ACTSC 446: Mathematical Economics Model

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Contents

Assignments, Midterm (March 4th) and Final are 10%, 25% and 65% respectively.

1 Introduction

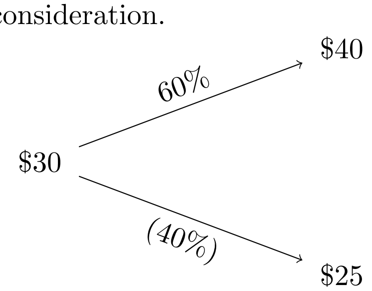
Pricing of financial derivatives mainly invokes finding the expected present value of future cash flows under a "risk neutral" probability measure. The expected present value concept is not new to us; we require three ingredients: a series of cash flows, the corresponding discount factors and probabilities. Given a series of known/estimated cash flows, the question that now arises is how do we determine the appropriate discount factors and probabilities?

$$\text{Price} = EPV = \text{Expected Present Value} = \frac{40(0.6) + 25(0.4)}{1 + \text{interest rate}}$$

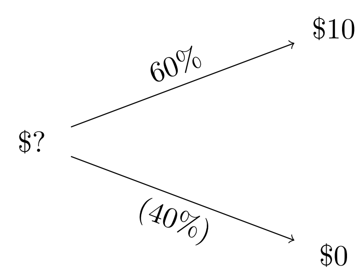
Since we have a stock models such as the CAPM or APT can be used to estimate this interest rate. Interest rate is of the form:

$$\text{Risk-free rate} + \text{Risk Premium}$$

where risk-free rate is the base rate and risk premium is more specific to the stock/asset in consideration.



Consider now a call option with time to maturity of 1 year and a strike of $30. Therefore we can transform our binomial tree into the following tree:



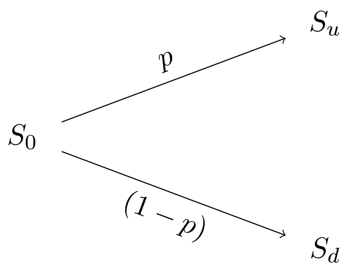
Therefore, the price is going to be

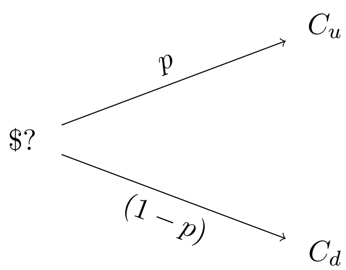
$$\text{Price} = \frac{10(0.6) + 0(0.4)}{1 + \text{desired return on the call option}}$$

Do not have an easy way to find the desired return on the call option. This method involves finding the expected cash-flow and then discounting it using appropriate rate of return, one that is adjusted for risk!.

1.1 Another Method (Risk-Neural Pricing)

Use probabilities that are "adjusted" for risk, and the risk-free rate as our discount rate. These "adjusted probabilities" are called risk-neutral probabilities. It is very important to note that the risk-neutral probabilities are not real-world probabilities. Think of an alternate world.dimension where all investors are risk-neutral and as a result, all assets yield the risk-free rate; yet, the prices of the assets are the same as what we observe in this world. Assume arbitrage is not possible. Suppose the following:





Idea:

1. Create a portfolio that mimics the payoffs of the call at time. No arbitrage. This implies price of the call if the price of the portfolio.

2. Assume that we can invest/borrow at the risk-free rate r.

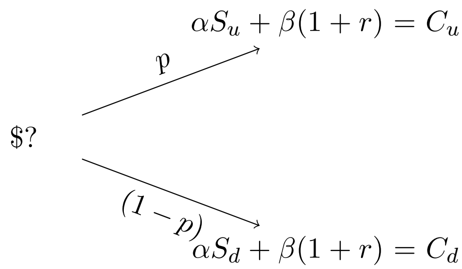
3. Let

$$\alpha = \text{number of share s of stock purchased at } t = 0$$

   and

$$\beta = \text{amount invested at } t = 0 \text{ at r}$$

Hence



Therefore, we can conclude that

$$\alpha = \frac{C_u - C_d}{S_u - S_d}$$

and

$$\beta = \frac{1}{1+r}[C_u - \frac{C_u - C_d}{S_u - S_d} S_u]$$

Therefore, the call price is just replicating portfolio price, that is,

$$\alpha S_0 + \beta = \frac{1}{1+r}[\frac{(1+r)S_0 - S_d)}{S_u - S_d} C_u + \frac{S_u - (1+r)S_0}{S_u - S_d} C_d]$$

Therefore, it is

$$\frac{1}{1+r}[qC_u + (1-q)C_d]$$

If $S_d - (1+r)S_0 < S_u$, then $0 < q < 1$. Here, q and $1-q$ are called risk-neutral probabilities and our discount rate is our risk-free rate of return r so we have the price as EPV under this risk-neutral probability measure.

$$S_d < S_0(1+r) < S_u \implies 0 < q < 1$$

Therefore,

$$\frac{S_d - S_0}{S_0} < r < \frac{S_u - S_0}{S_0}$$

1.2 Derivatives

With respect to options,

1. long position $\implies$ buy the option and
2. short position $\implies$ sell the option (write)

With respect to stock,

1. long position in stocks $\implies$ buy shares of stock and have ownership in a company

2. Short-sell stock - borrow stock (from a broker), sell it on the market and promise to return it at some point in the future.
   (a) Will short-sell stock if you expect the price to fall in the future
   (b) If any dividends are paid over this period of time, you have to pay it to the original lender of the stock.

1.3 Forwards and Futures Contracts

It is an obligation to buy/sell some underlying asset at some point in the future (the expiration date, at a price determined today (forward price)

The payoff on a forward contract is

$$\text{payoff} = \begin{cases} \text{spot price at expiration-Forward price, long position} \\ \text{Forward price - spot price at expiration, short position} \end{cases}$$

Since the initial premium is zero, the profit on a forward contract equals its payoff.

1.4 Call and Put options

A call (put) option gives the owner the right (but not the obligation) to buy (sell) an underlying asset at or before a pre-specified time (maturity or expiration date) for a price set today (called the strike or exercise price)

Types

- European - only able to exercise at the expiration date
- American - able to exercise at any time before expiration
- Bermudan - able to exercise during specified periods.

