ACTSC 331 Note : Life Contingency

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1 Review

1.1 Survival Model

This part of notes is in the study note for mlc.

1.2 Insurance

$$b_{T_x} = \text{benefit payable if the life dies at time } T_x$$

$$Z = \text{present value random var} = \text{PV of any payments on a contract}$$

Z depends on T_x or K_x or $K_x^{(m)}$ where $K_x = \lfloor T_x \rfloor$ and $K_x^{(m)} = \frac{1}{m}\lfloor mT_x \rfloor$

For insurance, we always have a max of one payment, such as, for benefits payable on death

1. $Z = v^{T_x} = e^{-\delta T_x}$ for whole life
2. $Z = \begin{cases} v^{T_x} & \text{if } T_x < n \\ 0 & \text{if } T_x \geq n \end{cases}$ for term
3. $Z = \begin{cases} 0 & \text{if } T_x < n \\ v^n & \text{if } T_x \geq n \end{cases}$ for pure endowment
4. $Z = \begin{cases} v^{T_x} & \text{if } T_x < n \\ v^n & \text{if } T_x \geq n \end{cases}$ for endowment insurance
5. $Z = \begin{cases} 0 & \text{if } T_x < n \\ \text{something depends on the contract} & \text{if } T_x \geq n \end{cases}$ for a deferred contract

If we annual benefits (payable at end of the year), use $K_x + 1$. For benefits paid of the end of the $\frac{1}{m}$ year of death, use $K_x^{(m)} + \frac{1}{m}$. For any Z, we can always find $E[Z] = EPV = AV$ by first principle ($E[Z] = \sum zP(Z = z) = \int v^t {}_tp_x \mu_t dt$) - the sum or integral over all dates of the amount paid x discount factor x probability of part.

If we have a discrete survival model given by a life table, it is tedious to calculate EPVs of whole life or long term contracts , so the values for whole life insurance are often included in the table.

$$E[Z] = \sum_{k=0}^{\infty} 1 \cdot v^{k+1} {}_{k|}q_x = A_x$$

Also, the relationship between A_x's in the table is a recursion:

$$A_x = A^1_{x:\overline{n}|} + {}_nE_x \cdot A_{x+n} = vq_x + E_x A_{x+1}$$

Trivial relationships could be observed

- term + pure endowment = Endowment
- term + deferred whole life = whole life
- deferred = pure endowment × the contract

The relationships all hold within payment timing options. Relationship between different timing (UDD) (pretty intuitive)

1.3 Annuities - same idea as insurance

Y = PVRV as before , a function of T_x, K_x or $K_x^{(m)}$. We can also have annuities payable at the end (due) or start (immediate) of each period. We can evaluate the EPVs with

- first principles,
- amount × discount× probability over all dates
- use the relationship to A's recall $\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}$. Then $Y = \ddot{a}_{\overline{K_x+1}|} = \frac{1-v^{K_x+1}}{d}$. Hence EPV $\ddot{a}_x = \frac{1-A_x}{d}$.

Similarly, $\ddot{a}_{x:\overline{n}|} = \frac{1-A_{x:\overline{n}|}}{d}$.
Relationships:

1. annuity due = 1 + annuity immediate. In other words, $\ddot{a}_x = 1 + a_x$, $\ddot{a}_x^{(m)} = \frac{1}{m} + a_x^{(m)}$
2. Term: $\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n}|} - v^n {}_np_x$
3. whole life = term + deferred, $\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x$
4. deferred = ${}_nE_x$ × any contract for age $x + n$

To get relationship between annual, and mthly cases, we need UDD. Idea: convert $\ddot{a}_x^{(m)}$ to $A_x^{(m)}$, use UDD on that, convert back to $\ddot{a}_x$. Result: $\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m)$, $\ddot{a}_{x:\overline{n}|}^{(m)} = \alpha(m)\ddot{a}_{x:\overline{n}|} - \beta(1 - {}_nE_x)$ where $\alpha(m) = \frac{id}{i^{(m)}d^{(m)}}, \beta = \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}$

1.4 Variance

- For insurance, it's easy to find the second moment of Z.

  $E[Z^2]$ = same calculation as $E[Z]$ but with v^2 instead of v = ${}^2A_{\text{something}}$. Then $Var(Z) = {}^2A - A^2$

- for annuities, it's not easy to do this way because payments are not independent relationship. Instead, we use $Y = \frac{1-Z}{d}$

  $Var(Y) = \frac{1}{d^2}Var(Z) = \frac{{}^2A-A^2}{d^2}$

1.5 Increasing contract

- $(IA)_x$ pays $k if they die in the kth year.
- $(I\ddot{a})_x$ pays $\$k + 1$ at the kth year.

1.6 Premium

Loss-at-issue RV $L_0 =$ PV future benefits $-$ PV future premiums

To find P by the equivalence principle, set $E[L_0] = 0$, i.e. set P so that EPV premiums is equal to the EPV benefits. We can also include expense and profit margin.

$L_0^g =$ PV future benefits $+$ expenses $-$ PV future premiums

Then set P^g such that the EPV premium is equal to EPV benefits plus EPV expenses. Apparently, the gross premium is always higher than the net premium. Expenses can be fixed or as a percentage of the premiums

2 Policy values

Definition. *The time t future loss RV is $L_t = PV$ at t of future benefits$-PV$ at t of future premiums conditional on the contract being in force at time t.*

If the contact has annual payments and t is an integer, then there may be a payment at t. The convention is to consider premium payment at time t to be in the future and benefits in the past. (i.e. P at t is in future premiums. S at t is not in future benefits.)

For annuities (where benefit is a series of payments) it can be either way. For endowment insurance or pure endowment, the payment is at time n. But L_n does not include the time n payment as future benefit so $L_t = 0$.

But what we do have is

$$\lim_{t \to n^-} L_t = S - 0 = S$$

for endowment insurance and pure endowment insurance. For term

$$\lim_{t \to n^-} L_t = 0$$

Example

5-year endowment insurance with annual premiums P, sum insured S payable at the end of year issued to (x).

$$L_0 = \begin{cases} Sv^{K_x+1} - P\ddot{a}_{\overline{K_x+1|}} & \text{if } K_x < 5 \\ Sv^5 - P\ddot{a}_{\overline{5|}} & \text{if } K_x \geq 5 \end{cases}$$

We also know $L_5 = 0$. What about L_1? If the policy is in forced at that point, the person is alive and age $x+1$. They have 4 years of benefit and 4 premiums left to pay.

$$L_1 = \begin{cases} Sv^{K_{x+1}+1} - P\ddot{a}_{\overline{K_{x+1}+1|}} & \text{if } K_{x+1} \leq 3 \\ Sv^4 - P\ddot{a}_{\overline{4|}} & \text{if } K_{x+1} \geq 4 \end{cases}$$

Similarly we can define L_2, L_3 and L_4. Now let's put in some number Makeham rule $\omega = 120, A = 0.0001, B = 0.00035, c = 1.075, i = 6\%, x = 50$

With these parameters, we get $P_{50} = 0.986493, A_{50} = 0.335868, A_{51} = 0.347203, A_{55} = 0.394409, {}_5E_{50} = 0.690562, {}_4E_{51} = 0.742018$.

To find P, set $E[L_0] = 0 = S(A_{50} - {}_5E_{50}A_{55} + {}_5E_{50}) - P(\ddot{a}_{50} - {}_5E_{50}\ddot{a}_{55})$.

Solving $P + 1735.55$

Then $E[L_1|K_{50} \geq 1] = SA_{51:\overline{4|}} - P\ddot{a}_{51:\overline{4|}} = S(A_{51} - {}_4E_{51}A_{55} + {}_4E_{51}) - P(\ddot{a}_{51} - {}_4E_{51}\ddot{a}_{55}) = 1727.95$

If $x+1$ is alive, their remaining premiums are not sufficient to cover their remaining benefits. The insurance company should hold 1727.95 capital in reserve to make up the short fall. Each year the expected value of L_t goes up for an endowment insurance.

Insurer can build up capital from early premium payments to cover the later benefit payments. Logically, the policy value at time t is the amount which, when combined with future premiums, will exactly cover the future benefits. In other words, ${}_tV + EPV_{\text{at t}}$future premiums $= EPV_{\text{at t}}$ future benefits.

Mathematically, ${}_tV = E[L_t|T_x > t]$

It's the amount that will, along with future premiums, cover future benefits. Where does the \$ come from? From other policies. say we have N identical policies (from last example - 5 year insurance). We collect 1735.55N at 0. It earns 6% $\rightarrow$ 1839.68N at 1, but some people die in age $50-51$ and each get $10,000$ at 1. The number who die on average is $Nq_{50} = 0.013507N$ so we have $1839.68N - 135.07N = 1704.61N$. There are 0.986493N policyholders still in force at time 1, so each has $\frac{1704.61N}{0.986493N} = 1727.95$ which is exactly what we had for $E[L_1|T_{50} > 1]$. Fomr this to work at, we needed

- the same interest rate earned as assumed
- mortality experience to be the same as expected.

In reality, there are two versions of the policy value

- Net Premium Policy Value (NPPV) - EPV of the future benefits minus the premiums on the policy value basis with an artificial premium recalculated using the equivalence principle (no exp) and the policy value basis
- Gross Premium Policy Value (GPPV) - EPV of future benefits minus premiums on the policy value basis with actual gross premiums and including expenses

Two differences for these two versions could be expenses vs non-expenses and actual vs artificial premiums

Basis : the set of interest, mortality, and expense assumptions used in an actuarial calculation.

Premium Basis - used to calculate P

Policy Value basis - used to calculate ${}_tV$

If they are, and they assume no expenses, then $GPPV = NPPV$. In general, the policy value basis is more conservative than the premium basis. Premium basis needs to be realistic but competitive. Policy values are about ensuring solvency so the basis is more pessimistic. (worse mortality, lower interest rates, higher expenses)

Example

Whole life \$10,000 issued to (50). Premiums payable for 15 years max. Basis: Markham rule, $\omega = 120, A = 0.001, B = 0.00035, C = 1.075$, 6% interest rate, 1% of premium plus \$100 initial.

P: $10000A_{50} + 100 + 0.01P\ddot{a}_{50:\overline{15}|} = P\ddot{a}_{50:\overline{15}|}$ then

$$P = \frac{10000A_{50} + 100}{0.99\ddot{a}_{50:\overline{15}|}} = 377.41$$

$$L_{10} = \begin{cases} 10,000v^{K_{60}+1} - P\ddot{a}_{\overline{K_{60}+1}|} & K_{60} \leq 4 \\ 10,000v^{K_{60}+1} - P\ddot{a}_{\overline{5}|} & L_{60} \geq 5 \end{cases}$$

$$L_{20} = 10,000v^{K_{70}+1} \text{ since no more premiums are due}$$

For gross,

$$L_{10}^g = \begin{cases} 10,000v^{K_{60}+1} - 0.99P\ddot{a}_{\overline{K_{60}+1}|} & K_{60} \leq 4 \\ 10,000v^{K_{60}+1} - 0.99P\ddot{a}_{\overline{5}|} & K_{60} \geq 5 \end{cases}$$

If policy value basis is the same

$${}_{10}V^g = E[L_{10}^g|T_{50} > 10] = 10,000A_{60} - 0.99P\ddot{a}_{60:\overline{5}|} = 4568.085 - 0.99 \times 377.41 \times 4.22367 = 2989.97$$

The gross for L_{20}^g is the same as the net.

$${}_{20}V^g = E[L_{20}^g|T_{50} > 20] = 10,000A_{70} = 5861.87$$

Now instead, assume the policy value basis is: same mortality, same expense, 5% interest. Then ${}_{10}V^g = 10,000A_{60:5\%} - 0.99 \times 377.41 \times \ddot{a}_{60:\overline{5}|} = 5107.311 - 0.99 \times 377.41 \times$

$4.29763 = 3501.56 > {}_{10}V^g_{5\%}$. More cautious assumption is going to result in higher policy values. Similarly, ${}_{20}V^g = 10,000A_{70:5\%} = 7687.99$

Back to Net, if we use policy value basis of 5%, we need to calculate P (artificial premium).P' is the theoretical premium that would have been charged at time 0 if we had used the policy value basis, and no expenses. We need $10,000A_{50} = P'\ddot{a}_{50:\overline{15}|5\%}$. Then

$$P' = \frac{3908.23}{9.764268} = 400.26$$

Then ${}_{10}V^n = E[L_{10}|T_{50} > 10] = 10,000A_{60} - P'\ddot{a}_{60:\overline{5}|} = 3387.15$; ${}_{20}V^n = \text{same} = 7687.99$ Again if policy value basis is premium basis,

$${}_{20}V^n = 5861.87$$

$${}_{10}V^n = 10,000A_{60} - P''\ddot{a}_{60:\overline{5}|}$$

where

$$P'' = \frac{10,000A_{50}}{\ddot{a}_{50:\overline{15}|}}$$

Why artificial premiums? Consider $E[L_{10}|T_5 > 10]$ where there were never any expenses is equal to

$$SA_{60} - P\ddot{a}_{60:\overline{5}|} = SA_{60:5\%} - \frac{SA_{60:6\%}}{\ddot{a}_{50:\overline{5}|6\%}}\ddot{a}_{60:\overline{5}|5\%}$$

There is no way to simplify this because the interest rates don't match. Theoretically, it's nicer if that the basis used to calculate P in the policy value is the same.

i.e

$$E[L_{10}|T_{60} > 10] = SA_{60} - \frac{SA_{50}}{\ddot{a}_{50:\overline{15}|}}a_{60:\overline{5}|}$$

Take a general case of \$1 whole life issued at (X). ${}_tV = E[L_t|T_X > t] = A_{X+t} - \frac{A_x}{\ddot{a}_x}\ddot{a}_{x+t}$ If everything is on the same basis

$$= (1 - d\ddot{a}_{x+t}) - (\frac{1 - d\ddot{a}_x}{\ddot{a}_x})\ddot{a}_{x+t} = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$$

Similarly for endowment insurance.

$${}_tV = A_{x+t:\overline{n-t}|} - \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}\ddot{a}_{x+t:\overline{n-t}|} = 1 - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}}$$

We can only use these simplifications if with the same basis and premium assumption with no expenses.

