-\lambda I)\vec{w} = \vec{u}$.

Here is another example, if λ has multiplicity 3 and 2 eigenvectors, $\vec{u}_1$ and $\vec{u}_2$, then 3 solutions are $\vec{u}_1e^{\lambda t}$, $\vec{u}_2e^{\lambda t}$, $\vec{v}te^{\lambda t} + \vec{w}e^{\lambda t}$, where $\vec{v}$ is some linear combination of $\vec{u}_1$ and $\vec{u}_2$, and $(A-\lambda I)\vec{w} = \vec{v}$. ($\vec{v}$ will become obvious when we write this down!).

Example: Solve $\vec{x}' = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}\vec{x}$. We find $\lambda = 1, -1, 4.$ and $\vec{v} = \begin{pmatrix} -1 \\ -6 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ so $\vec{x}(t) = c_1e^t\vec{v}_1 + c_2e^{-t}\vec{v}_2 + c_3e^{4t}\vec{v}_3$.

7.5 Inhomogeneous Linear Vector DEs

The method of undetermined coefficients can be applied to vector DEs.

Example: Consider: $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\vec{x} + \begin{pmatrix} te^{-2t} \\ 3e^{-2t} \end{pmatrix}$. We find $\vec{x}_h = c_1e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We guess $\vec{x}_p = [\vec{a} + \vec{b}t]e^{-2t}$. This gives $\vec{x}_p' = \vec{b}e^{-2t} - 2[\vec{a} + \vec{b}t]e^{-2t}$. The DE becomes

$$e^{-2t}[\vec{b} - 2\vec{a} - 2\vec{b}t] = A[\vec{a} + \vec{b}t]e^{-2t} + \vec{f} = e^{-2t}[A\vec{a} + A\vec{b}t] + \vec{f}$$

That is

$$e^{-2t}\begin{pmatrix} b_1-2a_1-2b_1t \\ b_2-2a_2-2b_2t \end{pmatrix} = e^{-2t}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + te^{-2t}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \begin{pmatrix} t \\ 3 \end{pmatrix} e^{2t}$$
$$= e^{-2t}\begin{pmatrix} a_1+a_2+(b_1+b_2+1)t \\ 2a_2+3+2b_2t \end{pmatrix}$$

Therefore, $a_2 = -\frac{3}{4}, a_1 = \frac{5}{36}, b_1 = -\frac{1}{3}, b_2 = 0$. Therefore, $\vec{x} = c_1e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-2t}\begin{pmatrix} \frac{5}{36} \\ -\frac{3}{4} \end{pmatrix} + te^{-2t}\begin{pmatrix} -\frac{1}{3} \\ 0 \end{pmatrix}$. Note:variation of parameters (next lecture) will usually be quicker, unless $\vec{f}(t)$ is of a simple form. For example, if I see $\vec{x}' = A\vec{x} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. I will always use undetermined coefficient.

If for linear homogeneous ODEs, Wronskian is zero if only if the solution is dependent.

7.6 Variation of Parameters for vector Des

Suppose $\vec{x}' = A\vec{x} + \vec{F}(t).If_h = c_1\vec{x}_1 + c_2\vec{x}_2$, then we try $\vec{x} = u_1\vec{x}_1 + u_2\vec{x}_2$ and plug this into DE:

$$u_1'\vec{x}_1 + u_1\vec{x}_1' + u_2'\vec{x}_2 + u_2\vec{x}_2' = Au_1\vec{x}_1 + Au_2\vec{x}_2 + \vec{F}$$

Therefore $u_1'\vec{x}_1 + u_2'\vec{x}_2 = \vec{F}$. In component form:

$$u_1'x_{11} + u_2'x_{21} = F_1$$
$$u_1'x_{12} + u_2'x_{22} = F_2$$

We will always be able to solve this for u_1' and u_2', because the columns of $\begin{pmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{pmatrix}$ are $\vec{x}_1$ and $\vec{x}_2$, which are linearly independent. So, we will get $u_1' = G_1(t), u_2' = G_2(t)$, $\vec{x} = [\int G_1(t)dt]\vec{x}_1 + [\int G_2(t)dt]\vec{x}_2$

Note: If we omit the constants of integration we will have a particular solution. If we include them, we get the general solution.

Example (MOUC example revisited)

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\vec{x} + e^{-2t}\begin{pmatrix} t \\ 3 \end{pmatrix}$$
$$\vec{x}_h = c_1e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We try $\vec{x} = u_1e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ which means

$$u_1'e^t + u_2'e^{2t} = te^{-2t}$$

$$u_2'e^{2t} = 3e^{-2t}$$

From the second one, we get $u_2' = 3e^{-4t}$, so $u_2 = \frac{-3}{4}e^{-4t} + c_2$. The first one becomes

$$u_1 = \int (t-3)e^{-3t}dt = -\frac{1}{3}(t-3)e^{-3t} + \int \frac{1}{3}e^{-3t}dt = \frac{8}{9}e^{-3t} - \frac{1}{3}te^{-3t} + c_1$$

so $\vec{x} = (\frac{8}{9}e^{-3t} - \frac{1}{3}te^{-3t} + c_1)e^t \begin{pmatrix}1\\0\end{pmatrix} + (c_2 - \frac{3}{4}e^{-4t})e^{2t}\begin{pmatrix}1\\1\end{pmatrix}$ (same as last time, but in a different form!)

Example: Solve $\vec{x}' = A\vec{x} + \vec{F}$, where $A = \begin{pmatrix}1 & 4\\1 & 1\end{pmatrix}$ and $\vec{F} = \begin{pmatrix}e^{-t}\\0\end{pmatrix}$

Solution: First, find $\vec{x}_h = c_1e^{3t}\begin{pmatrix}2\\1\end{pmatrix} + c_2e^{-t}\begin{pmatrix}-2\\1\end{pmatrix}$

Try $\vec{x} = u_1e^{3t}\begin{pmatrix}2\\1\end{pmatrix} + u_2e^{-t}\begin{pmatrix}-2\\1\end{pmatrix}$ and solve

$$u_1'(2e^{3t}) + u_2'(-2e^{-t}) = e^{-t}$$

and

$$u_1'(e^{3t}) + u_2'(e^{-t}) = 0$$

Hence solve $u_1' = -\frac{1}{4}e^{-4t} \implies u_1 = \frac{-1}{16}e^{-4t} + c_1$. Also $u_2 = \frac{-t}{4} + c_2$.