Set-up

- Call: option to buy
- Put: option to sell
- Let $S_T =$ spot price at expiration, $K =$ strike/exercise price
- Payoff of a purchased call option (long a call) $= \max(S_T - K, 0)$

$$\text{Profit} = \text{Payoff at Expiration} - \text{Future value of Premium}$$

- Payoff of a purchased put option (long a put) $= \max(K - S_T, 0)$.

$$\text{Profit} = \text{Payoff at Expiration} - \text{Future value of Premium}$$

- Payoff of a Written call option (short a call) $= -\max(S_T - K, 0)$

$$\text{Profit} = \text{Payoff at Expiration} + \text{Future value of Premium}$$

- Payoff of a Written put option (short a put) $= -\max(K - S_T, 0)$.

$$\text{Profit} = \text{Payoff at Expiration} + \text{Future value of Premium}$$

Examples (uses of derivatives)

1. Speculating on volatility: Straddle: buy a put and a call with some strike price. Make this investment if you believe that the stock is very volatile.
2. Floor: provides insurance against a fall in price.

$$\text{Floor} = \text{Long Stock} + \text{Long Put}$$

3. Cap: provider insurance against a rise in the price

$$\text{Cap} = \text{Long Call} + \text{Short Stock}$$

1.5 Financial Forwards and Futures Contracts

For the rest of this chapter, we will assume the price of the futures contracts is the same asa that of a forward contract.

1. Consider the purchase/sale of a stock for possible ways
   (a) Outright purchase: Pay S_0 for the stock today and receive it at time 0.
   (b) Fully leveraged purchase. Borrow S_0 at a risk-free rate r and purchase the stock today. Pay off the loan with interest at some time T, an amount equal to S_0e^{rT} (assuming r is continuously compounded)
   (c) Prepaid forward contract. Pay for the stock today and receive it at time T.
   (d) Forward contract: pay for the stock and receive it at time T, at a price set today.

   Interested in finding the price of c) and d). Notation:

   (a) r is risk-free interest rate (continuously compounded)
   (b) T is the expiration date
   (c) $F_{0,T}$ is the price of a T-year forward contract.
   (d) $F_{0,T}^P$ is the price of a T-year prepaid forward contract.

(e) σ = risk-adjusted interest rate. (continuously compounded)

2. Prepaid Forward (stock pays no dividends)

   (a) Pricing model 1: Pays to dividends so it doesn't matter. When the stock is delivered to the buyer (could be at time T, at any time in $(0, T)$, or even at time 0, it doesn't matter. This implies we should have $F^P_{0,T} = S_0$ (stock price at time 0)

   (b) Pricing method 2: Expected present value

   $$F^p_{0,T} = e^{-\alpha T} E_0(S_T)$$

   use risk-adjusted rate here. Now what is $E_0(S_T)$?

   $$E_0(S_T) = S_0 e^{\alpha T}$$

   Think of as the yield on the stock

   $$\Longrightarrow F^P_{0,T} = e^{-\alpha T}[S_0 e^{\alpha T}] = S_0$$

   (c) Pricing Method 3: pricing by arbitrage (risk-free profit).

   $$\text{portfolio} = \begin{cases} \text{buy a share of stock at } S_0 \\ \text{Sell a prepaid forward at} F^P_{0,T} \end{cases}$$

   i. Buy Stock: Cash flows: time 0: S_0 and time 1: S_T. S_t = time-t price of 1 share of stock.
   ii. Sell prepaid forward: time 0: $F^P_{0,T}$ and time 1: $-S_T$.

   Net: time 0: $F^P_{0,T} - S_0$ and time 1: 0

   No arbitrage implies $F^P_{0,T} - S_0 = 0$. Then $F^P_{0,T} = S_0$.

3. Prepaid Forward contract (stock pays dividends)

   In this case, we have

   $$F^P_{0,T} = S_0 - PV(\text{all dividends paid over (0, T)})$$

   If the stocks pays discrete dividends of D_{t_j} at time $t_j, j = 1, 2, \cdots, n$, where $t_j < T$, then

   $$F^P_{0,T} = S_0 - \sum_{j=1}^{n} PV_{0,t_j}(D_{t_j})$$

   Suppose the stock pays dividend at an annualized continuously compounded dividend yield of δ. Then, $F^P_{0,T} = S_0 e^{-\delta T}$.

At time 0, buy 1 share of stock. Then at the end of day 1. Receive a dividend equal to $\frac{\delta}{365}$(day 1 stock price). Reinvest interest from the stock $(1+\frac{\delta}{365}$ shares of stock. Therefore, at the end of year 1, have $(1+\frac{\delta}{365})^{365}$ shares of stock. At the end of year T, it turns into $(1+\frac{\delta}{365})^{365T} = e^{\delta T}$ shares of stock.

No arbitrage argument for pricing.

$$\text{portfolio}: \begin{cases} \text{buy } e^{-\delta T} \text{ share of stock and continuously reinvest dividends back into the stock} \\ \text{sell a prepaid forward at } F^P_{0,T} \end{cases}$$

Consider the following cash flows

(a) buy stock: time 0: $-e^{-\delta T}S_0$, time T: S_T

(b) sell prepaid forward: time 0: $F^P_{0,T}$, time T: $-S_T$

Net cash flow: time 0: $F^P_{0,T} - S_0 e^{-\delta T}$, time 1: 0.

Example 1: $S_0 = 100$, $D_{t_j} = \$1$, $r = 10\% \implies F^P_{0,1} = 100 = [\$1e^{-10\%\frac{1}{4}} + \$1e^{-10\%\frac{1}{2}} + \$1e^{-10\%\frac{3}{4}} + \$1e^{-10\%1}] = 96.2409$.