2.1 Retrospective Policy Value

If the policy value basis is equal to the premium basis and the experience matches the assumptions (actual interest, mortality, and expenses are as predicted) then we can also express ${}_tV$ in terms of cash flows from time 0 to time t.

$$ {}_tV = \frac{{}_0V + EPV \text{ at 0 of premiums in } (0,t) - EPV \text{ at 0 of benefits in } (0,t)}{{}_tE_x} $$

Compare to prospective policy value

$$ {}_tV = EPV \text{ at t of future benefits} - EPV \text{ at t of future premiums} $$

We don't actually use retrospective policy values much. We use asset shares instead which are based on the actual experience in time 0 to t.

Proof.

$$ \begin{aligned} {}_0V &= EPVB_0 - EPVP_0 \\ &= EPV \text{ at 0 of (0, t) benefits} + EPV \text{ at 0 of } (t,\infty) \text{ benefits} - EPV \text{ at 0 of (0, t) premiums} \\ &- EPV \text{ at 0 of } (t,\infty) \text{ premiums} \\ &= \text{EPV at 0 of (0, t) benefits} - (0,t) \text{ premiums} \\ &+ (\text{policy value at t brought back to time 0 for interest and survival} = {}_tE_{xt}V) \end{aligned} $$

Solving for ${}_tV$ gives the result. □

Recursion: we know $A_x = vq_x + vp_xA_{x+1}$ and $\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}$

$$ \begin{aligned} {}_tV &= EPV \text{ at t of benefits} - \text{Premiums} \\ &= EPV \text{ at t of } (t, t+s) \text{ benefits } - \text{premiums} + EPV \text{ at t of } (t+s,\infty) \text{ benefits} - \text{premiums} \\ &= EPV \text{ at t of } (t, t+s) \text{ benefit} - \text{Premiums} + {}_xE_{x+tt+s}V \end{aligned} $$

In particular , let $s = 1$,

$$ {}_tV = \text{EPV at t of next year of benefits} - \text{premiums} + vp_{x+tt+1}V $$

In general terms, let P_t = Premium at t, e_t = premium-related expenses at t, S_{t+1} = sum insured payable at if $K_x = t$ at $t+1$, E_{t+1} = benefit-related expenses at $t+1$, i_t = interest rate earned in $(t, t+1)$. then we can obtain the recursion

$$ ({}_tV + P_t - e_t)(1 + i_t) = q_{x+t}(S_{t+1} + E_{t+1}) + P_{x+tt+1}V $$

Policy value at t plus net income at t accumulated for 1 year is exactly enough to provide the death benefit for those who die and the $t+1$ policy value for survivors.

Proof. Assume whole life, no expense, constant interest rate i.

$$ {}_tV = SA_{x+t} - P\ddot{a}_{x+t} $$

$$ {}_{t+1}V = SA_{x+t+1} - P\ddot{a}_{x+t+1} $$

but we also know

$$ A_{x+t} = vq_{x+t} + vp_{x+t}A_{x+t+1} $$

$$ \ddot{a}_{x+t} = 1 + {}_vp_{x+t}\ddot{a}_{x+t+1} $$

so

$$ {}_tV = S(vq_{x+t} + vp_{x+t}A_{x+t+1}) - P(1 + vp_{x+t}\ddot{a}_{x+t+1}) $$

$$ {}_tV + P = Sq_{x+t}V + vp_{x+t}(SA_{x+t+1} - P\ddot{a}_{x+t+1}) $$

divide both sides by v. Hence we get the recursion formula. □

In general for any contract, we can always derive the recursive relationship by splitting out the cash flow in the next year (or $\frac{1}{m}$ year) from the cash-flow from the onwards.

What we have at t + what we get at t, accumulated for one period must equal what we need to provide then.

At time $t+1$ we must provide

- Policy value for $t+1$ (${}_{t+1}V$)
- enough extra to increase that to the benefit payable if the life dies

The extra amount is $S_{t+1} + E_{t+1} - {}_{t+1}V$ which is called NAAR (net amount at risk) or DSAR (death strain at risk). The $NAAP_{t+1}$ can be thought at as the sensitivity to mortality in the year $t \to t+1$.

Example: 5-year discrete endowment insurance to (50) usual mortality and 6% interest, no expenses. P was 1735.55 and ${}_1V$ was 1727.95. (Calculated between $10000A_{51:\overline{4|}} - P\ddot{a}_{51:\overline{4|}}$ but we could also get ${}_1V$ using recursion

$$ ({}_0V + 1735.55)(1.06) = q_{50}(10000) + p_{50}{}_1V $$

so ${}_1V = 1727.95$. Also then $NAAR_1 = 10000 - {}_1V = 8272.05$. Then for time 2,

$$ ({}_1V + 1735.55)(1.06) = q_{51}(10000) + p_{51}{}_2V $$

Hence ${}_2V = 3578.16$ and $NAAP_2 = 6421.81$. Policy value is higher than NAAR is lower in the second year. Similarly $NAAR_3 = 4436.57, {}_3V = 5563.43, {}_4V = 7698.41, NAAR_4 = 2301.59, {}_5V = 0$ but ${}_{5-}V = 10000$, so over the entre contract, reserve goes up and NAAR goes down. For a 5-year term insurance instead, $P = 146.16, {}_1V = 20.14, {}_2V = 31.69, {}_3V = 33.27, {}_4V = 23.31, {}_5V = 0 = {}_{5-}V$; NAAR's are huge compared endowment insurance.

Term insurance is much more sensitive to mortality risk then endowment insurance. Cash flow more frequent than usual. We could calculate (or approximate using UDD), the premium and the policy value at any payment date. At time $t+s$ ($t \in Z, 0 \leq s < 1$),

$$ {}_{t+s}V = EPV \text{ at } t+s \text{of future benefits} - \text{premiums (e.g.)} = SA^{(m)}_{x+t+s} - P\ddot{a}^{(m)}_{x+t+s} $$

but we don't usually have A's and $\ddot{a}$ for non-integer ages. This is where recursions are useful.

$$ ({}_tV + P_t - e_t)(1+i_t)^{1/m} = {}_{\frac{1}{m}}q_{x+t}(S_{t+\frac{1}{m}} + E_{t+\frac{1}{m}}) + {}_{\frac{1}{m}}p_{x+t}({}_{t+\frac{1}{m}}V) $$

so that gives us ${}_{t+\frac{1}{m}}V$ then

$$ ({}_{t+\frac{1}{m}}V + P_{t+\frac{1}{m}} - e_{t+\frac{1}{m}})(1+i)^{\frac{1}{m}} = {}_{\frac{1}{m}}q_{x+t+\frac{1}{m}})(S_{t+\frac{2}{m}} + E_{t+\frac{2}{m}}) + {}_{\frac{1}{m}}p_{x+t+\frac{1}{m}}({}_{t+\frac{2}{m}}V) $$

For endowment insurance, NAAR decreased over time (same for whole life). For term, NAAR is always large. Endowment insurance is less sensitive to mortality risk than term insurance is. On the other hand, endowment insurance is more sensitive to interest rate risk than term insurance is, because the reserves hold are much larger. In any year of a contract at time t, suppose there are N policies in force. The insurer has N_tV in reserves (they get $N(P_t - e_t)$) The expected extra amount needed at time $t+1$ to pay death benefits is $Nq_{x+t}NAAR_{t+1}$

The actual amount needed is the actual number of dollars $\times$ $NAAR_{t+1}$ .The difference $Nq_{x+t}NAARt+1$(actual number of dollars $-$ Nq_{x+t}) is the mortality loss (gain) in the year $t \to t+1$. We looked at $\frac{1}{m}$ly payment contracts and everything is the same as annual.

Recursion was

$$ ({}_{t+\frac{1}{m}}V + P_{x+\frac{1}{m}} - e_{x+\frac{k}{m}})(1+i)^{\frac{1}{m}} = {}_{\frac{1}{m}}q_{x+t+\frac{k}{m}}(S_{t+\frac{k+1}{m}} + E_{t+\frac{k+1}{m}} + {}_{\frac{1}{m}}P_{x+t+\frac{k}{m}}({}_{t+\frac{k+1}{m}}V) $$

But if benefit and premium payment frequencies are different, we need to leave out the corresponding term from the recursion on dates where no payment is made. e.g. premiums semiannual, benefits monthly

$$ ({}_tV + P_t)(1+i)^{\frac{1}{12}} = {}_{\frac{1}{12}}q_{x+t}S_{t+\frac{1}{12}} + {}_{\frac{1}{12}}p_{x+t}({}_{t+\frac{1}{12}}V) $$

$$ ({}_{t+\frac{1}{12}}V + 0)(1+i)^{\frac{1}{12}} = {}_{\frac{1}{12}}q_{x+t+\frac{1}{12}}S_{t+\frac{1}{12}} + {}_{\frac{1}{12}}p_{x+t+\frac{1}{12}}({}_{t+\frac{2}{12}}V) $$

...

What if $m \to \infty$ and we have continuous payment. No new principles! At any time t,

$$ {}_tV = \text{EPV at time t of future benefits} - \text{premiums given in force at time t} $$

We don't need to worry about when benefits or premiums are payable since it's all continuous T_x is continuous $\implies P(T_x = t) = 0$.

Example

Whole life, no expenses, continuous

$$_tV = S\bar{A}_{x+t} - P\bar{a}_{x+t}$$

$$L_t|T_x \geq t = Sv^{T_{x+t}} - P\bar{a}_{\overline{T_{x+t}}|} = Sv^{T_{x+t}} - P(\frac{1-v^{T_x}}{\delta}) = (S + \frac{P}{\delta})v^{T_{x+t}} - \frac{P}{\delta}$$

Hence we can find $P(L_t > l|T_x \geq t)$

$$\begin{aligned} P((S + \frac{P}{\delta})v^{T_{x+t}} - \frac{P}{\delta} > l) &= P(v^{T_{x+t}} > \frac{l + \frac{P}{\delta}}{S + \frac{P}{\delta}}) \\ &= P(-\delta T_{x+t} > \ln(\frac{l + \frac{P}{\delta}}{S + \frac{P}{\delta}})) \\ &= P(T_{x+t} < \frac{\ln(l + \frac{P}{\delta}) - \ln(S + \frac{P}{\delta})}{\delta}) = {}_{k^*}q_{x+t} \end{aligned}$$

We could do a similar procedure for a discrete contract but it's not as nice. Also

$$Var(L_t|T_x \geq t) = (S + \frac{P}{\delta})^2({}^2\bar{A}_{x+t} - \bar{A}^2_{x+t})$$

This approach also works for endowment insurance. But not for term, deferred, or other cases where the RV governing the premiums is different from the RV for benefits.