8 The 3 Most Famous PDEs

8.1 The Heat Equation (or Conduction Equation)

Consider a metal bar of length L, with uniform cross-sectional area and on insulated surface. Let $u(x,t)$ be the temperature at a distance x from one end at time t. It can be shown though a careful analysis of the physics of heat transfer that u should obey the equation.

$$\frac{\partial u}{\partial t} = \gamma\frac{\partial^2 u}{\partial x^2}$$

Rough Explanation: Suppose the temperature profile looks a wave curve at time t. Where this graph is concave up the temperature will initially increase, and where it is concave down it will decrease! To solve the Heat equation, we need on initial condition $u(x,0) = f(x)$. We will also need boundary conditions, and there are several possibilities for these.

1. If we imagine fixing the ends of the bar in ice, then we'ed have $u(0,t) = 0, u(L,t) = 0$.

2. If we insulate the ends, we need instead $u_x(0,t) = 0, u_x(L,t) = 0$.

3. If we leave the ends expressed, the conditions turn out to involve both u and u_x.
4. We could have different conditions at the two ends.
5. We could even imagine an infinitely long bar, in which case we require only that u be bounded as $x \to \pm\infty$.

Note: For a metal plate, we have the 2D Heat equation. $u_t = \gamma(u_{xx} + u_{yy})$

8.2 The Wave Equation

Now consider a string of length L, under tension. Let $u(x,t)$ be the vertical displacement of each point on the string at time t. Then $u_{tt} = \alpha^2 u_{xx}$.

Rough Explanation: the vertical acceleration of each point on the string is determined by the concavity at that point. With this, we need ICs

$$u(x,0) = \phi(x)$$

$$u_t(x,0) = \gamma(x)$$

(We could set the string in motion either by plucking it ($\phi(x) \neq 0$) or striking it ($\gamma(x) \neq 0$)).

For BCs, the simplest would be $u(0,t) = 0$, $u(L,t) = 0$.

8.3 Laplace's Equation

This is (in 2D) $u_{xx} + u_{yy} = 0$. This can be thought of as a kind of smoothness condition.

For example, In 1D it is just $f''(x) = 0$ ($\implies f(x) = Ax + B$).

The BCs must be closed.

For example, we might have $u(x,0) = 0, u(x,1) = 0, u(0,y) = 0, u(1,y) = \sin \pi y$.

8.4 First-order Linear PDEs and Partial Integration

Just as simple ODEs can be solved by integration (e.g. $y' = 2x \implies y = x^2 + c$). Some simple PDEs can be solved by "Partial integration". For example, suppose $u_y = -e^{-y}$ (where $u = u(x,y)$. We integrate with respect to y to obtain

$$u(x,y) = e^{-y} + g(x)$$

The "constant" of integration is an arbitrary function of x! To evaluate it, we need an initial condition for each value of x. There could all be given at $y = 0$, but they could be specified along a curve in the xy plane.

For example, If $u(x,0) = \frac{1}{1+x^2}$, then $\frac{1}{1+x^2} = 1 + g(x)$ so $g(x) = \frac{1}{1+x^2} - 1$, and $u(x,y) = \frac{1}{1+x^2} - 1 + e^{-y}$. We essentially have an ODE for each value of x here. The solution can be viewed as propagating along the lines $x = c$. When this happens, we call the lines "characteristics". Eg, if, instead, we've given the IC, $u = 0$ along $y = x^2$ (i.e. $u(x, x^2) = 0$) then $0 = e^{-x^2} + g(x)$ so $g(x) = -e^{-x^2}$ and $u(x, y) = e^{-y} - e^{-x^2}$.

Partial integration is only possible if both u_x and u_y appear in the PDE. However, we can always eliminate one of them through a change of variable if the equation is linear:

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y)$$

Example: Suppose $u_y = -u_x, u(x, 0) = e^{-x^2}$ If we let

$$\xi = x - 2y, \eta = y$$

then we may write

$$Z = u(x, y) = \hat{u}(\xi, \eta)$$
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial x}$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\frac{d\eta}{dy}$$

That is,

$$\frac{\partial u}{\partial x} = \frac{\partial \hat{u}}{\partial \xi}\frac{\partial \xi}{\partial x}$$
$$\frac{\partial z}{\partial y} = \frac{\partial \hat{u}}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \hat{u}}{\partial \eta}\frac{d\eta}{dy}$$

Since $\xi = x - 2y, \eta = y$, $u_x = \hat{u}_\xi$ and $u_y = -2\hat{u}_\xi + \hat{u}_\eta$. The PDE $u_y = 2u_x$, therefore becomes $(-2\hat{u}_\xi + \hat{u}_\eta = -2\hat{u}_\xi$ or just $\hat{u}_\eta = 0 \implies \hat{u}(\xi, \eta) = g(\xi)$. Returning to the original variables, $u(x, y) = g(x, 2y)$. This can be matched to the IC: $u(x, 0) = e^{-x^2} \implies e^{-x^2} = g(x)$. Hence $u(x, y) = e^{-(x-2y)^2}$. Check?

$$u_x = -2(x - 2y)e^{-(x-2y)^2}$$
$$u_y = 4(x - 2y)e^{-(x-2y)^2}$$

so $u_y = -2u_x$.

Question: How do we choose the new variables? Consider the ODE $\frac{dy}{dx} = f(x, y)$, If its general solution is written as $\phi(x, y) = k$ then $\frac{\partial x}{\partial y} = \frac{-dy}{dx}$ (that is $\frac{\phi_x}{\phi_y} = -f(x, y)$)

Proof. Just differentiate $\phi(x, y) = k$ implicitly! (w.r to x):

$$\phi_x + \phi_y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{-\phi_x}{\phi_y}$$

□

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y)$$

We will let $\xi = \xi(x, y)$, $\eta = \eta(x, y)$.