Example 2: $F^P_{0,1} = S_0 e^{-\delta \cdot 1} = 95.1229$

$$F^P_{0,T} = S_0 - PV(\text{dividends})$$

$$\text{Discrete } F^P_{0,T} = S_0 - \sum_{j=1}^{n} PV_{0,t_j}(D_{t_j})$$

Continuous: annualized continuously compounded dividend yield of δ.

$$\begin{aligned} F^P_{0,T} &= S_0 - PV(S_T(e^{\delta T}-1)) = S_0 - (e^{\delta T}-1)PV(s_T) \\ &= S_0 - S_0 e^{-\delta T}(e^{\delta T}-1) = S_0 e^{-\delta T} \end{aligned}$$

1.6 Pricing Forwards on Stocks

Assume a continuously compounded risk-free rate of borrowing/lending equal to r.

$$F_{0,T} = FV(F^P_{0,T}) = F^P_{0,T} e^{rT}$$

Cases:

1. No dividends

$$F_{0,T} = S_0 e^{rT}$$

2. Discrete dividends:

$$F_{0,T} = S_0 e^{rT} - \sum_{i=1}^{n} e^{-r(T-t_j)} D_{t_j}$$

3. Continuous dividends:

$$F_{0,T} = S_0 e^{-\delta T} e^{rT} = S_0 e^{(r-\delta)T}$$

Example: Let F_1 and F_2 be the forward prices of the same underlying stock with time to maturity T_1 and T_2 respectively, where $T_1 < T_2$ and let r be the annual continuously compounded risk-free interest rate. If $F_2 > F_1 e^{r(T_2-T_1)}$, then an arbitrage opportunity exists . Provide one.

Cashflows

Transactions	Time 0	Time T_1	Time T_2
Short T_2 year forward	0	0	$F_{0,T_1} - S_{T_2}$
Long forward T_1 year forward	0	$S_{T_1} - F_{0,T_1}$	0
buy one share of stock at time T_1	0	$-S_{T_1}$	S_{T_2}
borrow F_{0,T_1} at r at time T_1	0	F_{0,T_1}	$F_{0,T_1} e^{r(T_2-T-1)}$
Net Cash flow	0	0	$F_{0,T_2} - F_{0,T_1} e^{r(T_2-T_1)} > 0$

4. Continuous dividends:

$$F_{0,T} = S_0 e^{-\delta T} e^{rT} = S_0 e^{(r-\delta)T}$$

- Sometimes, you may be given the forward premium, which is defined as the ratio of current forward price to the current stock price, $\frac{F_{0,T}}{S_0}$. If you are given the forward premium and the forward price, you can figure out the current stock price. Sometimes, you may be given the annualized forward premium which is calculated as

$$\frac{1}{T} \ln\left(\frac{F_{0,T}}{S_0}\right)$$

- Synthetic Forwards:

  –

$$\text{Long forward} = S_T - F_{0,T}$$

$$\text{Forward} = \text{Stock} - \text{Bond}$$

$$\text{No dividends}, F_{0,T} = S_0 e^{rT}$$

Example 4: How would you create a long synthetic forward for a stock paying continuous dividends? Show that the payoff of the synthetic forward is the same as that of a long forward.

$$\text{Long forward} = S_T - F_{0,T} = S_T - S_0 e^{(r-\delta)T}$$

	Time 0	Time T
Buy $e^{-\delta T}$ share of stocks	$S_0e^{-\delta T}$	S_T
Borrow S_0e^{-rT} at the risk-free rate	$S_0e^{-\delta T}$	-$S_0e^{(r-\delta)T}$
	0	$S_T - S_0e^{(r-\delta)T}$

1.7 Currency Contracts

$$S_0e^{-\delta T} \implies S_0e^{(r-\delta)T}$$

$$F_{0,T} = x_0e^{(r_u-r_c)T}$$

2 Option: Parity and Other Relationship

$$C_0 + S_0 = P_0 + Ke^{-rT}$$

Call and Put with strike K and expiration time T

$$\text{Call Price} - \text{Put Price} = PV_{0,T}(\text{Forward Price} - \text{Strike Price})$$

$$C(K,T) - P(K,T) = PV_{0,T}(F_{0,T} - K)$$

$$C(K,T) = S_0 - PV(\text{Dividends} + K) + P(K,T)$$

Using the above equations, we can create the following synthetic securities: synthetic stock, synthetic call option, synthetic put option, synthetic T-bill

2.1 Generalized Put-Call Parity

$$C(S_t, Q_t, T) - P(S_t, Q_t, T) = F^P_{0,T}(S_t) - F^P_{0,T}(Q_t)$$

2.2 Properties of Option Prices

European option can be exercised only at the expiration date whereas an American option can be exercised at any time before expiration.

- Call option price
  1. Cannot be negative $\implies$ Call Price ≥ 0
  2. Parity equation implies Call price $\geq PV(F_{0,T}) - PV(K)$
  3. Call Price $\leq S_0$. Why? Payoff at time T is $\max\{S_T - K, 0\} \leq S_T$.
  4. $S_0 \geq C_{Amer}(S,K,T) \geq C_{Euro}(S,K,T) \geq \max(0, PV(F_{0,T}) - PV(K))$
- Put option price:
  1. Cannot be negative $\implies$ Put price ≥ 0

2. Parity equations $\implies$ Put price $\geq PV(K) - PV(F_{0,T})$
3. Put: Price $\leq K$ (or more strictly Ke^{-rT}) Why? Payoff at time T is

$$\max(K - S_T, 0) \leq K$$

$$K \geq P_{Amer}(S, K, T) \geq P_{euro}(S, K, T) \geq \max(0, PV(K) - PV(F_{0,T}))$$

Early exercise: For the American call option, at each point in time, we can

1. Hold on to the option
2. Sell it at time t for $C_{Amer}(S, K, T - t)$

   For a non-dividend paying stock, it is never optimal to exercise early.
3. Exercise at time t for $S_t - K$

For a non-dividend paying stock, it is never optimal to exercise early.