Similarly, for n-year endowment insurance with premiums for $\overline{n}|$. Let

$$H_{x+t} = \min\{T_{x+t}, n-t\}$$

and then $L_t|T_x \geq t = Sv^{H_{x+t}} - P\bar{a}_{\overline{H_{x+t}}|}$ so $Var(L_t|T_x \geq t) = (S + \frac{P}{\delta})^2({}^2\bar{A}_{x+t:\overline{n-t}|} - \bar{A}^2_{x+t:\overline{n-t}|})$ and

$$P(L_t > l|T_x \geq t) = P(H_{x+t} \geq k^*) = P(T_{x+t} \geq k^* \text{ and } n-t \geq k^*)$$

Hence it is ended up with $\begin{cases} k^*q_{x+t} & \text{if } k^* \leq n-t \\ 0 & \text{if } k^* > n-t \end{cases}$

But if benefits and premiums have different RVs, it's more complicated. For example, term insurance

$$L_t|T_x \geq t = \begin{cases} Sv^{T_{x+t}} - P\bar{a}_{\overline{T_{x+t}}|} & T_{x+t} \leq n \\ 0 - P\bar{a}_{\overline{n}|} & T_{x+t} > n \end{cases}$$

deferred

$$L_t|T_x \geq t = \begin{cases} 0 - P\bar{a}_{\overline{T_{x+t}}|} & T_{x+t} < n-t \\ v^{T_{x+t}} - P\bar{a}_{\overline{n-t}|} & T_{x+t} \geq n-t \end{cases}$$

Then for $Var(L_t|T_x \geq t)$ we could need $Var(PV$ benefits$), Var($ PV Premiums) and their covariance

$$L_t = SZ_t - PY_t | T_x \geq t$$

Then

$$Var(L_t|T_x \geq t) = S^2 Var(Z_t|T_x \geq t) + P^2 Var(Y_t|T_x \geq t) - 2SPCov(Z_t, Y_t|T_x \geq t)$$

$$Cov(Z_t, Y_t) = E[Z_t Y_t|T_x \geq t] - E[Z_t|T_x \geq t]E[Y_t \geq t]$$

We can now find the policy value (mean of L_t) or any payment date (integer t for annual, multiple of $\frac{1}{m}$ for $\frac{1}{m}$ly, and any real t for continuous contract. But what about in between payment date? No new principles! Still PV at time t of future $benefits - premiums$. But $A_{x:\overline{t}|}$ or $a_{x:\overline{t}|}$ don't exist if $t \notin \mathbb{Z}$ nor do $A_{x:\overline{t}|^{(m)}}$ or $a^{(m)}_{x:\overline{t}|}$ for t not a multiple of $\frac{1}{m}$.

We can use the policy value at the nearest payment date. Two approaches

1. start with ${}_{t+1}V$, discount back $(1-s)$ of a year for interest and survival; adjust for any income or outgo due to events in $(t+s, t+1]$.

$$_{t+s}V = {}_{t+1}V v^{1-s} {}_{1-s}p_{x+t+s} + Sv^{1-s} {}_{1-s}q_{x+t+s}$$

2. start with ${}_tV$ and accumulate for S of a year for interest and survival and adjust due to events in $(t, t+s]$

$$_{t+s}V = \frac{({}_tV + P_t)}{{}_sE_{x+t}} - \frac{Sv^{1-s} {}_sq_{x+t} v^{1-s}}{{}_sp_{x+t}}$$

equivalently

$$_{t+s}V = \frac{({}_tV + P_t)(1+i)^s}{{}_sp_{x+t]}} - \frac{S {}_sq_{x+t} V^{1-s}}{{}_sp_{x+t}}$$

In both cases we use an adjacent policy value, bring it to the correct time, and adjust for what did/did not happen in the time between.

Example

Use illustrative life table 6%. Whole life insurance for (40). Fully discrete $S = 1000$. $P = \frac{1000A_{40}}{\ddot{a}_{40}}$, Find ${}_{20.25}V$.

${}_{21}V = 1000A_{61} - P\ddot{a}_{61} = 264.061$, ${}_{20}V = 247.78$.

$$_{20.25}V = 264.061v^{0.75} {}_{0.75}p_{60.25} + {}_{0.75}p_{60.25} 1000v^{0.75} = 260.065$$

Under UDD ${}_{0.75}p_{60.25} = \frac{p_{60}}{{}_{0.25}p_{60}} = \frac{1-0.01376}{1-0.25\times 0.0376} = 0.989644$ and ${}_{0.75}q_{60.25} = 0.010356$

or

$$_{20.25}V = \frac{({}_{20}V + P) - {}_{0.25}q_{60} 1000v}{v^{0.25} {}_{0.25}p_{60}} = 260.065$$

Under UDD ${}_{0.25}p_{60} = 1 - 0.25 \times 0.01376 = 0.99656$ and ${}_{0.25}q_{60} = 0.00344$

${}_{20}V$ assumes that the P at time 20 is in the future ${}_{20+\epsilon}V$ assumes it is in the past. Over an entire whole life contract, the overall trend is increasing but there are discontinuities at every payment date.

We can approximate ${}_{t+s}V$ by using interpolation between ${}_{t}V$ and ${}_{t+1}V$. But we need to take into account the discontinuities caused by the premiums

$$({}_{t}V + P)(1-s) + ({}_{t+1}V)s = {}_{t+s}V$$

In our example,

$$ {}_{20.25}V \approx 0.75({}_{20}V + P) + 0.25({}_{21}V) = 260.016$$

For more accuracy, we can also incorporate interest in our interpolation.

$$ {}_{t+s}V = ({}_{t}V + P)(1+i)^s(1-s) + ({}_{t+1}V v^{1-s})s$$

In our example

$$ {}_{20.25}V = 0.75({}_{20}V + P)(1.06)^{0.25} + 0.25({}_{21}V)(\frac{1}{1.06})^{0.75} = 260.04$$

2.2 Thiele Differential Equation

We can calculate ${}_{t}V$ for any t in continuous integer t in annual $t + \frac{k}{m}$ in $\frac{1}{m}$ly. Any time t in annual or $\frac{1}{m}$ly, we also have 2 approximations. Discrete contracts have discontinuities due to payments, but not for the continuous case. We can derive a differential equation to find the rate of change of ${}_{t}V$. We can then use the DE to approximate the change in ${}_{t}V$ for a small interval $(t, t+dt)$.

Use the principle that ${}_{t}V =$ EPV of benefits − premiums

$$ {}_{t}V = \int_0^\infty (S_{t+n} + E_{t+n}) e^{-\int_t^{t+u}\delta_s ds} {}_{u}p_{x+t}\mu_{x+t+u} du - \int_0^\infty (P_{t+u} - e_{t+u}) e^{-\int_t^{t+u}\delta_s ds} {}_{u}p_{x+t} du$$

Note this allows δ to be a function of time if $\delta_s = \delta, \forall s$ then it's $e^{-\delta u}$. But if not, $e^{-\int_t^{t+u}\delta_s ds} = \frac{v(t+u)}{v(t)}$. Substitute $r = t+u$ and $du = dr$, $t = r-u$. $u = 0 \rightarrow r = t, u = \infty \rightarrow r = \infty$.

$$ {}_{u}p_{x+t} : {}_{t}p_{x\,u}p_{x+t} = {}_{t+u}p_x = {}_{r}p_x \text{ so } {}_{u}p_{x+t} = \frac{{}_{r}p_x}{{}_{t}p_x}$$

$$ {}_{t}V = \int_t^\infty ((S_r + E_r)\mu_{x+r} - (P_r - e_r))\frac{v(r)}{v(t)}\frac{{}_{r}p_x}{{}_{t}p_x} dr = \frac{1}{v(t){}_{t}p_x}(\int_t^\infty ((S_r + E_r)\mu_{x+r} - (P_r - e_r))v(r){}_{r}p_x dr$$

Differentiate ${}_{t}V v(t){}_{t}p_x$ with respect to t and it will be equal to

$$ v(t){}_{t}p_x(p_t - e_t - (S_t + E_t)\mu_{x+t})$$

If we assume compound interest, $v(t) = e^{-\delta t} = v^t$

$$\frac{d}{dt} e^{-\delta t} {}_tp_{xt}V = (-e^{-\delta t} {}_tp_x(\delta + \mu_{x+t}))_tV + e^{-\delta t} {}_tp_x(\frac{d}{dt} {}_tV)$$

equate, cancel out $e^{-\delta t} {}_tp - x$

$$\frac{d}{dt} {}_tV = \delta_t V + (P_t - e_t) - (S_t + E_t - {}_tV)\, u_{x+t}$$

Above represent the rat eof change of policy value at t. $\delta_t V$ is the amount held x instant interest rate and $(P_t - e_t)$ is the premium income instant rate minus expenses. For last part, if death occurs, provide NAAR to pay benefits. If we also have boundary conditions for the DE (${}_0V$ =?) then we can identify all the contract details. There is a one to one relationship between contracts and Thiele's DE's.

Example:

1. $\frac{d}{dt} {}_tV = -(1000 - {}_tV)\mu_{x+t} + \delta_t V$ with ${}_0V = P$. This is a whole life insurance with single premium of P at time 0.

2. $\frac{d}{dt} {}_tV = -(10000 - {}_tV)\mu_{x+t} + \delta_t V + P$ if $0 \leq t < n$. ${}_tV = 0$ for $t \geq n$ and $\lim_{t \to n^-} {}_tV = {}_{n^-}V = 20000$. continuously paid P throughout n years and benefit of 10,000 if dies before $\overline{n|}$, benefit of 20,000 if they survive.

   Identify DE from contact details and identify contract from DE and boundary conditions. If the De was identical except ${}_{n-}V = 0$, it would be a term insurance contract with 10,000 on death within $\overline{n|}$.

3. $\frac{d}{dt} {}_tV = (S - {}_tV)\mu_{x+t} = \delta_t V - X$ with ${}_0V = P$. There is a single premium paid at time 0 and death benefit of S paid on death. X is an annuity payment.

Using Thiele to approximate ${}_tV$, since a derivative is

$$\frac{d}{dt} {}_tV = \lim_{h \to 0} \frac{{}_{t+h}V - {}_tV}{h}$$

We can choose a small h and then use the right side of the DE to approximate the change in policy value over any interval of length h for a continuous contract.

This gives us:

$$(1 + \delta h)_t V + (P_t - e_t)h \approx {}_{t+h}V + NAAR_t h \mu_{x+t}$$

(more accurate if h is smaller)

It is essentially the continuous recursion for a small step size h. We can isolate ${}_tV$ or ${}_{t+h}V$ (whichever we don't know) and solve. We need a starting point (usually either 0 or time n in a term/endowment insurance contract) and then we can work iteratively to find the ${}_tV$ at any time that is a multiple of h.

Example: \$10000 $\overline{10|}$ endowment insurance contract to (40), 6% interest Usual Makeham mortality

$$\mu_{x+t} = A + Bc^{x+t}$$

where $A = 0.0001, B = 0.00035, c = 1.075$

We know the DE is

$$\frac{d}{dt}{}_tV = \delta_t V + P - (10,000 - {}_tV)\mu_{40+t}$$

$${}_{10^-}V = 10,000$$

where $\delta = \log(1.06)$. However for P, we will set P so that ${}_0V = 0$. Each step,

$$\frac{{}_{t+h}V - {}_tV}{h} \approx \delta_t V + P - (10,000 - {}_tV) \times \mu_{40+t}$$

Solving for ${}_tV$,

$${}_tV = \frac{{}_{t+h}V - Ph + 10,000h \times \mu_{40+t}}{h\delta + 1 + h\mu_{40+t}}$$

2.3 Advanced topic: Asset shares and analysis of surplus

Retrospective policy value(based on an assumptions) is

$${}_tV^R = \frac{{}_0V + EPV \text{ at 0 of premiums in } (0,t) - EPV \text{ at 0 of benefits in } (0,t)}{{}_tE_x}$$

If policy value basis = premium basis and equivalence P is used, then it equals ${}_tV^P =$ EPV of future benefits − EPV of future premium. It represents the amount the insure needs ti have per policy to cover future obligations.

Asset share is similar but based on actual experience in time $(0, t)$ and represents the amount the insurer actually has per policy. If $AS_t < {}_tV$, we have a loss. If $AS_t > {}_tV$, we have a profit. $AS_t = {}_tV$ is possible but it would only happen if experience exactly matched assumptions (unlikely).

Example: Usual Makeham model, 6% interest 15-year deferred discrete whole life insurance of \$100, 000 is issued to (50). Death benefit in deferred period is return of premiums without interest. Expense, 15% of first P and 2% of other premiums and \$100 on payment of benefit.

First, calculate P using equivalence principle

$$\text{EPV premiums} = \text{EPV benefits} + \text{Expense}$$

$$P\ddot{a}_{50:\overline{15|}} = 100,000 - 100_{15|}A_{50} + 13\%P + 2\%P\ddot{a}_{50:\overline{15|}} + P(IA)^1_{50:\overline{15|}} + 100A_{50}$$

Hence $P = 2038.16$. Next find ${}_5V$ (some basis)

- retrospective
- prospective
- recursion

$$ {}_5V = 11,612.70 $$

Retrospective:

$$ \frac{0.98P\ddot{a}_{50:\overline{5}|} - 0.13P - P(IA)^1_{50:\overline{50}|} - 100A^1_{50:\overline{5}|}}{{}_5E_{50}} = 11,612.70 $$

Prospective:

$$ P(IA)^1_{55:\overline{10}|} + SPA^1_{55:\overline{10}|} + 100,000{}_{10}E_{55}A_{65} + 0.02P\ddot{a}_{55:\overline{10}|} + 100A_{55} - P\ddot{a}_{55:\overline{10}|} $$

Recursively:

$$ ({}_0V + P - 0.13P)(1 + 0.06) = q_{50}(P + 100) + P_{50 1}V $$

$$ ({}_1V + P - 0.02P)(1.06) = q_{51}(2P + 100) + p_{51 2}V $$

Keep doing the same process. We will get the value for the year-5 reserve.