8.5 First-Order Linear PDEs

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y)$$

We will let $\epsilon = \epsilon(x,y)$ and $\eta = \eta(x,y)$ to replace $u(x,y)$ with $\hat{u}(\xi,\eta)$. By the chain rule,

$$u_x = \hat{u}_\xi \xi_x + \hat{u}_\eta \eta_x$$

$$u_y = \hat{u}_\xi \xi_y + \hat{\eta}\eta_y$$

so the PDE becomes

$$a(x,y)[\hat{u}_\xi \xi_x + \hat{u}_\eta \eta_x] + b(x,y)[\hat{u}_\eta \eta_y + \hat{u}_\eta \eta_y] + c(x,y)\hat{u} = f(x,y)$$

Rearranging, we have

$$[a(x,y)\xi_x + b(x,y)\xi_y]\hat{u}_\xi + [a(x,y)\eta_x + b(x,y)\eta_y]\hat{u}_\eta + c(x,y)\hat{u} = f(x,y)$$

The goal is to make either $\hat{u}_\xi$ or $\hat{u}_\eta$ disappear. Let's eliminate $\hat{u}_\eta$, by setting

$$a(x,y)\eta_x + b(x,y)\eta_y = 0$$

That is assuming that $\eta_y \neq 0$, $\frac{\eta_x}{\eta_y} = \frac{-b(x,y)}{a(x,y)}$

Our Lemma now suggests that we should choose η such that $\eta(x,y) = k$ is the general solution to the ODE $\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$. What about ξ?Our only other constraint is that the transformation $(x,y) \to (\xi,\eta)$ is invertible. Therefore its Jacobian must be non-zero.

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = det[\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}] = \xi_x\eta_y - \xi_y\eta_x \neq 0$$

If we just let $\xi = x$, then this is satisfied!

Example Solve the IVP $xu_x + yu_y = 3u$ and $u(x,1) = 1 - x^2$, where $x > 0, y \geq 1$.

Solution: We start by solving $\frac{dy}{dx} = \frac{y}{x} \implies \int \frac{dy}{y} = \int \frac{dx}{x} \implies \ln y = \ln x + C_1$ so $\ln y - \ln x = C_1$.

This tells us that we may use $\xi = x, \eta = \ln y - \ln x$. Then $u_2 = \hat{u}_\xi \xi_x + \hat{u}_\eta \eta_x = \hat{u}_\xi - \frac{1}{x}\hat{u}_\eta$ and $u_y = \hat{u}_\xi \xi_y + \hat{u}_\eta \eta_y = \frac{1}{y}\hat{u}_\eta$.

The PDE becomes $(x\hat{u}_\xi - \hat{u}_\eta) + \hat{u}_\eta = 3\hat{u} \implies \xi\hat{u}_\xi = 3\hat{u}$

Returning to the original variables, we have

$$u(x,y) = f(\ln y - \ln x)x^3 = f(\ln \frac{y}{x})x^3 = g(\frac{y}{x})x^3 = x^3 \ln \frac{x}{y}$$

Finally, since $u(x,1) = 1 - x^2$, we have $1 - x^2 = x^3 \ln x$ so $h(x) = \frac{1-x^2}{x^3}$. Hence

$$u(x,y) = x^3[\frac{1 - (\frac{x}{y})^2}{(\frac{x}{y})^3}] = y^3(1 - \frac{x^2}{y^2}) = y^3 - x^2y$$

Comment: This is known as the method of characteristics. In these problems, information from the K can be viewed as being carried along the curves $\phi(xy) = K$.

8.6 Second-Order Linear PDEs

Consider

$$a(x,y)U_{xx} + b(x,y)U_{xy} + c(x,y)U_{yy} + d(x,y)U_x + e(x,y)U_y + f(x,y)U = g(x,y)$$

We will introduce $\xi - \xi(x,y)$ and $\eta - \eta(x,y)$ to convert this to

$$A(\xi,\eta)\hat{U}_{\xi\xi} + B(\xi,\eta)\hat{U}_{\xi\eta} + C(\xi,\eta)\hat{U}_{\eta\eta} + D(\xi,\eta)\hat{U}_{\xi} + E(\xi,\eta)\hat{U}_{\eta} + F(\xi,\eta)\hat{U} = G(\xi,\eta)$$

The goal is to eliminate one or more of the second order terms (make A, B, or C = 0) What are those functions?

This is tedious. I will get you started. We need U_x, U_y, U_{xx}, etc. in terms of $\hat{U}_\xi$, etc.

$$U_x = \hat{U}_\xi \xi_x + \hat{U}_\eta \eta_x$$

Therefore

$$\begin{aligned} U_{xx} &= \frac{\partial}{\partial x}(U_x) = \frac{\partial}{\partial x}[\hat{U}_\xi \xi_x + \hat{U}_\eta \eta_x] \\ &= \frac{\partial}{\partial x}[\hat{U}_\xi \xi_x + \hat{U}_\xi \frac{\partial}{\partial x}[\xi_x] + \frac{\partial}{\partial x}[\hat{U}_\eta]\eta_x + \hat{U}_\eta \frac{\partial}{\partial x}[\eta_x] \\ &= [\hat{U}_{\xi\xi}\xi_x + \hat{U}_{\xi\eta}\eta_x]\xi_x + \hat{U}_\xi \xi_{xx} + [\hat{U}_{\xi\eta}\xi_x + \hat{U}_{\eta\eta}\eta_x]\eta_x + \hat{U}_\eta \eta_{xx} \\ &= (\xi_x)^2\hat{U}_{\xi\xi} + 2\xi_x\eta_x\hat{U}_{\xi\eta} + (\eta_x)^2\hat{U}_{\eta\eta} + \hat{U}_\xi \xi_{xx} + \hat{U}_\eta \eta_{xx} \end{aligned}$$