Proof. Want to show that $C_{Amer}(S, K, T - t) \geq S_t - K$. Recall the parity equation.

$$C_{Eur}(S_t, K, T-t) - P_{Eur}(S_t, K, T-t) = S_t - Ke^{-r(T-t)} = S_t - K + K(1 - e^{-r(T-t)}) \geq S_t - K$$

$$C_{Eur} \geq P_{Eur} + (S_t - K) \geq S_t - K$$

Since $C_{Amer} \geq C_{Eur}$, we have $C_{Amer} \geq S_t - K$ □

Strike price: $K_1 < K_2 < K_3$

1. $C(K_1) \geq C(K_2)$
2. $P(K_1) \leq P(K_2)$
3. $C(K_1) - C(K_2) \leq K_2 - K_1$
4. $C(K_1) - C(K_2) \leq K_2 - K_1$ and $P(K_2) - P(K_1) \leq K_2 - K_1$

(The absolute value of the slope of the option price with respect to strike is ≤ 1)
Example 5: $K_2 - K_1 = 5$. $C(50) - C(55) = 6.75$ (violated) and $P(55) - P(50) = 7.75$ (violated)

1. Sell call with $K = 50$
2. Buy call with $K = 55$.

$t=0$	$S_T<50$	$50 \le S_T < 55$	$S_T > 55$
16	0	$-(S_T-50)$	$-(S_T-50)$
-10	0	0	S_T-55
6	0	$50-S_T \ge -5$	-5

Receive \$6 at time 0 and lose at most \$5 at time T. This implies arbitrage opportunity does exist.

Profit at time T is $(\$6e^{rT})-5>0$. To make a risk-free profit

1. Sell call with strike of 50
2. Buy call with strike of 55

2.3 Convexity

$$\frac{C(K_1)-C(K_2)}{K_2-K_1} \ge \frac{C(K_2)-C(K_1)}{K_2-K_1}$$

$$\frac{P(K_2)-P(K_1)}{K_2-K_1} \le \frac{P(K_3)-P(K_2)}{K_3-K_2}$$

Based on the graph on the left, we should have

$$\frac{K_3-K_2}{K_3-K_1}C(K_1)+\frac{K_2-K_1}{K_3-K_1}C(K_3) \ge C(K_3)$$

Based on the second graph, we also have

$$\frac{K_3-K_2}{K_3-K_1}P(K_1)+\frac{K_2-K_1}{K_3-K_1}P(K_3) \ge P(K_2)$$

Remember that the option prices are convex.

Example 6:

$$\frac{K_3-K_2}{K_3-K_1}=\frac{5}{25}=0.2$$

$$0.2C(80)+0.8C(105)-8.4<9=C(100)$$

$$0.2P(80)+0.8P(105)=20.64<P(100)=21$$

1. Buy 2 calls with $K=80$
2. Buy 8 calls with $K=105$
3. Sell 10 calls with $K=100$

Time 0	$S_T<80$	$80 \le S_T < 100$	$100 \le S_T < 105$	$S_T \ge 105$
-44	0	$2(S_T-80)$	$2(S_T-80)$	$2(S_T-80)$
-40	0	0	0	$8(S_T-105)$
90	0	0	$-10(S_T-100)$	$-10(S_T-100)$
6	0	$2(S_T-80)$	$8(105-S_T) \ge 0$	0

2.4 Swaps

$$P(0,t) = \text{Price of a t-year zcb} = [1 + r(0,t)]^{-t}$$

Forward rate is the rate locked in today to borrow/lend at some time in the future $r(t, t+k)$ is the forward rate (set today) for borrowing/lending over $(t, t+k)$. Assume the swap price is level at R.

$$\text{PV of the swap obligations} = \frac{40}{1.04} + \frac{45}{1.05^2}$$

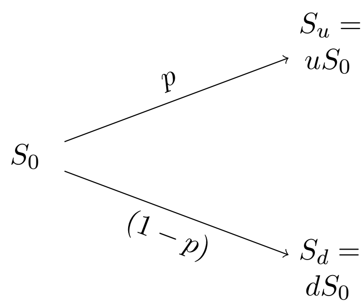
But this should equal

$$\frac{R}{1.04} + \frac{R}{1.05^2}$$

Overall, $\frac{40}{1.04} + \frac{45}{1.05^2} = \frac{R}{1.04} + \frac{R}{1.05^2} \implies R = 42.4271$

2.5 Binomial Option Pricing

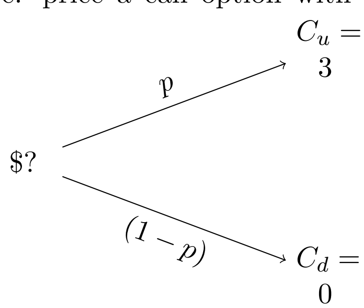
One Period Binomial Tree



where $S_0 = 30$, $S_u = 33$ and $S_d = 27$.

Assume no dividends.

Objective: price a call option with a trike of \$30 and maturity in one-year. Call's

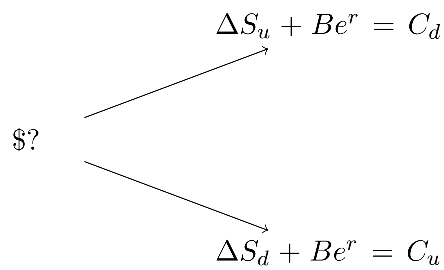


Payoffs:

Risk-free rate: $r = 5\%$ (continuously compounded). Assume you can borrow/lend at this rate. Construct a replicating portfolio. (to re-create the payoffs of the call)

1. Purchase Δ shares of stock at time 0.

2. To invest an amount of money B at risk-free rate.



3. Want to have (at time 1) Solve simultaneously to find Δ and B. This implies

$$\Delta = \frac{C_u - C_d}{S_u - S_d} = \frac{3-0}{33-27} = 0.5$$

$$B = e^{-r}[C_u - \delta S_u] = -12.8416$$

This implies

$$\begin{cases} \text{Purchase half share of stock} \\ \text{Borrow \$12.8916 at the risk-free rate} \end{cases}$$

Note: Δ is called the delta of the option. Price (based on a no-arbitrage argument) is (at time 0)

$$C_0 = \Delta S_0 + B = \frac{C_u - C_d}{S_u - S_d} S_0 + e^{-r}[C_u - \Delta S_u] = e^{-r}[\frac{e^r - d}{u-d} C_u + \frac{u - e^r}{u-d} C_d]$$

Let $q = \frac{e^r - d}{u-d}$. Then $1 - q = \frac{u-e^r}{u-d}$. Note that $0 < q < 1$ if $d < e^r < u$ (No arbitrage condition) If satisfied, q and $1-q$ are called risk-neutral probabilities.