Now suppose actual interest was 6%, 5.5%, 6.5%, 6%, 7% per year. Actual expense were 10% of first P, 1% of rest, and \$50 on death. Actual mortality was $q_x = 0.014$ for $x = 50, 51, \cdots, 54$. Then we can calculate $AS_5 = 11,979.98$. In total, profit of 365.28 per policy.

t	amount at $t-1$	policy value at $t-1$	expenses at at $t-1$	acc value at t	amount paid at t	remaining	survivors	AS_t
1	0	$P \times 1$	$10\%P_1 \times 1$	$(a_1 + P_1 - e_1)(1 + 6\%)$	$(P + 50) \times 0.014$	$accv1 - db1$	$1 - 0.014$	$\frac{rem_1}{surv_1}$
2	rem_1	$P \times surv_1$	$1\%P_2$	$(a_2 + P_2 - e_2)(15.5\%)$	$(2P + 50)(surv_1) \times (0.014)$	$accv_2 - db_2$	$surv_1(1 - 0.014)$	$\frac{rem_2}{surv_2}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

We have a surplus of $AS_5 - {}_5V = 365.28$. Interest was higher, expense were lower, mortality was close. Analysis of surplus can tell us how much of the surplus (or loss) is caused by each of: expenses, mortality, interest. The idea is to change one factor at a time from assumed to actual. Order matters!.

$$_tV \text{ (all assumed)} \implies AS_5^{\text{expense}} \text{ (actuall expense assumed, mort, int)}$$
$$\implies AS_5^{\text{mortality expense}} \text{ (actuall expense, mort assumed int)} \implies AS_5$$

$AS_5^{\text{expense}} \implies {}_5V$ = effect of expense and $AS_t^{\text{mortality expense}} - AS_5^{\text{expense}}$ = effect of mortality. Lastly $AS_5 - AS_5^{\text{mortality expense}}$ = effect of interest.

By calculation, 365 surplus = 250 expenses − 10 mortality + 125 interest.

2.4 Advanced topics: Contracts where benefit is a% of $_tV$

Sometimes the benefit in an insurance contract be a percentage of the policy value at the time of death. We can't calculate P directly so we have to use recursion (or Thiele). Example: fully discrete, death benefit is $c_{t+1}V$ for $t = 0, 1, \cdots, n-1$, premium P_t payable at times $t = 0, 1, \cdots, n-1$. Benefit in the very last year is $_nV = S$.

Recursion:

$$({}_tV + P)(1+i) = q_{x+t}(c_{t+1}V) + p_{x+t}({}_{t+1}V) = p^*_{x+t}{}_{t+1}V$$

where $p^*_{x+t} = p_{x+t} + cq_{x+t}$

$$p^*_{x+t}v_{t+1}V - {}_tV = P$$

multiply by $_tE^*_x = v^t p^*_x p^*_{x+1} \cdots p^*_{x+t-1}$

$$v^{t+1}{}_{t+1}p^*_x{}_{t+1}V - {}_tE^*_x{}_tV = P_tE^*_x$$

Sum from $t = 0$ to $n-1$, middle terms cancel.

$$_nE^*_x{}_nV - {}_0E^*_x{}_0V = P\sum_{t=0}^{n-1} v^t{}_tp^*_x$$

then

$$_tV = \frac{{}_0V + P\ddot{a}^*_{x:\overline{n}|}}{{}_tE^*_x}$$

If $_nV$ = known number, then

$$P = \frac{{}_nV{}_nE^*_x}{\ddot{a}^*_{x:\overline{n}|}}$$

If $c = 1$,

$$_tV = \frac{P\ddot{a}_{\overline{t}|}}{v^t} \; (p^*_x = 1)$$

2.5 Advanced topics: policy alternations

Sometime ofter inception, the policyholder may request to

- cancel (lapse) the policy.
- change premium terms
- Change the sum insured
- change the benefit type (from whole life/endowment insurance to something else)

By law, if the policy has been in force for 2^+ years, the insurer must provide some surrender value if the policy is lapsed. The surrender value (or cash value) can be

- agreed upon ahead of time (at $t = 0$)
- some calculation based on the policy value at time t

For example some% of ${}_tV$ ($< 100\%$) possibly fixed expense subtracted. Some % of AS_t. Let the cash value at time t be C_t. If the policy lapses, policyholder receives C_t at t. Otherwise, we ca easily calculate the new (modified) terms of the contract using:

C_t + EPV t of future (modified) premiums = EPV t of future (modified) benefit + expense

Example

Whole life to (40) $S = 10,000$, interest rate in the long term is 6%. No expenses (fully discrete). From the ILT, $A_{40} = 0.161324, A_{60} = 0.369131, A_{65} = 0.4397965, l_{60} = 81880.73, l_{65} = 75339.64$. Find $P = \frac{10{,}000 A_{40}}{\ddot{a}_{40}} = 108.8$. Suppose at time 20, policyholder wants to modify the policy by

1. Surrender
2. stop policy premium but keep whole life
3. turn it into a $\left.5\right|$ deferred annuity of X per year with a 10, 000 death benefit in deferred period, and pay 5 more premium.

The insurer calculates surrender value as $C_t = 90\%$ of ${}_tV - \$100$ expense.

$$_{20}V = 10,000A_{60} - 108.88\ddot{a}_{60} = 2477.8$$

$$C_{20} = 90\% \times 24778 - 100 = 2130.02$$

1. They get \$2130.02 (not great for policyholder, less than contributors)

2. $C_{20} + 0 = S' A_{60} + 0$ (have now + future premiums = future benefit + future expense)

$$\therefore S' = \frac{C_{20}}{A_{60}} = 5770.36$$

3. $C_{20} + 108.88\ddot{a}_{60:\overline{5}|} = 10,000A^1_{60:\overline{5}|} + X\,{}_{5|}\ddot{a}_{60} + 0$

$$C_{20} = 108.88(\ddot{a}_{60} - {}_5E_{60}\ddot{a}_{65}) = 10,000(A_{60} - {}_5E_{60}A_{65}) + X({}_5E_{60}\ddot{a}_{65})$$

Solving for X, we get 284.39.

2.6 Review for Test 1

- time t loss rv L_t = PV at t of future benefit − premiums
- $E[L_t|T_x > t] = {}_tV$ = EPV at t of benefits − premiums
- gross v.s. net policy value. If the policy value basis = Premium basis and P is equivalence principle premiums, then net policy value has simplified calculation ${}_tV = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_t}$
- Prospective v.s. Retrospective

$$ {}_tV = \frac{\text{EPV of premium (0, t)} - \text{EPV of benefit (0, t)}}{{}_tE_x}$$

equal to prospective if same basis and equivalence principle

- recursive equation

$$({}_tV + P_t - e_t)(1 + i) = q_{x+t}(S_{t+1} + E_{t+1}) + p_{x+t}({}_{t+1}V)$$

or

$$({}_tV + P_t - e_t)(1 + i) = {}_{t+1}V + NAAR_{t+1}q_{x+t}$$

where $NAAR_{t+1} = S_{t+1} + E_{t+1} - {}_{t+1}V$ is mortality risk. Some principles if $\frac{1}{m}$ly contract.

- Values between payment dates.
  - accurate method
  - linear interpolation with or without interest
- Continuous contracts: same principles for everything. easier to find the distance of L_t differential equation instead of recursion

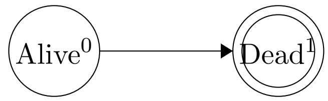
$$\frac{d}{dt}{}_tV = \delta_tV + (P_t - e_t) - NAAR_t\mu_{x+t}$$

use as approximation for a small h:

$$\frac{{}_{t+h}V - {}_tV}{h} = \cdots \text{same RS}$$

3 Multiple state models

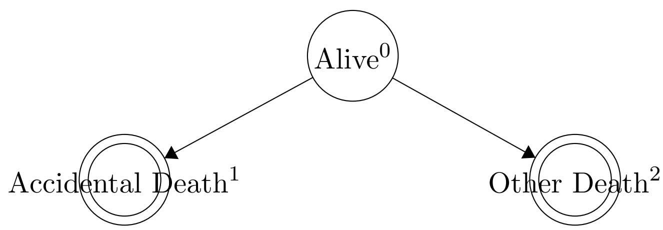
We are used to the model



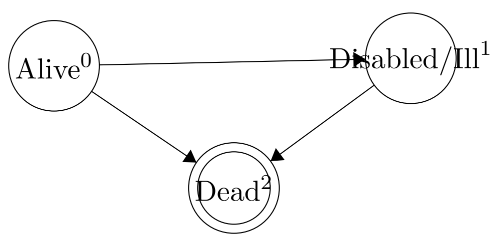
We use the RV T_x = time until death of (x). But really, we have a continuous process $Y(t) = \begin{cases} 0 & \text{if (x) is alive at time t} \\ 1 & \text{if (x) is dead at time t} \end{cases}$. Then $T_x = \max\{t : Y(t) = 0\}$ (i.e. latest time t where $Y(t)$ is still in the "alive" state). In general, there are policies that cannot be modelled with 2 states.

Examples:

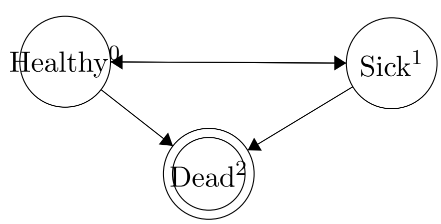
1. Insurance which pays a different amount for "accidental" death than other death. We need



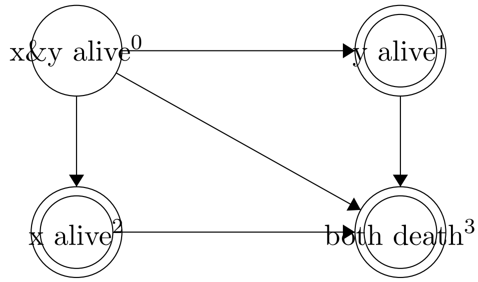
2. Disability (permanent)/ Critical illness insurance pays a lump sum benefit on becoming permanently disabled or diagnosis. We need



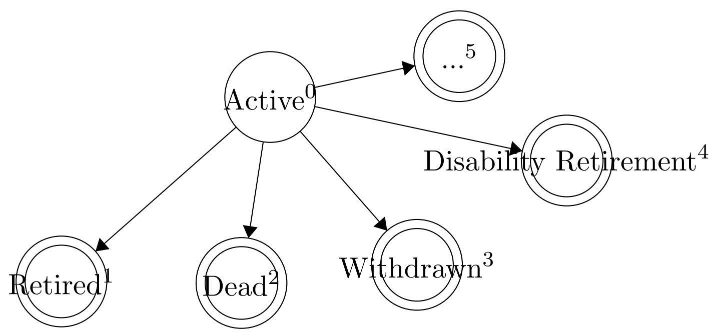
3. Disability income policy may pay (non-permanent) or annuity while policyholder is disabled. We need



4. Joint life insurance that depends on the survival or death of more than one policy-holders. We need



5. Pension plan - employees may entitled to different benefits base on how they exit the plan. We need



Assumptions

1. Always in star 0 when the policy is purchased. All possibilities are counted in the sates of the model. (i.e. it's not possible to leave the model entirely). Can't be in 2 states at once. That means at any time t, $Y(t) = \{0, 1, \cdots\}$.

2. Markov property (future movements of $Y(t)$ depend only on the present state, not on the history before the presents). In some cases, this assumption may not be realistic.

3. $P(2$ or more transitions in $(t, t+h)) = o(h)$ (a function is $o(h)$ if $\lim_{h\to 0} \frac{o(h)}{h} = 0$). can't have 2 transitions at exactly the same time.

4. time until transition for all states is differentiable functions.