Repeating this for U_{xy} and U_{yy}, plugging these into the original PDE and rearranging, we get

$$A(\xi,\eta) = a(x,y)(\xi_x)^2 + b(x,y)\xi_x\xi_y = c(x,y)(\xi_y)^2$$

$$B(\xi,\eta) = 2a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y$$

$$C(\xi,\eta) = a(x,y)(\eta_x)^2 + b(x,y)\eta_x\eta_y = c(x,y)(\eta_y)^2$$

To eliminate the $\hat{U}_{\xi\xi}$ term, we set $A = 0$,

$$A(\xi,\eta) = a(x,y)(\xi_x)^2 + b(x,y)\xi_x\xi_y = c(x,y)(\xi_y)^2 = 0$$

Assuming that $\xi_y \neq 0$, we can rewrite this as

$$a(\frac{\xi_x}{\xi_y})^2 + b(\frac{\xi_x}{\xi_y}) + c = 0$$

Therefore

$$\frac{\xi_x}{\xi_y} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Similarly we can eliminate the $\hat{U}_{\eta\eta)}$ term by requiring that $\eta_y \neq 0$ and $\frac{\eta_x}{\eta_y} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Three cases arise, dividing second-order linear PDEs into 3 classes

1. Hyperbolic Equations: If $b^2 - 4ac > 0$, then we can choose $\xi = \phi$ and $\eta = \phi_2$ where $\phi_1 = k_1$, is the solution to $\frac{x}{y} = \frac{-b-\sqrt{b^2-4ac}}{2a}$;

   $\phi_2 = k_2$ is the solution to $\frac{x}{y} = \frac{-b+\sqrt{b^2-4ac}}{2a}$

   This will eliminate both $\hat{U}_{\xi\xi}$ and $\hat{U}_{\eta\eta}$ leaving after divisor by B.

$$\hat{U}_{\xi\eta} + \Phi(\xi, \eta, \hat{U}, \hat{U}_\xi, \hat{U}_\eta) = 0$$

   This is the canonical form of hyperbolic PDE

   E.G. $U_{xy} = U$ is hyperbolic

2. Parabolic Equations

   If $b^2 - 4ac = 0$, then we can choose $\xi = \phi$, where $\phi = k$ is the solution to $\frac{dy}{dx} = \frac{b}{2a}$. We cannot also eliminate $\hat{U}_{\eta\eta}$. However since $\frac{\xi_x}{\xi_y} = \frac{-b}{2a}$, the $\hat{U}_{\xi\eta}$ term disappear. Hence $B = 0$. Lastly, the canonical form of a parabolic PDE is

$$\hat{U}_{\eta\eta} + \Phi(\xi, \eta, \hat{U}, \hat{U}_\xi, \hat{U}_\eta) = 0$$

The classification of a PDE has implications on the kind of boundary conditions required and the method of solutions used.

8.7 Method of Characteristics

8.7.1 Example

$u_{xy} + u_y = xy, u(x, 0) = 0, u(0, y) = \sin y$

Can just integrate with respect to y. Hence $u_x + u = \frac{1}{2}xy^2 + f(x)$, then this is just ODE, integrating factor is e^x.

Hence

$$e^x u = \int \frac{1}{2}xe^x y^2 + e^x f(x) = \frac{1}{2}y^2(xe^x - e^x + h(y)) + \int e^x f(x)dx$$

$$u(x, y) = \frac{1}{2}y^2(x - 1 + \frac{h_1(y)}{e^x}) + e^{-x}[F(x) + h_2(y)]$$

where $F' = e^x f$.

Therefore $u(x, y) = \frac{1}{2}y^2(x - 1) + g(x) + e^{-x}h(y)$ is the general solution.

$u(x, 0) = 0 = g(x) + e^{-x}h(0)$, $u(0, y) = \sin y = -\frac{1}{2}y^2 + g(0) + h(y)$. so $g(x) = -e^{-x}h(0)$ and $h(y) = \sin y + \frac{1}{2}y^2 - g(0) \implies g(0) = -h(0)$.

so $u(x, y) = \frac{1}{2}y^2(x - 1) + e^{-x}\sin y + \frac{1}{2}y^2e^{-x}$

8.7.2 Example

Consider the wave equation $u_{tt} = \alpha^2 u_{xx}$ subject to $u(x, 0) = \gamma(x), u_t(x, 0) = 0$.

Starting by solving

$$\frac{dt}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = \pm\frac{1}{\alpha}$$

so $dx = \pm\alpha dt$ Hence $x = \pm\alpha t + c$ so $\xi = x + \alpha t$ and $t = x - \alpha t$

$$u_{tt} = \alpha^2 \hat{u}_{\xi\xi} - 2\alpha^2 \hat{u}_{\xi t} + \alpha^2 \hat{u}_{tt}$$

$$u_{xx} = \hat{u}_{\xi\xi} + 2\hat{u}_{\xi t} + \hat{u}_{tt}$$

then $u_{tt} = \alpha^2 u_{xx} \implies {}_{\xi t} = 0$

Continuing with the 1D Wave equation problem

$$U_{tt} = \alpha^2 U_{xx}$$

$$U(x, 0) = \gamma(x)$$

$$u_t(x, 0) = 0$$

A clarification: unless we restrict the domain of $\gamma(x)$ and add BCs, this problem is for an infinitely long string!.