Note: If $e^r < d$, we can invest in the stock (long) and lend at the risk-free rate to make an arbitrage profit (because we can always earn a higher return on the stock then on the bank account)

On the other hand, if $e^r > u$, then you will always earn a higher return on the bank account than on the stock.

Example

Assume that the stock pays dividends at a continuous annualized dividend yield of δ. Consider a call option which matures at time h. The payoff at time h is C_u if $S_h = uS_0$ and C_d otherwise.

1. Assume a continuously compounded risk-free rate r, show that the no-arbitrage price of this call option is

$$C_0 = \Delta S_0 + B$$

where

$$\Delta = e^{-\delta h}\frac{C_u - C_d}{S_0(u-d)} \text{ and } B = e^{-rh}\frac{uC_d - dC_u}{u-d}$$

Portfolio consists of purchasing Δ shares of stock and investing B in the bank account.

2. Determine the risk-neutral probabilities and the no-arbitrage condition. At time h

$$\begin{cases} \Delta e^{\delta h} S_u + Be^{rh} = C_u \\ \Delta e^{\delta h} S_d + Be^{rh} = C_d \end{cases}$$

Note: Δ shares of stock at time 0 will give $e^{\delta h}\Delta$ shares at time h. (assuming you reinvest the dividends received back into the stock)

Solve simultaneously to get

$$\Delta = e^{-\delta h}\frac{C_u - C_d}{S_u - S_d} = e^{-\delta h}\frac{C_u - C_d}{S_0(u-d)}$$

This implies

$$B = e^{-rh}[C_u - \Delta e^{\delta h} S_h] = e^{-rh}\frac{uC_d - dC_u}{u-d}$$

This implies

$$C_0 = \Delta S_0 + B$$

Subset: Δ and B into C_0 and rearrange to get

$$C_0 = e^{-rh}[\frac{e^{(r-\delta)h} - d}{u-d}C_u + \frac{u - e^{(r-\delta)h}}{u-d}C_d]$$

Note that $0 < q < 1$ if $d < e^{(r-\delta)h} < u$ (No-arbitrage condition).
If satisfied q and $1-q$ are risk-neutral probabilities.

2.6 Constructing the Binomial Tree

$$F^p_{t,t+h} = S_t e^{-\delta h}$$

$$F_{t,t+h} = S_t e^{(r-\delta)h}$$

Turns out that $F_{t,t+h} = E[S_{t+h}]$ under the risk-neutral probability measure.

Proof.

$$E[S_{t+h}|S_t] = quS_t + (1-q)dS_t = \frac{e^{(r-\delta)h} - d}{u-d}uS_t + \frac{u - e^{(r-\delta)h}}{u-d}dS_t = S_t e^{(r-\delta)h} = F_{t,t+h}$$

Introduce uncertainty via a volatility coefficient σ at time $t+h$: $\begin{cases} uS_t = F_{t,t+h}e^{\sigma\sqrt{h}} \\ dS_t = F_{t,t+h}e^{-\sigma\sqrt{h}} \end{cases}$

□

2.7 American Option

Replace the call option's value under the European setting (i.e. $\Delta_h S_h + B_h$) with

$$max(\text{Exercise Value at h}, \Delta_h S_h + B_h)$$

at all nodes prior to expiration (except time 0)

2.7.1 Example 15

Using the binomial model from last example, find the price on American put option with 2 year to maturity and strike price 30. The continuous compounded rate is 5%.

In this case,

$$q = \frac{e^{rh} - d}{u - d} = 0.7569$$

At time 1:

1. Stock price goes up and holding value $= [q \cdot 0 + (1-q)(0.3)]e^{-r} = 0.0695$. Exercise value is 0. This implies $P_n = \max(0.0659, 0) = 0.0695$

2. Stock price goes down: holding value $e^{-r}[q \cdot 0.3 + (1-q)5.7] = 1.5369$. Exercise value is $\max(30 - 2 \geq 0) = 3$.

2.8 Model Extension (Dividend paying stocks)

$$S_{t+h} = F_{t,t+h} e^{\pm\sigma\sqrt{h}}$$

Continuous dividends then

$$F_{t,t+h} = S_t e^{(r-\delta)h}$$

$$\begin{cases} u \cdot S_t = S_t e^{(r-\delta)h + \sigma\sqrt{h}} \\ d \cdot S_t = S_t e^{(r-\delta)h - \sigma\sqrt{h}} \end{cases}$$

At each node, $\Delta = e^{-\delta h} \frac{C_u - C_d}{S_u - S_d}$ and $B = e^{-rh}[C_u - \Delta e^{\delta h} S_u]$ and the option value is $\Delta S + B$.

2.8.1 Example 16

Assume a continuous dividend paying stock $S_0 = 30$. Using the CRR model with time steps and length of a year, and $u = 1.1$ and $d = 0.9$. construct the binomial tree for the stock price over the next 2 years. Find the price of a European call option on the stock, with 2 years to maturity and strike price of \$30. The continuously compounded rate is $r = 5\%$.

2.9 Discrete Dividends

Assume over $(t, t+h)$, receive dividends (with certainty) with a future value at time $t+h$ of D.

Therefore, $F_{t,t+h} = F^p_{t,t+h} e^{rh} = [S_t - De^{-rh}]e^{rh} = S_t e^{rh} - D$. Therefore,

$$\begin{cases} u \cdot S_t = (S_t e^{rh} - D)e^{\sigma\sqrt{h}} \\ d \cdot S_t = (S_t e^{rh} - D)e^{-\sigma\sqrt{h}} \end{cases}$$

$$\Delta = \frac{C_u - C_d}{S_u - S_d}, B = e^{-rh}[C_u - \Delta(S_u + D)] = e^{-rh}[\frac{S_u C_d - S_d C_u}{S_u - S_d}] - \Delta D e^{-rh}$$

Non-recombining tree

2.10 Model Analysis

Risk-neutral probabilities: for the non-dividend paying stock, the R-N probabilities are

$$q = \frac{e^r - d}{u - d} \text{ and } 1 - q = \frac{u - e^r}{u - d}$$

Recall for one period model,

$$C_0 = e^{-rh}[qC_u + (1-q)(-C_d)] = EPV$$

Can generalize to models with greater than one periods using the recursive algorithm.