3.1 Notations

$$\begin{aligned} {}_tp_x^{ij} &= P(Y(x+t)=j|Y(x)=i) \\ &= P(\text{policyholders in state j at age } x+t \text{ given that they were in state i at x}) \\ i &\text{ can be } =j \text{ or } \neq j \end{aligned}$$

$$\begin{aligned} {}_tp_x^{\overline{ii}} &= P(Y(x+s)=i, 0\leq s\leq t|Y(x)=i) \\ &= P(\text{policyholder stays in state i throughout age } x\to x+t \text{ given in i at x }) \\ &\text{depending on the model and the nature of state i, } {}_tp_x^{\overline{ii}} \text{ may or may not } = {}_tp_x^{ii} \end{aligned}$$

They would be equal if

- It is impossible to leave state i (i is an absorbing state e.g. death), then ${}_tp^{\overline{ii}}$ and ${}_tp_x^{ii}=1$.
- It is impossible to re-enter state i after leaving it.

Hazard Rate:

$$\begin{aligned} \mu_x^{ij} &= \lim_{h\to 0^+}\frac{{}_tp_x^{ij}}{h} \text{ for } i\neq j \\ &= \text{force of transition from state i to state j at age x} \end{aligned}$$

In the alive-dead model, ${}_tp_x^{00} = {}_tp_x = {}_tp_x^{\overline{00}}$; ${}_tp_x^{01} = {}_tq_x$; ${}_tp_x^{11} = 1 = {}_tp_x^{\overline{11}}$; ${}_tp_x^{10} = 0$; $\mu_x^{01}=\mu_x$. At any time t, $Y(t)$ must equal one of its states, so $\sum_j^n {}_tp_x^{ij}=1$ for any i.

Lemma 1. ${}_np_x^{ij} = h\mu_x^{ij}+o(h)$

Proof. $\mu_x^{ij} = \frac{{}_np_x^{ij}}{h} = \frac{{}_hp_x^{ij}-o(h)}{h}$ since $\frac{o(h)}{h}\to 0$, so $h\mu_x^{ij} = {}_np_x^{ij}-o(h)$ so for a small h, the probability of going from i to j in a time interval of length h is $h\mu_x^{ij}$ plus some error. □

Lemma 2. ${}_tp_x^{ii} = {}_tp_x^{\overline{ii}}+o(t)$.

Proof.

$$\begin{aligned} {}_tp^{ii} &= P(\text{ in i at age } x+t|\text{in i at x}) \\ &= P(\text{stay in i}|\text{in i at x}) + P(\text{leave and not come back}|\text{in i at x}) \\ &= {}_tp_x^{\overline{ii}} + P(2^+ \text{ transitions in } (0,t)) \\ &= {}_tp_x^{\overline{ii}} + o(t) \end{aligned}$$

If we cannot leave i or cannot reenter i, then we don't need the $o(t)$ since ${}_tp_x^{ii} = {}_tp_x^{\overline{ii}}$ □

Lemma 3. ${}_hp_x^{\overline{ii}} = 1 - h\sum_{j\neq i}\mu_x^{ij} + o(h)$ where $\sum_{j\neq i}\mu_x^{ij}$ is the total force of exit from i.

Proof.

$$\begin{aligned}
1 - {}_hp_x^{\overline{ii}} &= 1 - P(\text{do not leave i} \mid \text{in i at x}) \\
&= P(\text{do leave i by } x+t | \text{in i at x}) \\
&= \sum_{j\neq i} {}_hp_x^{ij} + P(\text{leave and come back}|\text{in i at x}) \\
&= \sum_{j\neq i}[h\mu_x^{ij} + o(h)] + P(2^+ \text{ transitions in } (0,h)) \\
&= h\sum_{j\neq i}\mu_x^{ij} + o(h)
\end{aligned}$$

so ${}_hp_x^{\overline{ii}} = 1 - h\sum_{j\neq i}\mu_x^{ij} + o(h)$. □

$\sum_{j\neq i}\mu$ represents the total force of transition at of state i at age x.

Recall in the 2-state model that

$${}_tp_x = e^{-\int_0^t \mu_{x+r}dr}$$

Similarly for MSM, to start, Notice

$$\begin{aligned}
{}_{t+n}p_x^{\overline{ii}} &= {}_tp_x^{\overline{ii}}\,{}_hp_{x+t}^{\overline{ii}} \\
{}_{t+h}p_x^{\overline{ii}} &= {}_tp_x^{\overline{ii}}(1 - h\mu_{x+t}^{i\cdot} + o(h) \\
\frac{{}_{t+h}p_x^{\overline{ii}} - {}_tp_x^{\overline{ii}}}{h} &= -{}_tp_x^{\overline{ii}}\mu_{x+t}^{i\cdot} + \frac{{}_tp_x^{\overline{ii}}o(h)}{h} \\
\text{Let } h &\to 0 \\
\frac{d}{dt}{}_tp_x^{\overline{ii}} &= -{}_tp_x^{\overline{ii}}\mu_{x+t}^{i\cdot} + 0 \\
\text{using the chain rule} \\
\frac{\frac{d}{dt}{}_tp_x^{\overline{ii}}}{{}_tp_x^{\overline{ii}}} &= \frac{d}{dt}(\log({}_tp_x^{\overline{ii}})) = -\mu_{x+t}^{i\cdot}
\end{aligned}$$

integrate from 0 to t and exponentiate

$${}_tp_x^{\overline{ii}} = e^{-\int_0^t \mu_{x+r}^{i\cdot}dr + c}$$

but since ${}_0p_x^{\overline{ii}} = 1$, so $c = 0$.

Make sure when compared to the 2-state model because with ${}_tp_x^{\overline{ii}}$, we only care about i and "not i".

3.2 Kolmogorov forward equations

In a general MSM for any two states i and j (where i can equal to j)

$$ {}_{t+h}p_x^{ij} = {}_tp_x^{ij}\,{}_hp_{x+t}^{ij} + \sum_{k\neq j} {}_tp_x^{ik}\,{}_hp_{x+t}^{kj} $$

Apply those lemmas

$$ \begin{aligned} {}_{t+h}p_x^{ij} &= {}_tp_x^{ij}({}_hp_x^{\overline{jj}} + o(h)) + \sum_{k\neq j} {}_tp_x^{ik}(h\mu_{x+t}^{kj} + o(h)) \\ &= {}_tp_x^{ij}(1 - h\mu_{x+t}^{j\cdot} + o(h)) + h\sum_{k\neq j}({}_tp_x^{ik}\,{}_x\mu_{x+t}^{kj} + o(h)) \\ &= {}_tp_x^{ij} - h{}_tp_x^{ij}\,{}_x\mu_{x+t}^{j\cdot} + h\sum_{k\neq j} {}_tp_x^{ik}\mu_{x+t}^{kj} + o(h) \\ \frac{{}_{t+h}p_x^{ij} - {}_tp_x^{ij}}{h} &= \frac{-h{}_tp_x^{ij}\mu_{x+t}^{j\cdot}}{h} + \frac{h\sum_{k\neq j} {}_tp_x^{ik}\mu_{x+t}^{kj}}{h} + \frac{o(h)}{h} \\ \frac{d}{dt}{}_tp_x^{ij} &= -{}_tp_x^{ij}\mu_{x+t}^{j\cdot} + \sum_{k\neq j} {}_tp_x^{ik}\mu_{x+t}^{kj} \end{aligned} $$

but $\mu_{x+t}^{j\cdot} = \sum_{k\neq j}\mu_{x+t}^{jk}$ by definition so we can write

$$ \frac{d}{dt}{}_tp_x^{ij} = \sum_{k\neq j}({}_tp_x^{ik}\mu_{x+t}^{kj} - {}_tp_x^{ij}\mu_{x+t}^{jk}) $$

where this measures the change in probability i to j; for ${}_tp_x^{ik}\mu_{x+t}^{kj}$, it measures that go somewhere other than j and then transition to j; for ${}_tp_x^{ij}\mu_{x+t}^{jk}$, it goes to j but transition elsewhere at last second.

Let's consider the healthy-disabled-and-death example.

$$ {}_{t+h}p_x^{01} = {}_tp_x^{01}\,{}_hp_{x+t}^{11} + {}_tp_x^{00}\,{}_hp_{x+t}^{01} $$

Since 0 and 1 cannot be re-entered once left,

$$ {}_tp_x^{00} = {}_tp_x^{\overline{00}} $$

and

$$ {}_tp_x^{11} = {}_tp_x^{\overline{11}} $$

so

$$ {}_{t+h}p_x^{01} = {}_tp_x^{01}\,{}_hp_{x+t}^{\overline{11}} + {}_tp_x^{\overline{00}}(h\mu_{x+t}^{01} + o(h)) $$

$$ {}_{t+h}p_x^{01} = {}_tp_x^{01}(e^{-\int_0^h \mu_{x+t+r}^{12}dr}) + (e^{-\int_0^t \mu_{x+r}^{0\cdot}dr})h\mu_{x+t}^{01} + o(h) $$

where $\mu_x^{i\cdot} = \mu_x^{12}$ and $01 + 01$.

$$ \frac{d}{dt}{}_tp_x^{02} = \sum_{k=0}^{1}({}_tp_x^{0k}\mu_{x+t}^{k2} - {}_tp_x^{02}\mu_{x+t}^{2k}) $$
$$ = {}_tp_x^{00}\mu_{x+t}^{02} + {}_tp_x^{01}\mu_{x+t}^{12} $$

$$ \frac{d}{dt}{}_tp_x^{11} = \sum_{k=0,k\neq 1}^{2}({}_tp_x^{1k}\mu_{x+t}^{k1} - {}_tp_x^{11}\mu_{x+t}^{1k}) = -{}_tp_x^{11}\mu_{x+t}^{12} $$

but ${}_tp_x^{10} = \mu_x^{21} = 0, \mu_x^{10} = 0$.

Similarly, we can find the KFEs for any probability and solve the DEs simultaneously for the ${}_tp_x^{ij}$'s. This is double but tedious. So we can use another approach $\Longrightarrow$ condition on the time of first transition (say r) and then integrate r from 0 to t.

Let 0 to 1 be $\sigma = 0.05$, 1 to 2 be $\nu = 0.01$ and 0 to 2 be $\mu = 0.02$. Constant force do not depend on age x.

${}_tp_x^{00} = {}_tp_x^{\overline{00}} = e^{-\int_0^t(\sigma+\mu)dr} = e^{-t(\sigma+\mu)} = e^{-0.07t}$

${}_tp_x^{11} = {}_tp_x^{\overline{11}} = e^{-t\nu} = e^{-0.01t}$

Obviously, ${}_tp_x^{22} = 1$ and ${}_tp_x^{20}, {}_tp_x^{21}$, and ${}_tp_x^{10} = 0$. Since at time t, the policyholder must be in some state $\{0,1,2\}$, ${}_tp_x^{12} = 1 - e^{-0.01t}$

$$ {}_tp_x^{01} = \int_0^t {}_rp_x^{00}\sigma_{t-r}p_{x+r}^{11}dr = \int_0^t e^{-0.07r}0.05e^{-0.01(t-r)}dr = 0.05e^{-0.01t}\int_0^t e^{-0.06r}dr = \frac{5}{6}(e^{-0.01t} - e^{-0.07t}) $$

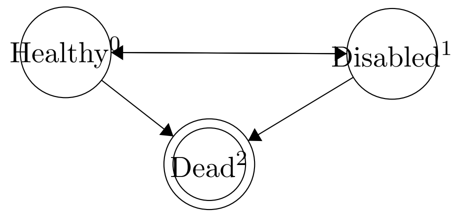
so again since we must be in some state at time t, ${}_tp_x^{00} + {}_tp_x^{01} + {}_tp_x^{02} = 1$ so ${}_tp_x^{02} = 1 - e^{-0.07t} - \frac{5}{6}(e^{-0.01t} - e^{-0.07t})$ or we could calculate ${}_tp_x^{02}$ directly ${}_tp_x^{02} = P(0 \text{ to } 2 \text{ directly}) + P(0 \text{ to } 1 \text{ to } 1) = \int_0^t {}_rp_x^{\infty}\mu \cdot 1dr + P(0 \to 1 \to 2)$.

$$ P(0 \to 1 \to 2) = \int_0^t {}_rp_x^{00}\sigma_{t-r}p_{x+r}^{12}dr \text{ or } \int_0^t {}_rp_x^{01}\nu \cdot 1dr $$

both integrals must evaluate to the same thing.then $P(0 \to 2 \text{ directly }) + P(0 \to 1 \to 2)$ adds up to the same thing we had before for ${}_tp_x^{02}$.

With some models, we can get expressions for ${}_tp_x^{1j}$ by conditioning on the time of first transition. We can find an analytical expression in terms of t if the transition forces are constant, but with more complex models and or transition forces that depends on $x + t$, the KFEs give us valuable info.