We let $\xi = x + \alpha t$, $t = x - \alpha t$ and this converts the PDE to

$$\hat{u}_{\xi t} = 0$$

We can integrate this:

$$\hat{U}(\xi, t) = F(\xi) + G(t)$$

That is,

$$u(x, t) = F(x + \alpha t) + G(x - \alpha t)$$

Enforce the ICs:

$$u(x, 0) = \gamma(x) \implies \gamma(x) = F(x) + G(x)$$

$$u_t(x, 0) = 0 \implies 0 = \alpha F'(x) - \alpha G'(x)$$

Now

$$\gamma'(x) = F'(x) + G'(x)$$

Therefore

$$F'(x) = G'(x) = \frac{1}{2}\gamma'(x)$$

Therefore,

$$F(x) = \frac{1}{2}\gamma'(x) + c_1$$

$$G(x) = \frac{1}{2}\gamma'(x) + c_2$$

$$F + G = \gamma$$

so

$$c_1 + c_2 = 0$$

Finally, then $u(x,t) = \frac{1}{2}[\gamma(x+\alpha t) + \gamma(x - \alpha t)]$

We have $u(x, \frac{2}{\alpha}) = \frac{1}{2}[\gamma(x+2) + \gamma(x-2)]$

This method (method of characteristics) worlds on many hyperbolic equations (but not all, consider $u_{xy} + u_x + u_y = 1$); on some parabolic equations (e.g. $U_{xx} + U_x = 1$) but not on the heat equation

$$u_t = \gamma u_{xx}$$

; on no elliptic equations.

8.8 Separation of Variables

possibly the most powerful method in general we assume that $u(x,y)$ can be factorized as $u = F(x)G(y)$

Example: Solving the heat equation.

Consider the initial/boundary value problem

$$U_t = \gamma U_{xx}$$

$$u(x,0) = 20\sin(3\pi x)$$

$$u(0,t) = 0, u(L,t) = 0$$

Consider a bar of length L, with its ends fixed in ice, and initial temperature profile.

We assume that $u(x,t) = F(x)G(t)$. Then

$$u_t = F(x)G'(t)$$

$$u_{xx} = F''(x)G(t)$$

so

$$u_t = \gamma u_{xx} \implies F(x)G'(t) = \gamma F''(x)G(t)$$

Next separate the variables:

$$\frac{G'(t)}{\gamma G(t)} = \frac{F''(x)}{F(x)}$$

The key realization: For this equation to hold true, for all x and all y, both sides must equal a constant.

$$\frac{G'(t)}{\gamma G(t)} = \frac{F''(x)}{F(x)} = c, c \in \mathbb{R}$$

This gives two ODEs:

$$G'(t) = c\gamma G(t)$$
$$F''(x) = cF(x)$$

We will start with the second one. We need side conditions: $u(0,t) = 0 \implies F(0)G(t) = 0$

Also, $u(1,t) = 0 \implies F(1)G(t) = 0 \implies F(1) = 0$. We have a BVP for $F(x)$:

$$F''(x) - cF(x) = 0, F(0) = F(1) = 0$$

You can verify that nontrivial solutions only exist if $c < 0$, so let $c = -\lambda^2$.

Hence $F'' + \lambda^2 F = 0, F(0) = F(1) = 0$ and then $F(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$. Hence from $F(0) = 0, F(1) = 0$, we know, $c_1 = 0, c_2 \sin \lambda = 0 \implies \lambda = n\pi \; (c = -n^2\pi^2)$

so nontrivial solution exist only when $c = -n^2\pi^2$, and these solution are

$$F_n(x) = C_n \sin n\pi x$$

Next, recall that $G'(t) = c\gamma G(t)$ so $G'(t) = -n^2\pi^2\gamma G(t)$ Hence

$$G_n(t) = A_m e^{-n^2\pi^2\gamma t}$$

Finally, then, we have

$$u_n(x,t) = F_n(x)G_n(t) = B_n e^{-n^2\pi^2\gamma t} \sin(n\pi x)$$

The is a solution to the equation and BG for each value of n. With our initial condition $u(x,0) = 20 \sin 3\pi x$, we have $n = 3$ and $B_3 = 20$ so $u(x,t) = 20e^{-9\pi^2\gamma t} \sin 3\pi x$

Question: What if we have some other initial condition? First, note that $u_t = \gamma u_{xx}$ is linear (end homogeneous $_{xx} - u_t = 0$). So the principle of superposition applies: any linear combination of the function $e^{-n^2\pi^2\gamma t} \sin n\pi x$ will be a solution.

In fact, the infinite sum

$$\sum_{n=1}^{\infty} B_n e^{-n^2\pi^2\gamma t} \sin n\pi x$$

will be a solution, if it converges. Applying, the K, $u(x,0) = f(x)$ gives

$$f(x) = \sum_{n=1}^{\infty} B_n \sin n\pi x$$

It is possible to expand any piecewise-defined function with a finite domain into a series of this type! This is the Fourier Sine Series of $f(x)$. Find the B_n's, and you have the solution for u.

For other boundary conditions, we might read the Fourier Cosine Series instead:

$$f(x) = \sum_{n=1}^{\infty} A_n \cos n\pi x$$

On an interval $[0,L]$, this would be $\sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}$

- We might instead need the Full Fourier Series.

$$f(x) = \sum_{n=1}^{\infty} [A_n \cos n\pi x + B_n \sin n\pi x]$$

- If we have an infinitely long bar, we end up with a Fourier Integral instead:

$$f(x) = \int_0^{\infty} [a(w) \cos wx + b(w) \sin wx] dx$$

which can be expressed in complex form as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(w) e^{iwx} dx$$

There are formulas for all of the coefficients, $A_n, B_n, a(w), b(w), c(w)$. Specifically,

$$c(w) = \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

8.9 The Fourier Transform

Definition. *The Fourier Transform of a function $f : \mathbb{R} \to \mathbb{R}$ is defined as*

$$F(f(x)) = \int_{-\infty}^{\infty} f(x) e^{iwx} dx$$

provided that this integer converges. We will also use the notation $\hat{f}(w)$.

Example:

Find $F(f(x))$, if $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-ax} & \text{if } x \geq 0 \end{cases}$ where $a > 0$.

Solution: $F(f(x)) = \int_{-\infty}^{\infty} f(x) e^{iwx} dx = \int_0^{\infty} e^{-ax} e^{-iwx} dx = \int_0^{\infty} e^{-(a+iw)x} dx = \frac{1}{a+iw}$

Comment: The FT is just one of many integral transforms. The expression

$$\hat{f}(w) = \int_{\alpha}^{\beta} f(x) K(w, x) dx$$

The function $L(w, x)$ is called the kernel of the transformation.