In general, we want to price using the risk-neutral probabilities. It is easier than the replicating portfolio. How can we compute them easily?

Under the risk-neutral measure, we must have

$$E_Q[S_{t+h}|S_t] = F_{t,t+h}$$

under the R-IV probability measure. For the non-dividend paying stock, we have

$$\Delta = \frac{C_u - C_d}{S_u - S_d}$$

2.11 Binomial model as an approximation of the log-normal model

A random variable is said to be log-normally distributed with parameters μ and σ. If it is of the form e^X, where $X \sim N(\mu, \sigma^2)$, The binomial model can be shown to approximate a log-normal distribution (continuous). To see this, we need to consider very small time steps (makes the model more realistic).Consider time steps of size $\frac{1}{n}$, where n is large. Consider an interval of time where we have $n \cdot t$ steps.

$$S_{nt} = S_0 u^{(\text{number of up steps})} d^{(\text{number of steps down})}$$

Suppose, for simplicity, that $u = e^{\frac{\sigma}{\sqrt{n}}}$ and $d = e^{-\frac{\sigma}{\sqrt{n}}}$

Note: number of up steps plus number of down steps is equal to $n \cdot t$. Let $Nu + Nd = n \cdot t$. This implies $S_{nt} = S_0(e^{\frac{\sigma}{\sqrt{n}}})^{Nu}(e^{-\frac{\sigma}{\sqrt{n}}})^{Nd} = S_0 \cdot e^{\frac{\sigma}{\sqrt{n}}[Nu+(-1)Nd]}$.

Suppose the R-N probabilities are $q = 0.5 = 1 - q$. Want to re-write $1 \cdot Nu + (-1)Nd$

Let $X_i = \begin{cases} -1 & \text{with probability } 0.5 \\ 1 & \text{with probability } 0.5 \end{cases}$

$$1 \cdot Nu + (-1)Nd = \sum_{i=1}^{n \cdot t} X_i$$

where X_i is i.i.d.. This implies $S_{nt} = S_0 e^{\frac{\sigma}{\sqrt{n}} \sum_{i=1}^{nt} X_i}$. Take the limit as $n \to \infty$. Want to use the CLT. Now, $E[X_i] = 0$ and $Var(X_i) = 1$. This implies $\frac{\sum_{i=1}^{nt} X_i - 0}{\sqrt{nt} 1} \to^D N(0, 1)$. Therefore,

$$S_t = S_0 e^{\sigma \sqrt{t} \frac{\sum_{i=1}^{nt} X_i}{\sqrt{nt}}}$$

as $n \to \infty$ this R.V. tends to $N(0, \sigma^2 t)$. This implies S_t is log-normally distributed with parameters 0 and $\sigma^2 t$. If we change $u = e^{(r-\delta)h + \sigma\sqrt{h}}$, $d = e^{(r-\delta)h - \sigma\sqrt{h}}$ where $h = \frac{1}{n}$. We would have $S_t = S_0 e^Y$ where $Y \sim N((r - s)t, \sigma^2 t)$. Equivalently, we can write

$$\ln(\frac{S_t}{S_0}) \sim N((r - s)t, \sigma^2 t)$$

Let $t = 1 \implies \sigma^2$ is the variance of the continuously compounded returns over 1 year.

Now if σ^2_{yearly} is the variance of the continuously compounded returns. Suppose we want to find the $\sigma^2_{monthly}$ (for monthly returns)

$$r_{yearly} = \sum_{i=1}^{12} r_{monthly,i} \implies Var(r_{yearly}) = 12 Var(r_{monthly})$$

That is,

$$\sigma_y^2 = 12\sigma_M^2$$

Thus

$$\sigma_M = \frac{\sigma_y}{\sqrt{12}}$$

S_0, in general $\sigma\sqrt{h}$ represents one standard deviation of the continuously compounded returns.

3 Discrete-time Securities Market

N securities. Time-0 value: $S(0) = \begin{pmatrix} S_1(0) & S_2(0) & \cdots & S_N(0) \end{pmatrix}$. M possible states at time 1.

$$S(1,\Omega) = \begin{pmatrix} S_1(1,w_1) & \cdots & S_N(1,w_1) \\ S_1(1,w_2) & \cdots & S_N(1,w_2) \\ \vdots & \ddots & \vdots \\ S_1(1,w_M) & \cdots & S_N(1,w_N) \end{pmatrix}$$

$$\theta_j = \text{number of units held in the jth asset}, j = 1, 2, \cdots, N$$

$$e = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{pmatrix}$$

Value at time 0 of your portfolio is $S(0)\theta = \theta_1 S_1(0) + \theta_2 S_2(0) + \cdots + \theta_N S_N(0)$ and the possible values at time 1 are given by

$$S(1,\Omega)e = \begin{pmatrix} \theta_1 S_1(1,w_1) + \cdots + \theta_N S_N(1,w_1) \\ \vdots \\ \theta_1 S_1(1,w_M) + \cdots + \theta_N S_N(1,w_M) \end{pmatrix}$$

An arbitrage opportunity will exist e if:

$$S(0)e \leq 0 \text{ and } S(1,\Omega)e > 0$$

The fundamental theorem of asset pricing: The single-period securities market model is arbitrage-free if and only if there exists a state price vector.

3.1 Price via 1 of 2 ways

1. Under the assumption of no arbitrage, the time-0 value of the security in consideration is equal to the time-0 value of a portfolio that replicates its payoffs "disadvantage": need to recalculate the replicating portfolio at each node and this portfolio will differ depending on the security you are pricing

2. Using the risk-neutral probability measure: Calculate the risk-neutral probability from the current assets on the market store in memory and use to price any other security.

Creating the R-N probability f(N securities, 1 period model): Assume that S_1 is our bank account earning interest at an annual effective risk-free rate of i: Example (motivating)

Binomial model: $N = 2, M = 2$, then $\psi_1 = \frac{q}{1+i}, \psi_2 = \frac{1-q}{1+i}$. Hence R-N probabilities are $q = \psi_1(1+i)$ and $1 - q = \psi_2(1+i)$.