Example:



because we can renter state 0 and 1.

$$ {}_tp_x^{00} \neq {}_tp_x^{\overline{00}} $$

$$ {}_tp_x^{11} \neq {}_tp_x^{\overline{11}} $$

then ${}_tp_x^{01} = \int_0^t {}_sp_x^{\overline{00}}\mu_{x+s}^{01}{}_{t-s}p_{x+s}^{11}ds = \int_0^t {}_rp_x^{00}\mu_{x+r}^{01}{}_{t-r}p_{x+r}^{\overline{11}}dr$

Instead we use KFEs

$$ \frac{d}{dt}{}_tp_x^{01} = \sum_{k=0,\neq 1}^{2}({}_tp_x^{0k}\mu_{x+t}^{k1} - {}_tp_x^{01}\mu_{x+t}^{1k}) = {}_tp_x^{00}\mu_{x+t}^{01} - {}_tp_x^{01}(\mu_{x+t}^{10} + \mu_{x+t}^{12}) $$

Similarly, we can obtain the KFEs for other ${}_tp_x^{ij}$'s and solve. We can use the KFs and a small time step h to approximate ${}_tp_x^{ij}$'s.

(Why? If mortality was Makeham or other models that depend on age, we could need to use numerical integration to evaluate the integrals) We have

$$ \frac{d}{dt}{}_tp_x^{ij} = \sum_{k\neq j}({}_tp_x^{ik}\mu_{x+t}^{k+j} - {}_tp_x^{ij}\mu_{x+t}^{jk}) $$

approximate with $\frac{{}_{t+h}p_x^{ij} - {}_tp_x^{ij}}{h}$ for some h so

$$ {}_{t+h}p_x^{ij} = {}_tp_x^{ij} + h\sum_{k\neq j}({}_tp_x^{ik}\mu_{x+t}^{kj} - {}_tp_x^{ij}\mu_{x+t}^{jk}) $$

Start at $t = 0$, ${}_0p_x^{ii} = 1$ and ${}_0p_x^{ij} = 0$ for $j \neq i$. then we can find ${}_hp_x^{ij}$ if we have the μ_x^{ij}'s. Hence we can get ${}_{2h}p_x^{ij}$ from those ${}_hp_x^{ij}$'s and the μ_{x+h}^{ij}'s, and then for $3h, 4h, \cdots$.

Smaller h $\implies$ more accurate, well $h = \frac{1}{12}$ gives decent accuracy.

3.3 Benefits in MSM

$\bar{a}_x^{ij}$ = EPV of an annuity that pays 1 per year payable continuously whenever in state j, given that they start at age x in state i.

$\bar{A}_x^{ij}$ = EPV of 1 paid at the instant of each transition into state j, given that they start at age x in state i.

We could modify these benefits by making them discrete (annual or $\frac{1}{m}$ly) or add a term of $\overline{n|}$ to either benefit (no payments after time n) or add require payment only on the first transition to j for A or only on the first visit to j (for a).

To calculate the EPV of any benefit in a MSM, we use the same principle

$$\int (\sum)\text{amt paid at t} \times \text{discount factor} \times \text{prob of payment at t}$$

$$\bar{a}_x^{ij} = \int_0^\infty e^{-\delta t} {}_tp_x^{ij} dt$$

$$A_{x:\overline{n|}}^{1\ \ ij} = \sum_{k=0}^{n-1} v^k \sum_{i \neq j} {}_kp_x^{il} p_{x+k}^{lj}$$

$$\bar{A}_{x:\overline{n|}}^{1\ \ ij} = \int_0^n e^{-\delta t} \sum_{l \neq j} ({}_tp_x^{il} \mu_{x+t}^{lj}) dt$$

In both cases for A, we start in i but the transition into j (that triggers payment) can be from any state $l \neq j$.

3.4 Premiums a Policy Values in MSM

Some principles

- set P so EPV premiums = EPV benefits + expenses (at inception)
- ${}_tV$ is EPV of future benefits + expense − EPV of future premiums (at time t).

In alive and dead, ${}_tV$ was conditional on the policy being in force at t. Which is just being alive at age $x + t$, but with MSM, there may be more than one way for a policy to be in force at time t. so ${}_tV^{(1)}$ = EPV benefits − premiums at t, given policyholder is in state i at t.

${}_tV^{(i)} = 0$ if i is an absorbing state such as dead.

Example

consider the health, disable and dead model again. From 0 to 1, $\sigma = 0.05$; from 1 to 2, $\nu = 0.1$; from 0 to 2, $\mu = 0.02$ for age x to $x + 20$.

We had expressions for all ${}_tp_x^{ij}$'s. Contract: premiums P payable continuously while healthy; annuity of B payable while disabled.

death benefit $S = 100,000$ payable on death from either state 0 or 1. All 20 year term. $\delta = 6\%$ no expense.

Calculate P:

$$P\bar{a}_{x:\overline{20|}}^{00} = B\bar{a}_{x:\overline{20|}}^{01} + S\bar{A}_{x:\overline{20|}}^{1\ \ 02}$$

$$\bar{a}^{00}_{x:\overline{20}|} = \int_0^{20} e^{-\delta t} {}_tp^{00}_x dt = \int_0^{20} e^{-0.06t} e^{-0.07t} dt = 7.120972$$

$$\bar{a}_{x:\overline{20}|} = \int_0^{20} e^{-\delta t} {}_tp^{01}_x dt = \frac{5}{3} \int_0^{20} e^{-0.06t}(e^{-0.07t} - e^{-0.1t}) dt = 1876227$$

$$\bar{A}^{1}_{x:\overline{20}|}{}^{02} = \int_0^{20} e^{-\delta t}({}_tp^{00}_x \mu^{02}_{x+t} + {}_tp^{01}_x \mu^{12}_{x+t}) dt = 0.330042$$

so

$$P = \frac{B\bar{a}^{01}_{x:\overline{20}|} + S\bar{A}^{1}_{x:\overline{20}|}{}^{02}}{\bar{a}^{00}_{x:\overline{20}|}} = 7269.58$$

at time 10, calculate ${}_tV^{(i)}$ for all i.

$${}_{10}V^{(2)} = 0$$

$${}_{10}V^{(0)} = B\bar{a}^{01}_{x+10:\overline{10}|} + S\bar{A}^{1}_{x+10:\overline{10}|}{}^{02} - P\bar{a}^{00}_{x+10:\overline{10}|}$$

$${}_{10}V^{(1)} = B\bar{a}^{11}_{x+10:\overline{10}|} + S\bar{A}^{1}_{x+10:\overline{10}|}{}^{12}$$

$$\bar{a}^{11}_{x+10:\overline{10}|} = \int_0^{10} e^{\delta t} {}_tp^{11}_{x+10} dt = 4.988147$$

$$\bar{A}^{1}_{x+10:\overline{10}|}{}^{12} = \int_0^{10} e^{-\delta t} {}_tp^{11}_{x+10} \mu^{12}_{x+10+t} dt = 0.4988147$$

$${}_{10}V^{(0)} = -9229.35$$

$${}_{10}V^{(1)} = 99762.94$$

Huge difference in ${}_{10}V$ depending on the state of the policyholder healthy lives are subsidizing disabled lives.

3.5 Thiele DE for MSM

$$\frac{d}{dt}{}_tV^{(i)} = \delta_t V^{(i)} + (P^{(i)}_t - e^{(i)}_t) - B^{(i)}_t - \sum_{j \neq i} \mu^{ij}_{x+t}(S^{(ij)} + {}_tV^{(j)} - {}_tV^{(i)})$$

In our example we have from last time,

$$\frac{d}{dt}{}_tV^{(0)} = \delta_t V^{(0)} + 7269.58 - 0 - \mu^{01}_{x+t}(0 + {}_tV^{(1)} - {}_tV^{(0)}) - \mu^{02}_{x+t}(100,000 + 0 - {}_tV^{(0)})$$

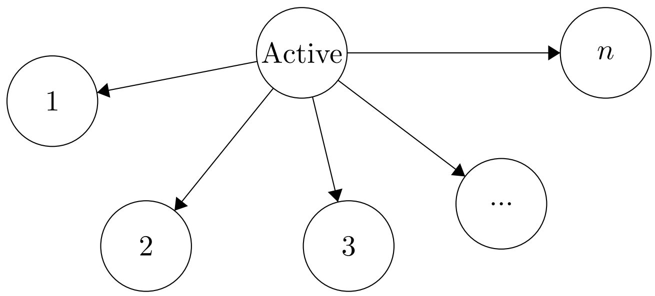
$$\frac{d}{dt}{}_tV^{(1)} = \delta_t V^{(1)} + 0 - 10,000 - \mu^{12}_{x+t}(100,000 + 0 - {}_tV^{(1)})$$

Just like before, we might be able to solve analytically for ${}_tV^{(i)}$'s, or we can use a small time interval h and use the DEs to approximate. Boundary conditions usually at the end of contract.

4 Multiple decrement model, multiple life model

4.1 Multiple Decrement Models (MDM)

Special case of MSM, one active state (state 0) and n absorbing states $(1, \cdots, n)$. So only one transition (max) can occur!



At time t, policyholder must still be in 0 or have made one transition to j $(1, 2, \cdots, n)$ and then still be in d.

$$ {}_tp_x^{00} = {}_tp_x^{\overline{00}} = e^{-\int_0^t \mu_{x+r}^{0\cdot} dr} = e^{-\int_0^t \sum_{j=1}^n \mu_{x+t}^{0j} dr} $$

$$ {}_tp_x^{0j} = \int_0^t {}_sp_x^{00} \mu_{x+s}^{0j} ds $$

and as always, ${}_0p_x^{ii} = 1, {}_0p_x^{ij} = 0$

4.1.1 KFEs for a MDM

$$ \frac{d}{dt} {}_tp_x^{00} = -\sum_{i=1}^n {}_tp_x^{00} \mu_{x+t}^{0i} $$

(we can solve this DE to get ${}_tp_x^{00} = e^{-\int_0^t \sum_{j=1}^n \mu_{x+r}^{0j} dr}$).

$$ \frac{d}{dt} {}_tp_x^{0j} = {}_tp_x^{00} \mu_{x+t}^{0j} $$

We define ${}_tp_x^{0\cdot} = 1 - {}_tp_x^{00} = \sum_{j=1}^n {}_tp_x^{0j}$ (not in 0 at t) (similar to $\mu_{x+t}^{0\cdot} = \sum_{j=1}^n \mu_{x+t}^{0j}$)

${}_tp_x^{0\cdot}$ is the total probability of leaving active state by t. if the transition forces are simple enough, we can evaluate the integrals and get exact expressions ${}_tp_x^{0j}$. But if we want accuracy, we can use a life table just like in alive $\to$ dead.

Recall: x_0 = starting age(integer), l_{x_0} = number of starting lives, l_x = number of still alive at age x = $l_{x_0 x-x_0}p_{x_0}$ and $d_x = l_x - l_{x+1}$

From the MDM case, we need

$$ l_x = \text{avg number of still in state 0 at age x} = l_{x_0}({}_{x-x_0}p_{x_0}^{00}) $$

$$d_x^{(j)} = l_x p_x^{0j} = \text{number who go to j in age x and } x+1 = l_{x_0}({}_{x-x_0|}p_{x_0}^{0j})$$

Then $d_x = \sum_{j=1}^n d_x^{(j)} = l_x - l_{x+1}$. We need a fractional age assumption to use the table.

1. UDD: Assume that for each decrement in the table, ${}_tp_x^{0j} = tp_x^{0j}, 0 \le t \le 1$ for $j = 1, 2, \cdots, n$.