Note: This often fails to exist!

For example: If $f(x) = 1$, then

$$F\{f\} = F\{1\} = \int_{-\infty}^{\infty} e^{-iwx} dx = \int_0^{\infty} e^{-iwx} dx + \int_{-\infty}^{0} e^{-iwx} dx$$

Above does not exist.

Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$. If*

1. *f is piecewise continuous on $\mathbb{R}$, and*
2. *f is absolutely integerable on $\mathbb{R}$. (That means $\int_{-\infty}^{\infty} |f(x)|dx$ converges)*

Then $\hat{f}(w)$ exists.

Proof. First note that $\int_{-\infty}^{-\infty} |f(x)e^{-iwt}|dx = \int_{-\infty}^{\infty} |f(x)|dx$ since $|e^{-iwt}| = 1$. If this converges, then we can apply the triangle inequality:

$$|\int_{-\infty}^{\infty} f(x)e^{-iwx}dx| \leq \int_{-\infty}^{\infty} |f(x)e^{-iwx}|dx$$

then it converges. □

Note that for f to be absolutely integrable we must have $\lim_{x\to\pm\infty} f(x) = 0$, we must have $\lim_{x\to\pm x} f(x) = 0$. In fact, $x^n, \sin x, \cos x, e^x, \ln x$, etc, do not have Fourier Transforms.

Theorem. *(Linearity) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Let $\alpha, \beta \in \mathbb{R}$. If $\hat{f}$ and $\hat{g}$ exist, then $F\{\alpha f + \beta g\} = \alpha\hat{f} + \beta\hat{g}$.*

Proof. By definition, it can be written as an integral form so by the linearity of integration, it is true. □

Theorem. *If f is continuous and absolutely integrable, then the Fourier transform is invertible and in fact*

$$f(x) = F^{-1}\{\hat{f}(w)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw$$

(this is the formula for the Fourier Integral representation of $f(x)$!)

Theorem. *Let f be differentiable on $\mathbb{R}$, with transform $\hat{f}(w)$, Then $F\{f'(x)\} = iw\hat{f}(w)$.*

Proof. $F\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)e^{-iwx}dx = f(x)e^{-iwt}|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} iwf(x)e^{-iwx}dx$

Now since $\hat{f}(w)$ exists, $\int_{-\infty}^{\infty} f(x)e^{-iwx}dx$ converges, so $\lim_{x\to\pm\infty} f(x)e^{-iwx} = 0$ so $F\{f'(x)\} = iw\int_{-\infty}^{\infty} f(x)e^{-iwx}dx = iw\hat{f}(w)$ □

In general, $F\{f(x)\} = (iw)^n\hat{f}(w)$ and we can even say that $F\{\int_a^x f(t)dt\} = \frac{1}{iw}\hat{f}(w)$. These results suggest that the F.T. can convert ODEs to algebraic equations.

8.9.1 Demonstration of Concept

Suppose $y'' + ay' + by = f(x)$. Apply F.T.,

$$F\{y'' + ay' + by\} = \hat{f}(w)$$

Hence $F\{y''\} + aF\{y'\} + bF\{y\} = \hat{f}(w)$. Therefore, $-w^2\hat{y} + aiw\hat{y} + b\hat{y} = \hat{f}(w)$. Lastly, $\hat{y} = \frac{-\hat{f}(w)}{w^2 - iaw - b}$. That implies $y = F^{-1}\{\frac{-\hat{f}(w)}{w^2 - iaw - b}\}$.

Theorem. *The Time-Shifting Property: Let $f(t)$ be continuous and absolutely integrable with $F\{f(t)\} = \hat{f}(w)$. Then $F\{f(t-a)\} = \hat{f}(w)e^{-iwa}$ and so $F^{-1}\{\hat{f}(w)e^{-iwa}\} = f(t-a)$*

Proof. $F\{f(t-a)\} = \int_{-\infty}^{\infty} f(t-a)e^{-iwt}dt = \int_{-\infty}^{\infty} f(\tau)e^{-iw(\tau+a)}dt = e^{-iwa}\hat{f}(w)$ □

Example: Consider the "unidirectional wave equation",

$$u_t + \alpha u_x = 0$$

The strategy is to eliminate the x derivative by using Fourier Transform in the x variable. That is, we define $F\{u(x,t)\} = \hat{u}(w,t) = \int_{-\infty}^{\infty} u(x,t)e^{-iwx}dx$

With this, $F\{u_x\} = iw\hat{u}$. Meanwhile,

$$F\{u_t\} = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}e^{-iwx}dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} ue^{iwx}dx = \hat{u}_t$$

Apply f to the PDE:

$$u_t + \alpha_x = 0$$

$$\implies F\{u_t + \alpha u_x\} = F\{0\}$$

$$\implies \hat{u}_t + \alpha(iw)\hat{u} = 0$$

This is essentially an ODE for $\hat{u}(t)$: the solution is

$$\hat{u}(w,t) = \hat{G}(w)e^{-iwt}$$

Now we just need the inverse transform:

$$u(x,t) = F^{-1}\{\hat{G}(w)e^{-i\alpha wt}\} = G(x - \alpha t)$$

by the time shifting properties.

Note:

1. We've assumed that $\hat{u}$ exists, which probably isn't true for this PDE! (We need $u \implies 0$ as $x \to \pm\infty$). Nevertheless, we've got the correct result.

2. Evaluating F^{-1} is usually harder. We will need this.

   Definition: The convolution of two functions f and g is $f \star g(x) = \int_{-\infty}^{\infty} f(x-\tau)g(\tau)d\tau$. Note this won't always exist, but a sufficient condition is that f and g be causal, meaning that $f(x) = 0$ when $x < 0$.