Let's generalize this. ψ is $\exists, S(0) = \psi S(1, \Omega)$. Take the first column of $S(1, \Omega)$, we have that $S_1(0) = \psi \begin{pmatrix} S_1(1, w_1) \\ \vdots \\ S_1(1, w_M) \end{pmatrix}$. In scalar form, we have $S_1(0) = \sum_{j=1}^{n} \psi_j S_1(1, w_j)$. This implies

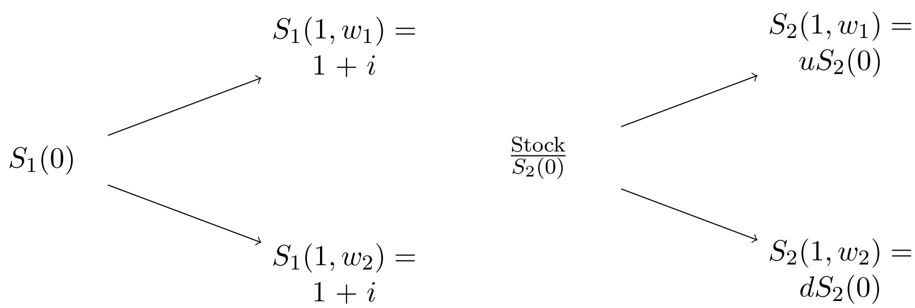
$$\frac{1}{1+i} = \sum_{j=1}^{M} \psi_j$$

$$\implies 1 = \sum_{j=1}^{M} \psi_j (1+i)$$

Define $Q(w) = (1+i)\psi(w)$ for $w \in \Omega$. Note $\sum_{w \in \Omega} Q(w)(1+i) = 1$.

To see why these are R-N probabilities, consider the following:

By definition $S_j(0) = \sum_{w \in \Omega} \psi(w) S_j(1, w) \implies S_j(0) = \frac{1}{1+i} \sum_{w \in \Omega} Q(w) S_j(1, w)$. Aside Binomial Model:



Let $Q(w_1) = q$ and $Q(w_2) = 1 - q$. This implies $S_2(0) = \frac{1}{1+i} \sum_{w \in \Omega} Q(w) S_2(1, w) = \frac{1}{1+i}[quS_2(0) + (1-q)dS_2(0)]$. This implies $q = \frac{1+i-d}{u-d}, 1 - q = \frac{u-(1+i)}{u-d}$

In general, $S_j(0) = E[\frac{S_i(1)}{1+i}]$

(FTAP) The following are equivalent (assuming S_1 is the bank account)

1. Market is arbitrage-free
2. There exists a state price vector
3. There exists a risk-neutral probability measure.

3.2 Pricing

Random payoff of X at time 1 $X = \begin{pmatrix} X(w_1) \\ X(w_2) \\ \vdots \\ X(w_M) \end{pmatrix}$. Aim: find the time-0 value of X.

3.2.1 Methods

1. Method 1: find the trading strategy θ that replicates the cash-flow X. That is, find θ such that

$$S(1, \Omega)\theta = X$$

   in other words,

$$\sum_{j=1}^{N} S_j(1, w)\theta_j = X(w), \forall w \in \Omega$$

   If we are able to find a θ which replicates X, we say X is attainable. Then, if the model is arbitrage-free, we compute the time-0 value of X as the time-0 value of the portfolio that replicates X. Therefore,

$$S(0) \cdot \theta$$

   gives the time-0 value of X.

2. Method 2: Suppose X is attainable and the first security is a bank account earning interest at an annual effective risk-free rate of i. Then the time-0 value of X is

$$\psi \cdot X = \sum_{w \in \Omega} \frac{Q(w)X(w)}{1+i}$$

3.3 Completeness

Completeness : An arbitrage-free securities market model is said to be complete if every cash flow X is attainable.

Theorem An arbitrage-free securities market model is complete if and only if there is a unique state price vector.

Corollary If the first security is a bank account, then the model is arbitrage-free and complete if and only if the risk-neutral probability measure is unique.

4 Stochastic Calculus

4.1 Motivation for the Itô Integral

Consider an investment horizon of length T. Invest in the stock at points $0 = t_0 < t_1 < t_2 < \cdots < t_{N-1}$ (end of period is $t_N = T$).

1. $\delta(t)$ is the number of shares of stock purchased at time t
2. $B(t)$ will denote the price per share of stock at time t. Assume $B = \{B(t), t \geq 0\}$ is a Brownian Motion process.
3. Overall, gain over $(0, T)$, ignoring interest is
$$\sum_{i=0}^{N-1} \delta(t_i)(B(t_{i+1} - B(t_i))$$
4. Take the limit as $N \to \infty$. (equivalent to saying $\max(t_{i+1} - t_i) \to 0$). We have
$$\int_0^T \delta(t) dB(t)$$
(sometimes, it may be written as $\int_0^T \delta(t,w) dB(t,w), w \in \Omega$, but we will omit the '$w$' for simplicity of notation).

4.1.1 Quadratic Variation

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} |B(t_{i+1}) - B(t_i)|^2$$

The quadratic variation for a standard BM process over $(0, T)$ is T.

$$\lim_{N\to\infty} E[\sum_{i=0}^{N-1} |B(t_{i+1} - B(t_i)|^2] = T$$

To show this, $E[QV(0,T)] = T$ and $Var(QV(0,T) - L) \to 0$.

Remark:

1. $$QV[0,T] = \sum_{i=0}^{N-1} (B(t_{i+1}) - B(t_i))^2 \leq \max_{0 \leq i \leq N-1} |B(t_{i+1}) - B(t_i)| \times \sum_{i=0}^{N-1} |B(t_{i+1}) - B(t_i)|$$
2. Finite QV implies that it is not possible to have finite first/total variation
3. If $QV = 0$, then the first/total variation is finite.

4.2 Conditional Expectations and Filtration

We speak of a filtration which can be thought of as the continuous-time analog of P_k. Our conditional expectation will be written in the form $E[X_t|F_s]$ where $X = \{X_t, t \geq 0\}$ is a stochastic process and $\{\mathcal{F}_t, t \geq 0\}$ is a filtration. The filtration models the information available over time.