2. CFT: Assume that each force of transition within each year is constant.

$$\mu_{x+t}^{0j} = \mu_x^{0j}, 0 \le t < 1$$

for $j = 1, 2, \cdots, n$. With CFT,

$$\mu_{x+t}^{0\cdot} = \sum_{j=1}^{n} \mu_{x+t}^{0j} = \sum_{j=1}^{n} \mu_x^{0j}, 0 \le t < 1 = \mu_x^{0\cdot}$$

Thus, ${}_tp_x^{0\cdot} = 1 - {}_tp_x^{00} = 1 - e^{-\int_0^t \mu_{x+r}^{0\cdot} dr} = 1 - e^{-t\mu_x^{0\cdot}}$

In particular, $p_x^{0\cdot} = 1 - e^{-\mu_x^{0\cdot}}$. Consider

$${}_sp_x^{0j} = \int_0^s {}_rp_x^{00} \mu_{x+r}^{0j} dr = \int_0^s e^{-r\mu_x^{0\cdot}\mu_x^{0j} dr} = \frac{\mu_x^{0j}}{\mu_x^{0\cdot}}(1 - e^{-s\mu_x^{0\cdot}})$$

$0 \le s \le 1, j = 1, \cdots, n$

But if we let $e^{-s\mu_x^{0\cdot}} = {}_sp_x^{00}$ from before, so we have ${}_sp_x^{0j} = \frac{\mu_x^{0j}}{\mu_x^{0\cdot}}(1 - ({}_1p_x^{00})^s)$ (logically, it makes sense to be proper trivial to the force from 0 to j)

If we let $s = 1$, we get

$${}_tp_x^{0j} = \frac{\mu_x^{0j}}{\mu_x^{0\cdot}}(1 - p_x^{00})$$

$$p_x^{0j} = \frac{\mu_x^{0j}}{\mu_x^{0\cdot}}(p_x^{0\cdot})$$

So finally,

$${}_sp_x^{0j} = \frac{p_x^{0j}}{p_x^{0\cdot}}(1 - (p_x^{00})^s)$$

We can obtain $p_x^{0j}, p^{0\cdot}$ and p_x^{00} from the MD table since they are annual.

$$p_x^{00} = \frac{l_{x+1}}{l_x}, p_x^{0\cdot} = 1 - p_x^{00} = \frac{l_x - l_{x+1}}{l_x} = \frac{d_x}{l_x}$$

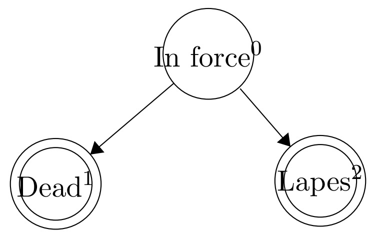
$$p_x^{0j} = \frac{d_x^j}{l_x}$$

Awesomely, we get the exact same result for ${}_sp^{0j}_x$ if we assume UDD in the MDM (proof is in the supplementary notes). Compare to CFM result in Alive to Dead.

$$ {}_sp^{00}_x = {}_sp_x = (p_x)^s = (p^{00}_x)^s $$

$$ {}_sp^{01}_x = {}_sq_x = 1 - (p_x)^s = (1 - (p^{00}_x)^s) $$

Example: Suppose we have $n = 2$



x	l_x	d^1_x	d^2_x	d_x
65	1000	20	50	70
66	930	27.9	55.8	83.7
67	846.3	33.9	59.2	93.1
68	753.2			

Find

1. probability (65) is still in force at 67

$$ {}_2p^{00}_{65} = \frac{l_{67}}{l_{65}} = 0.8463 $$

2. probability (65) lapses between 66 and 67

$$ {}_{1|}p^{02}_{65} = (p^{00}_{65})(p^{02}_{66}) = 0.0558 $$

3. contract over by 68

$$ {}_3p^{01}_{65} = {}_3p^{01}_{65} + {}_3p^{02}_{65} = 1 - {}_3p^{00}_{65} = 1 - \frac{l_{68}}{l_{65}} = 0.2468 $$

4. (65) dies before age 66.2

$$ {}_{1.2}p^{01}_{65} = p^{01}_{65} + (p^{00}_{65})({}_{02}p^{01}_{66}) = \frac{d^1_{65}}{l_{65}} + \frac{l_{66}}{l_{65}} $$

From UDD, ${}_{0.2}p^{01}_{66} = 0.2(p^{01}_{66})$ and From CFT, ${}_{0.2}p^{01}_{66} = \frac{p^{01}_{66}}{p^{0\cdot}_{66}}(1 - (p^{00}_{66})^{0.2})$

4.2 Premiums and Policy Value in MDM

Some principles, policy values are the same as in the alive-dead model since there is only one active state.

4.2.1 Example

3 year fully discrete contract for (65), 10,000 at the end of year of dead. If they lapse or survive 3 years, return of $\frac{1}{2}$ of total premiums paid. Calculate the annual premium P.

No expenses, multiple decrement table given, $i = 8\%$

$$\text{EPV of premiums} = P(\frac{l_{65} + l_{66}v + l_{67}v^2}{l_{65}}) = 2.5867P$$

$$\text{EPV of benefits} = 10,000(\frac{d_{65}^{(1)}v + d_{66}^{(1)}v^2 + d_{67}^{(1)}v^3}{l_{65}} = 693.4918$$

$$\text{EPV ROP benefit} = \frac{\frac{P}{2}d_{65}^{(2)}v + Pd_{66}^{(2)}v^2 + \frac{3P}{2}(d_{67}^{(2)} + l_{68})v^3}{l_{65}} = 1.03835P$$

so using the equivalent principle,

$$2.5867P = 693.4918 + 1.03835P$$

Hence $P = 447.90$.

4.3 Dependent and Independent Probabilities

${}_tp_x^{0j}$'s are known as dependent probability since they assume that the other decrements are present in the MDM

Define the independent probability

$${}_tp_x^{j} = e^{-\int_0^t \mu_{x+r}^{0j} dr}$$

(the exact result you should get in a model with only decrement j)

Also define ${}_tp_x^{j} = 1 - {}_tp_x^{j}$ Comparing ${}_tq_x^{j}$ and ${}_tp_x^{0j}$; both are the probability of going from 0 to j in t years but ${}_tp_x^{0j}$ depends on the other decrements being available whereas ${}_tq_x^{j}$ as sues j is the only possible decrement. If we have expressions for μ_x^{0j}'s, we can obtain exact independent probabilities.

But to go from a discrete MD table to associated Single Decrement Tables, we need a fractional age assumption. Assume CFT, we know

$$\frac{{}_sp_x^{0j}}{{}_sp_x^{0\cdot}} = \frac{\mu_x^{0j}}{\mu_x^{0\cdot}}$$

where $0 \leq s \leq 1$ and with respect to SDMs

$$_tp_x^{00} = \prod_{d=1}^{n} {}_tp_x^d$$

(Another justification

$$_tp_x = e^{-\int_0^t \mu_{x+r}^{0\cdot} dr} = \prod_{d=1}^{n} {}_tp_x^d$$

)

Let $t = 1$, $p_x^{00} = e^{-\mu_x^{0\cdot}}$ and $p_x^j = e^{-\mu_x^{0j}}$, thus

$$\frac{\mu_x^{0j}}{\mu_x^{0\cdot}} = \frac{-\log p_x^j}{\log p_x^{00}} = \frac{_sp_x^{0j}}{_sp_x^{0\cdot}}$$

so

$$p_x^j = (p_x^{00})^{p_x^{0j}/p_x^{0\cdot}}$$

4.3.1 Example

$$q_{65}^1 = 1 - p_{65}^1 = 1 - (\frac{l_{66}}{l_{65}})^{(\frac{d_{65}^{(1)}}{l_{65}})/(1-\frac{l_{66}}{l_{65}})} = 1 - 0.97947$$

4.4 Building a MDM from SDMs

We can get exact expressions if we know μ_{x+t}^{0j}'s for all j, but if we have discrete tables, we need a fractional age assumption.

Recall

$$_tp_x^j = e^{-\int_0^t \mu_{x+r}^{0j} dr}$$

$$_tq_x^j = 1 - {}_tp_x^j$$

$$_tp_x^{0j} = \int_0^t {}_rp_x^{00} \mu_{x+r}^{0j} dr$$

$$_tp_x^{0\cdot} = 1 - {}_tp_x^{00} = 1 - e^{-\int_0^t \mu_{x+r}^{0\cdot} dr}$$

If we assumed UDD in the multiple secrete table or CFT, we got

$$p_x^j = (p_x^{00})^{p_x^{0j}/p_x^{0\cdot}}$$

so we can take apart a MDM. To assemble a MDM from SDMs, we can just reverse the relationship

$$p_x^{0j} = \frac{\log p_x^j}{\log p_x^{00}} p_x^{0\cdot}$$

but remember ${}_tp_x^{00} = \prod_{i=1}^n {}_tp_x^j$ so in particular $p_x^{00} = \prod_{j=1}^n p_x^j$ and $p_x^{0\cdot} = 1 - \prod_{i=1}^n p_x^i$ so

$$p_x^{0j} = \frac{\log p_x^j}{\log \prod_{i=1}^n p_x^i}(1 - \prod_{i=1}^n p_x^i$$

This relationship holds if we assume CFT or UDD in the MDM. On the other hand, we can assume each SDM has the UDD property. (i.e. ${}_tq_x^j = tq_x^j \implies {}_tp_x^j\mu_{x+t}^{0j} = q_x^j$)

Then

$$\begin{aligned}{}_tp_x^{0j} &= \int_0^t {}_tp_x^{00}\mu_{x+r}^{0j}dr = \int_0^t {}_rp_x^1{}_rp_x^2\cdots{}_rp_x^n\mu_{x+r}^{0j}dr = q_x^j\int_0^t\prod_{i=1}^n {}_rp_x^idr \\ &= q_x^j\int_0^t\prod_{i\neq j}(1-{}_rq_x^idr = q_x^j\int_0^t\prod_{i\neq j}(1-rq_x^i)dr\end{aligned}$$

${}_tp_x^{0j}$ = depends on n, $q_x^j\times$ degree n polynomial with coefficients from q_x^i's

4.4.1 Example

Take apart the MDM into two SDMs and increase lapses by 20%. Lastly, reassemble into a MDM.

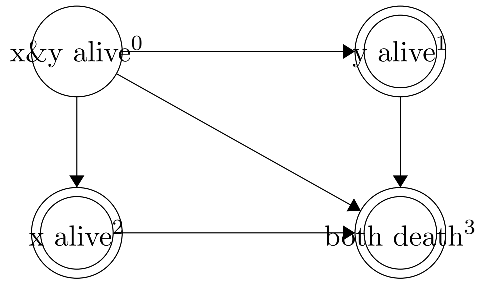
We have been assuming that the time until transition was differentiable (hence continuous), but it's possible in real life to have transitions occur discretely. For example, retirement at age 65 exactly lapses at policy anniversary etc.

How do we incorporate discrete transitions into a MDM? Easy! The discrete decrement(s) is not "competing" with other decrements for lives during each year. So if we have the independent probabilities for the discrete decrement, we include them at the end of the year based on the lives remaining from continuous decrements.

Use the same example and the same decrement table, but we are adding a third decrement retirement with the following probabilities: 50% of people reaching age 66 retire exactly then. 70% of people reaching 67 retire exactly then; the rest definitely will retire reaching age of 68.

x	l_x	d_x^1	d_x^2	d_x^3
65	1000	20	50	0
exact 66	930	0	0	465
66	465	13.95	27.9	0
exact 67	423.15	0	0	296.205
67	126.945	5.08	8.89	0
exactly 68	112.98	0	0	112.98
68	0	0	0	0

4.5 Multiple Life Functions



We have ${}_tp_{xy}^{11} = {}_tp_{xy}^{\bar{1}1}$ for all state i because no state can be reentered but it is possible to have 0, 1, or 2 transitions in a time period of length t. Future lifetime RV framework,

$$T_x = \text{time until death of (x)}$$

$$T_y = \text{time until death of (y)}$$

Define $T_{xy} = \min\{T_x, T_y\}$ (the joint life status (earliest death of (x) and (y)))

$$S_{xy}(t) = Pr(T_{xy} > t) = {}_tp_{xy} = P(\min\{T_x, T_y\} > t) = P(T_x > t, T_y > t)$$

If we assume T_x and T_y are independent, then ${}_tp_{xy} = {}_tp_x {}_tp_y$

We have ${}_tp_{xy}$ (both survive t years) and ${}_tq_{xy} = 1 - {}_tp_{xy}$ (at least one dies within t years)

$${}_{u|t}q_{xy} = ({}_up_{xy})({}_tq_{x+u:y+u}) \text{ (both survive to u, at least one dies by } u+t)$$

Note that “xy” is a status.

Define $T_{\overline{xy}} = \max\{T_x, T_y\}$ (the last survivor status (latest death of (x) and (y)))

Notice that $T_x + T_y = T_{xy} + T_{\overline{xy}}$ so ${}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy}$ and ${}_tq_{\overline{xy}} = {}_tq_x + {}_tq_y - {}_tq_{xy}$

$${}_tp_{xy} = {}_tp_{xy}^{00}$$

$${}_tq_{xy} = {}_tp_{xy}^{01} + {}_tp_{xy}^{02} + {}_tp_{xy}^{03} = {}_tp_{xy}^{0\cdot}$$

$${}_tp_{\overline{xy}} = {}_tp_{xy}^{00} + {}_tp_{xy}^{01} + {}_tp_{xy}^{03}$$

$${}_tq_{\overline{xy}} = 1 - {}_tp_{xy}^{03}$$

Also some new probabilities based on the order in which the deaths occur

$${}_tq_{\overset{1}{x}y} \implies \text{ x dies first within } \bar{t}| = \int_0^t {}_tp_{xy}^{00} \mu_{x+r:y+r}^{02} dr$$

$${}_tq_{\overset{2}{x}y} \implies \text{ x dies second within } \bar{t}| = \int_0^t {}_tp_{xy}^{01} \mu_{x+r:y+r}^{13} dr$$

$$
{}_tq^{\;1}_{xy} + {}_tq^{\;2}_{xy} = {}_tq_x
$$

$$
{}_tq_{x\overset{1}{y}} + {}_tq_{x\overset{2}{y}} = {}_tq_y
$$

$$
{}_tq^{\;1}_{xy} + {}_tq_{x\overset{2}{y}} = {}_tq_{xy}
$$

$$
{}_tq^{\;2}_{xy} + {}_tq_{x\overset{2}{y}} = {}_tq_{\overline{xy}}
$$

$$
{}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy}
$$

for p's, q's, A's, a's, etc.