   Like multiplication, convolution is commutative and distributive over addition.

$$f \star g = g \star f$$

   where $(\alpha f + \beta g) \star h = \alpha(f \star h) + \beta(g \star h))$

Example: Let $f(x) = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases}$ and let $g(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x \geq 0 \end{cases}$. Then $f \star g(x) = \int_{-\infty}^{\infty} f(x-\tau)g(\tau)d\tau = \int_{0}^{\infty} f(x-\tau)e^{-\tau}d\tau = \int_{0}^{x}(x-t)e^{-\tau}d\tau = -(x-\tau)e^{-\tau}|_{0}^{x} - \int_{0}^{x} e^{-\tau}d\tau = x + e^{-x} - 1$

8.9.2 The Convolution Theorem

If f, g are piecewise continuous and absolute integrable,

$$F\{f \star g(x)\} = \hat{f}(w)\hat{g}(w)$$

so

$$F^{-1}\{\hat{f}(w)\hat{g}(w)\} = f \star g(x)$$

Proof. $F\{f \star g(x)\} = \int_{-\infty}^{\infty} f \star g(x)e^{-iwx}dx = \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} f(x-\tau)g(\tau)d\tau]e^{-iwx}dx = \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} f(x-t)e^{-iwx}dx]g(\tau)d\tau$ (reversing order of integration)

Let $\xi = x - t$ so $d\xi = dx$.

Therefore $= \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} f(\xi)e^{-iw(\xi+\tau)}d\xi]g(\tau)d\tau = \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} f(\xi)e^{-iw\xi}d\xi]e^{-iw\tau}g(\tau)d\tau = [\int_{-\infty}^{\infty} f(\xi)e^{-iw\xi}d\xi][\int_{-\infty}^{\infty} g(\tau)e^{-iw\tau}dt] = \hat{f}(w)\hat{g}(w)$ □

8.10 Solution of the Heat Equation for an Infinitely-Long Bar

$$u_t = \gamma u_{xx}, (\gamma > 0, x \geq 0)$$

$$u(x, 0) = f(x)$$

$$u \to 0 \text{ as } x \to \pm\infty$$

1. Since the BCs are at ∞, separation of variables does not work here.
2. However, if $\hat{f}(w)$ exists, then it is reasonable to assume that $\hat{u}(w, t)$ will exist, for all t.

So, let $\hat{u}(w,t) = F\{u(xt)\} = \int_{-\infty}^{\infty} u(x,t)e^{-iwx}dx$. Therefore $u_t \to \hat{u}_t, u_{xx} \to -w^2\hat{u}$. Also, the IC $u(x,0) = f(x)$ implies that $\hat{u}(w,0) = \hat{f}(w)$. (we've already used the BCs in assuming that $\hat{u}$ exists). This implies $\hat{u}_t = -\gamma w^2 \hat{u}, \hat{u}(w,0) = \hat{f}(w)$. Therefore, $\hat{u}(w,t) = H(w)e^{-w^2\gamma t}$. Enforce the IC: $\hat{u}(x,0) = \hat{f}(w) \implies H(w) = \hat{f}(w)$, so $\hat{u}(w,t) = \hat{f}(w)e^{-w^2\gamma t}$

Now, $u(x,t) = F^{-1}\{\hat{f}(w)e^{-w^2\gamma t}\} = F^{-1}\{\hat{f}(w)\} \star F^{-1}\{e^{-w^2\gamma t}\} = f(x) \star F^{-1}\{e^{-w^2\gamma t}\}$

Use the formula for F^{-1}:

$$\begin{aligned}
F^{-1}\{e^{-w^2\gamma t}\} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-w^2\gamma t}e^{iwx}dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(\gamma t w^2 - ixw)}dw \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\gamma t[w^2 - \frac{ix}{\gamma t}w]}dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\gamma t[(w-\frac{ix}{2\gamma t})^2 + \frac{x^2}{4\gamma^2 t^2}]}dw \\
&= \frac{1}{2\pi}e^{-\frac{x^2}{4\gamma t}}\int_{-\infty}^{\infty} e^{-\gamma t(w-\frac{ix}{2\gamma t})^2}dw \text{ Let } v = \sqrt{\gamma t}(w - \frac{ix}{2\gamma t}) \\
&= \frac{1}{2\pi}e^{-\frac{x^2}{4\gamma t}}\int_{-\infty}^{\infty} e^{-v^2}\frac{dv}{\sqrt{\gamma t}} \\
&= \frac{1}{2\pi\sqrt{\gamma t}}e^{-\frac{x^2}{4\gamma t}}\int_{-\infty}^{\infty} e^{-v^2}dv
\end{aligned}$$

Now $\int_{-\infty}^{\infty} e^{-v^2}dv$ is know to be $\sqrt{\pi}$ so this is $\frac{1}{2\sqrt{\pi\gamma t}}e^{-x^2/4\gamma t}$. You can find the proof on my website.

Finally, $u(x,t) = f(x) \star \frac{1}{2\sqrt{\pi\gamma t}}e^{-x^2/4\gamma t} = \frac{1}{2\sqrt{\pi\gamma t}}\int_{-\infty}^{\infty} f(x-t)e^{-t^2/4\gamma t}dt$

We can simplify this a bit: Let $y = \frac{-\tau}{2\sqrt{\gamma t}}$ so $dy = \frac{-dt}{2\sqrt{\gamma t}}$ and $y \to -\infty$ as $\tau \to \infty$. Therefore

$$u(x,t) = \frac{1}{2\sqrt{\pi\gamma t}}\int_{-\infty}^{\infty} f(x + 2\sqrt{\gamma t}y)e^{-y^2}(-2\sqrt{\gamma t}dy) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2\sqrt{\gamma t}y)e^{-y^2}dy$$

Example: Suppose $f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$. Then $f(x + 2\sqrt{\gamma t}y) = \begin{cases} 1 & |x + 2\sqrt{\gamma t}y| < 1 \\ 0 & O.W. \end{cases}$