- $P = \{P_0, P_1, \cdots, P_T\}$ is discrete time
- $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$ is continuous time, called a filtration
- Each $\mathcal{F}_t$ is called a σ-filed, and models the information up to time t. "Given $\mathcal{F}_t$" can be thought of as being "given $\{B_u, 0 \leq u \leq t\}$".

4.2.1 Properties of conditional expectation

1. $E[E[X|\mathcal{F}_t]] = E[X]$
2. If X is $\mathcal{F}_t$-measurable, then $E[X|\mathcal{F}_t] = X$
3. $E[E[X|\mathcal{F}_s]|\mathcal{F}_t] = E[E[X|\mathcal{F}_t]|\mathcal{F}_s] = E[X|\mathcal{F}_s], s \leq t$

4.3 Martingale

Suppose the state space is Ω, and that we have a probability measure, a filtration $\{\mathcal{F}_t, t \geq 0\}$ and an adapted stochastic process $M = \{M_t, t \geq 0\}$.

- $E[|M_t|] < \infty, \forall t$
- $E[M_t|\mathcal{F}_s] = M_s, \forall s \leq t.$

then M is a martingale. Note: from the second property, if $s = 0$, we have $E[M_t|\mathcal{F}_0] = M_0$.

Example 36: Show that the standard Brownian motion $W = \{W_t, t \geq 0\}$ is a continuous martingale with respect to its own filtration.

$$E[W_t|\mathcal{F}_s] = E[W_t - W_s|\mathcal{F}_s] + E[W_s|\mathcal{F}_s], (s < t) = E[W_t - W_s] + W_s = E[W_t] - E[W_s] + W_s = W_s$$

where $W_t = W_t - W_s + W_s$

Example 37, Show that $\{W_t^2 - t, t \geq 0\}$ is a martingale with respect to the filtration generated by $\{W_t, t \geq 0\}$.

$$\begin{aligned}
E[W_t^2 - t|\mathcal{F}_s] &= E[\{(W_t - W_s) + W_s\}^2 - t|\mathcal{F}_s] \\
&= E[(W_t - W_s)^2|\mathcal{F}_s] + E[W_s^2|\mathcal{F}_s] + 2E[(W_t - W_s)W_s|\mathcal{F}_s] - t \\
&= Var(W_t - W_s) + W_s^2 + 2W_s E[W_t - W_s|\mathcal{F}_s] - t \\
&= t - s + W_s^2 + 0 - t = W_s^2 - s
\end{aligned}$$

Example 38, Let X be a random variable and define $M_t = E[X|\mathcal{F}_t], 0 \leq t \leq T$, show that $\{M_t, 0 \leq t \leq T\}$ is a martingale with respect to $\{F_t, 0 \leq t \leq T\}$. $(s < t)$

$$E[M_t|\mathcal{F}_s] = E[E[X|\mathcal{F}_t]|\mathcal{F}_s] = E[X|\mathcal{F}_s] = M_s$$

from the last property of C.E.

4.4 Itô Integral

Properties of $I(T) = \int_0^T \delta(t) dW(t)$

1. Adapted w.r.t. $\mathcal{F} = \{\mathcal{F}, t \geq 0\}$
2. Linerity
3. Martingale property:

$$E[\int_0^T \delta(t) dB(t)|\mathcal{F}_s] = \int_0^s \delta(t) dB(t)$$

Proof.

$$\begin{aligned} E[\int_0^T \delta(t) dB(t)|\mathcal{F}_s] &= E[\int_0^s \delta(t) dB(t)|\mathcal{F}_s] + E[\int_s^T \delta(t) dB(t)|\mathcal{F}_s] \\ &= \int_0^s \delta(t) dB(t) + E[\int_s^T \delta(t) dB(t)] = \int_0^s \delta(t) dB(t) \end{aligned}$$

Remark: $E[I(T)] = I(0) = 0 \implies E[\int_0^T \delta(t) dB(t)] = 0.$ □

4.5 Itô's Lemma

If $Y_t = f(t, X_t)$,

$$dY_t = f_t(t, X_t) dt + f_X(t, X_t) dX_t + \frac{1}{2} f_{XX}(t, X_t)(dX_t)^2$$

$$dW_t dt = dt dW_t = dt dt = 0$$

$$dW_t dW_t = dt$$

4.6 Vasicek Model

$$dr(t) = a(b - r(t)) dt + \sigma dW(t)$$

Problem possible to have two interest rates.

4.7 Cox-Ingersoll-Ross Model (CIR)

$$dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dW(t)$$

where $\sqrt{r(t)}$ is the extra term compared to the Vasicek Model. Model has the mean reversion feature and keeps the interest rate positive (will not show, but it can be shown that $r(t)$ will have a non-central χ^2 distribution. When $r(t)$ is small, the volatility is also small, but the drift will be positive and relative large.

What is the mean of $r(t)$?
Integral form :

$$r(t) = r(0) + a\int_0^t (b - r(u))du + \sigma\int_0^t \sqrt{r(u)}dW(u)$$

Take expectations:

$$E[r(t)] = r(0) + a\int_0^t (b - E(r(u)))du + 0$$

(mean of the Itô integral is zero)

$$\implies \frac{d}{dt}E[r(t)] = a[b - E[r(t)] = ab - aE[r(t)]$$

$$\implies \frac{d}{dt}E[r(t)] + aE[r(t)] = ab$$

$$\implies \frac{d}{dt}[e^{at}E[r(t)]] = e^{at}ab$$

By integrating throughout, we have

$$e^{at}E[r(t)] - r(0) = b(e^{at} - 1)$$

$$\implies E[r(t)] = r(0)e^{-at} + b(1 - e^{-at}$$

If we look at $E[r(t)]$ as $t \to \infty$ we have $E[r(t)] \to b$.

4.8 Black-Sholes Merton Model

$$S_t = S_0e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}, t \geq 0$$

(see the geometric brownian motion last class)
where $S_0 > 0$, and

- μ is the instantaneous mean
-