Then $u(x,t) = \frac{1}{\sqrt{\pi}}\int_{\frac{-1-x}{2\sqrt{\gamma t}}}^{\frac{1-x}{2\sqrt{\gamma t}}} e^{-y^2}dy = \frac{1}{\sqrt{\pi}}[\int_0^{\frac{1-x}{2\sqrt{\gamma t}}} e^{-y^2}dy - \int_0^{\frac{-1-x}{2\sqrt{\gamma t}}} e^{-y^2}dy] = \frac{1}{2}[erf(\frac{1-x}{2\sqrt{\gamma t}}) - erf(\frac{-1-x}{2\sqrt{\gamma t}})]$ (could also write this in term of $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{x^2}{2}}dx$)

8.11 The Black-Scholes PDE

A European call option is the right to buy a commodity for an agreed-upon price K at an agreed-upon time T. The value of an option depends on the price of the stock, S, but it also depends explicitly on t. (more naturally it depends on the time remaining before the strike time, $T - t$). We will denote the value of the option as $F = (S,t)$. Our mathematical model will depend on several significant assumptions.

1. We will assume that trading is continuous and that assets are infinitely divisible.
2. We will ignore transaction costs.
3. We permit "short selling"
4. We will assume that our assets pay no dividend
5. Two more we will need soon.
   (a) We will assume that it is always possible to invest money at a constant interest rate r.
   (b) We will assume that no arbitrage exists. That is, the values of financial instruments will automatically (and instantaneously) adjust to put them in balance with investments at rate r.

Stock price:
S itself depends on time: $S = S(t)$. Furthermore, this dependence is partly random.

Change in stock price in a given time interval made up with two parts, deterministic growth/decay and a random part. That is $\Delta S = f_{\text{deterministic}} + f_{\text{random}}$. We will assume f_{det} to be Malthusian:

$$f_{\text{det}} = \mu S \Delta t$$

, so that without the random component we'd have $\Delta S = \mu S \Delta t$. Therefore $S = Ce^{\mu t}$.

We will assume f_{random} to be stochastic:

$$f_{\text{random}} = \sigma S \Delta W(t)$$

where $\Delta W(t)$ is a random variable with mean 0 and variance Δt (Wiener Process) so $\Delta S = \mu S \Delta t + \sigma S \Delta w(t)$. We call u the growth rate and σ is the volatility.

Customarily this is written (letting $\Delta t \to 0$). Hence

$$dS = \mu S dt + \sigma S dW(t)$$

(this is a stochastically differential equation).

Assume that the value of the option is $F(S, t)$. Assume that the value of the stock is $S(t)$, satisfying

$$dS = uSdt + \sigma S dW(t)$$

(i.e. $\Delta S = \mu S \Delta t + \sigma S \Delta W(t)$) where $\Delta W(t)$ is a random variable with a time-dependent probability density function such that the mean is 0 and the variance is Δt. It an be shown that $[\Delta W(t)]^2$ has mean Δt and variance $2(\Delta t)^2$. Now recall that a function $f(x, y)$ cam be expanded in a Taylor series as $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0 + \cdots$

Assuming that $F(S,t)$ is smooth, then we can take an arbitrary point (S_0, t_0) and write

$$\Delta F = \frac{\partial F}{\partial S}\Delta S + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\Delta S^2 + \frac{\partial^2 F}{\partial S \partial t}\Delta S \Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}\Delta t^2 + \cdots$$

Now since $\Delta S = \mu S \Delta t + \sigma S \Delta W(t)$. $\Delta F = F_S \mu S \Delta t + F_S \sigma S \Delta W(t) + F_t \Delta t + \frac{1}{2} F_{SS} \mu^2 (\Delta t)^2 + F_{SS} \mu \sigma^2 \Delta t \Delta W(t) + \frac{1}{2} F_{SS} \sigma^2 S^2 [\Delta W(t)]^2 + F_{St} \mu S (\Delta t)^2 + F_{St} \sigma S \Delta t \Delta W(t) + \frac{1}{2} F_{tt} (\Delta t)^2 + \cdots$

Next, if Δt is small, then we neglect terms of order $(\Delta t)^2$. Furthermore, since $\Delta W(t)$ has mean 0 and variance Δt, we can also neglect, the term of order $\Delta t \Delta W(t)$. Also since $[\Delta W(t)]^2$ has mean Δt and variance $2(\Delta t)^2$, we can say that $[\Delta W(t)]^2 \approx \Delta t$, and so

$$\Delta F \approx F_S \mu S \Delta t + F_S \sigma S \Delta W(t) + F_t \Delta t + \frac{1}{2} F_{SS} \sigma^2 S^2 \Delta t$$

8.11.1 Eliminating the Stochastic Term

The key step: if we assume that arbitrage is always instantly eliminated then it must be possible to cancel out the random contribution to F. Here is the Black-Scholes argument:

Consider a portfolio constructed by buying one option and selling ϵ unit of stocks. (assuming short-selling and divisible stocks!)

The portfolio has value

$$\pi(t) = F - \epsilon - \epsilon S$$

Over the interval Δt, the change in π is $\Delta \pi = \Delta F - \epsilon \Delta S = [\sigma S F_S - \epsilon \sigma S]\Delta W(t) + [F_t + \mu S F_S + \frac{1}{2}\sigma^2 F_{SS} - \epsilon \mu S]\Delta t$

Now, if we set $\epsilon = F_S$, then the stochastic component disappears. We are left with

$$\Delta \pi = [F_t + \frac{1}{2}\sigma^2 S^2 F_{SS}]\Delta t$$

so $\frac{d\pi}{dt} = F_t + \frac{1}{2}\sigma^2 S^2 F_{SS}$

Finally, if arbitrage cannot exist, then the value of π must equal the value of a simple investment at interest rate r, so $\frac{d\pi}{dt} = r\pi$

Hence $r\pi = F_t + \frac{1}{2}\sigma^2 S^2 F_{SS}$ so

$$\frac{1}{2}\sigma^2 S^2 F_{SS} + rSF_S + F_t - rF = 0$$

Hence we get our Blakc-Scholes equation for European call option.