4.6 Benefits for Multiple Lives

Insurance: Joint lit insurance (pays an first death); last survivor insurance (pays on second death); contingent insurance (depends on order).

(all can be whole life, term, endowment insurance, and continuous, annual, or $\frac{1}{m}$ly)

Annuities: joint life annuity (pay until the first death); last survivor annuity (pays until second death); reversionary annuity (pays to x while alive, starting after the death of y)

In the continuous case, we can develop expression for the EPVs of any of these benefits.

joint whole life annuity $\bar{a}_{xy} = \int_0^\infty e^{-\delta t}{}_tp_{xy}dt = \int_0^\infty e^{\delta t}{}_tp^{00}_{xy}dt$

last survivor whole life annuity $\bar{a}_{\overline{xy}} = \int_0^\infty e^{-\delta t}{}_tp_{\overline{xy}}dt = \int_0^\infty e^{-\delta t}({}_tp^{00}_{xy} + {}_tp^{01}_{xy} + {}_tp^{02}_{xy})dt = \int_0^\infty e^{\delta t}{}_tp^{00}_{xy}dt + \int_0^\infty e^{\delta t}{}_tp^{01}_{xy}dt + \int_0^\infty e^{\delta t}{}_tp^{02}_{xy}dt = -\bar{a}_{xy} + \bar{a}_x + \bar{a}_y$

Logical Explanation $\bar{a}_{\overline{xy}}$ pays until the last death, i.e. pays while either is alive. $\bar{a}_x + \bar{a}_y$ pays to each while alive (2 while both are alive) so subtract the extra $\bar{a}_{xy}$

Reversionary annuity $\bar{a}_{x|y}$ (means to pay to y after death of x) $= \int_0^\infty e^{-\delta t}{}_tp^{02}_{xy}dt$ (x must be dead and y alive to get paid) $= \int_0^\infty e^{-\delta t}({}_tp^{02}_{xy} + {}_tp^{00}_{xy}dt - \int_0^\infty e^{-\delta t}{}_tp^{00}_{xy}dt$. Hence $\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}$

Logical Explanation $\bar{a}_{x|y}$ pays y after x is dead. $\bar{a}_y$ pays while y is alive (1 while both alive) so subtract the extra $\bar{a}_{xy}$.

Joint life insurance $\bar{A}_{xy} = \int_0^\infty e^{-\delta t}{}_tp^{00}_{xy}(\mu^{01}_{x+t:y+t} + \mu^{02}_{x+t:y+t})dt$ (either x or y dying triggers payment)

Last survivor insurance $\bar{A}_{\overline{xy}} = \int_0^\infty e^{-\delta t}({}_tp^{01}_{xy}\mu^{13}_{x+t} + {}_tp^{02}_{xy}\mu^{23}_{y+t})dt$ (either x or y dying last triggers the payment)

But if we have the 2-life model with a "common shock" (the force of transition from $0 \to 3$ directly). We would just add ${}_tp^{00}_{xy}\mu^{03}_{x+t:y+t}$ to the probability of payment at t inside both the integrals.

$$
\bar{A}_{\overline{xy}} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}
$$

Contingent Insurance $\bar{A}_{\overset{1}{x}y} = \int_0^\infty e^{-\delta t}{}_tp_{xy}^{00}\mu_{x+t:y+t}^{02}dt$. Similarly, $\bar{A}_{x\overset{1}{y}} = \int_0^\infty {}_tp_{xy}^{00}\mu_{x+t:y+t}^{01}dt$.

We can see

$$\bar{A}_{\overset{1}{x}y} + \bar{A}_{x\overset{1}{y}} = \bar{A}_{xy}$$

and $\bar{A}_{\overset{2}{x}y} = \int_0^\infty e^{-\delta t}{}_tp_{xy}^{01}\mu_{x+t}^{13}dt$ and $\bar{A}_{x\overset{2}{y}} = \int_0^\infty e^{-\delta t}{}_tp_{xy}^{02}\mu_{y+t}^{23}dt$. We can see

$$\bar{A}_{\overset{2}{x}y} + \bar{A}_{x\overset{2}{y}} = \bar{A}_{\overline{xy}}$$

We can add a term to any of these benefits t integrate up to n instead of ∞. We can have annual or $\frac{1}{m}$ly payments to sum instead of integrals.

Special Case independent lives; not necessarily realistic because spouses usually have similar health and activity habits; also they spend more time together than any 2 random people, so they can be subject to accidents.

If lives are independent, that means $\mu_{xy}^{03} = 0$ (no common shock).

$$\mu_{xy}^{01} = \mu_y = \mu_y^{23}$$

$$\mu_{xy}^{02} = \mu_x = \mu_x^{13}$$

Then $\mu_{xy} = \mu_{xy}^{0\cdot} = \mu_{xy}^{01} + \mu_{xy}^{02} = \mu_x + \mu_y$ so ${}_tp_{xy} = {}_tp_{xy}^{00} = e^{-\int_0^t \mu_{x+r:y+r}^{0\cdot}dr} = e^{-\int_0^t(\mu_{x+r}+\mu_{y+r})dr} = {}_tp_x{}_tp_y$

Example: $q_x = 0.02, q_{x+1} = 0.025, q_{x+2} = 0.03, q_y = 0.03.q_{y+1} = 0.035, q_{y+2} = 0.04$. Assuming (x) and (y) are independent, calculate

1. ${}_2p_{xy} = {}_2p_x{}_2p_y = (1-0.02)(1-0.025)(1-0.03)(1-0.035) = 0.8944$
2. ${}_2p_{\overline{xy}} = {}_2p_x + {}_2p_y - {}_2p_{xy} = 0.9972$
3. ${}_{1|}q_{xy} = p_{xy}q_{x+1:y+1} = 0.0526$
4. ${}_{1|}q_{\overline{xy}} = 0.0022$

4.7 Gompertz and Makeham Mortality

Gompetz $\mu_x = Bc^x$ so if (x) and (y) are independent with same mortality, then $\mu_{xy} = \mu_x + \mu_y = Bc^x + Bc^y$. Define w such that $c^w = c^x + c^y$ (i.e. $w = \frac{\log(c^x+c^y)}{\log c}$ then $\mu_{xy} = Bc^w$ and the joint life status xy can be replaced by the single life status w. w is the equivalent single age)

Makeham $\mu_x = A + Bc^x$ so $\mu_{xy} = A + Bc^x + A + Bc^y = 2A + B(c^x + c^y)$. Define v such that $c^x + c^y = 2c^v$. Hence $\mu_{xy} = 2(A + Bc^v) = \mu_v + \mu_v$ so v is the equivalent equal age xy can be replaced with vv.

These simplification assume that x and y are independent. If x and y independent, we have ${}_tp_{xy} = {}_tp_{x}{}_tp_y$. The assumptions behind independence may not hold in practice so we ned ways to cover the dependence situation.

4.8 Common Shock Model

Assume lives are independent except for a common probability of accident which affects both lives. We use the same state model for joint life but for the transition from both alive to both dead will be a constant, λ, no matter what ages they are.

μ_x^* - unshocked force of mortality for (x)

μ_y^* - unshocked force of mortality for (y)

λ - force of accident/shock (constant)

T_x^* - future lifetime unshocked of (x)

T_y^* - future lifetime unshocked of (y)

Z- time until death from shock $\sim Exp(\lambda)$

$T_x = \min\{T_x^*, Z\}$ and $T_y^* = \min\{T_y^*, Z\}$

Since (x) and (y) can each die from mortality or shock, the min of those two times will be the actual time of death.

$$T_{xy} = \min\{T_x, T_y\} = \min\{T_x, T_y, Z\}$$

or we can think of the MSM probabilities ${}_tp_{xy}^{00} = e^{-\int_0^t \mu_{x+r:y+r}^{0\cdot} dr}$

We have $\mu_{xy}^{0\cdot} = \mu_x^* + \mu_y^* + \lambda$. Therefore ${}_tp_{xy}^{00} = {}_tp_x^* {}_tp_y^* e^{-\lambda t}$

From this relationship we can get the others

$${}_tp_x = {}_tp_x^* e^{-\lambda t}$$

Same for (y).

$${}_tp_{\overline{xy}} = e^{-\lambda t}({}_tp_x^* + {}_tp_y^* - {}_tp_x^* {}_tp_y^*) = e^{-\lambda t} {}_tp_{\overline{xy}}^*$$

but unfortunately, there is dependency on both lives.

Commonly used with Makeham mortality for both life. Sometimes we consider A to be the shock so $\mu_x^* = Bc^*$ and $\mu_y^* = Bc^y$ and $\lambda = A$ then we would have $\mu_{xy} = A + Bc^x + Bc^y$.

4.8.1 Example

(x) and (y) have unshocked mortality as follows $q_x^* = 0.02, q_{x+1}^* = 0.025, q_{x+2}^* = 0.03, q_y^* = 0.03. q_{y+1}^* = 0.035, q_{y+2}^* = 0.04$. In addition both lives experience a shock with constant force $\lambda = 0.0005$.

Recalculate

1. ${}_2p_{xy}$
2. ${}_2p_{\overline{xy}}$
3. ${}_{1|}p_{xy}$
4. ${}_{1|}p_{\overline{xy}}$

Now for EPVs

$$\bar{a}_{xy} = \int_0^\infty e^{-\delta t} {}_tp_{xy} dt = \int_0^\infty e^{-\delta t} {}_tp_x^* {}_tp_y^* e^{-\lambda t} dt = \int_0^\infty e^{-(\delta+\lambda)t} {}_tp_x^* dt = \int_0^\infty e^{-\delta' t} {}_tp_{xy}^* dt = \bar{a}_{xy}^*$$

where $\delta' = \delta + \lambda$. We can evaluate common shock EPVs by using independent unshocked mortality EPVs and a modified interest rate

$$i' = e^{\delta+\lambda} - 1$$

Will $\bar{a}_{xy}^*$ be higher or lower than $\bar{a}_{xy}^*$? Lower because the shock could cause the lives to die sooner $\rightarrow$ lower EPV.

$$\bar{A}_{xy} = \int_0^\infty e^{-\delta t} ({}_tp_{xy}(\mu_{x+t:y+t}^{01} + \mu_{x+t:y+t}^{02} + \mu_{x+t:y+t}^{03}) dt = \int_0^\infty e^{-\delta t} - t p_x^* {}_tp_y^* e^{-\lambda t} (\mu_{y+t}^* + \mu_{x+t}^* + \lambda) dt$$
$$= \int_0^\infty e^{-(\delta+\lambda)t} {}_tp_{xy}^* (\mu_{y+t}^* + \mu_{x+t}^*) dt + \int_0^\infty e^{-(\delta+\lambda)t} {}_tp_{xy}^* \lambda dt$$

so $\bar{A}_{xy} = \bar{A}_{xy@\delta'}^* + \lambda \bar{a}_{xy@\delta'}^*$ or we can use the following result for any status "u" that is either whole life or endowment insurance type.

$$\bar{A}_u = E[v^{T_u}]$$

where T_u = time until "u" expires

$$\bar{a}_u = E[\bar{a}_{\overline{T_u}|}] = E[\frac{1 - v^{T_u}}{\delta}] = \frac{1 - E[v^{T_u}]}{\delta}$$

so $\bar{a}_u = \frac{1 - \bar{A}_u}{\delta}$ and $\bar{A}_u = 1 - \delta \bar{a}_u$ so in particular for the common shock model

$$\bar{A}_{xy} = 1 - \delta \bar{a}_{xy} = 1 - \delta \bar{a}_{xy@\delta'}^*$$

Examples: Interpret the following EPV and express it as a combination of simple EPVs:

1. $\bar{A}_{x\overline{yz}} = \bar{A}_{xy} + \bar{A}_{xz} - \bar{A}_{xyz}$

2. $\bar{a}_{x|y:\overline{n}|}$ means that y gets paid if x dies within n years

$$\bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|}$$

and $\bar{a}_{x:\overline{n}|y}$ means that if x dies before n then y get paid; if n years passed and x is still alive, y will still get paid too

$$\bar{a}_y - \bar{a}_{xy:\overline{n}|}$$

5 Advanced Topics

(MLC material - not on or final)

5.1 Interest Rate Risk

We have been implicitly assuming a flat yield curve for our interest rates. i.e. the same annual effective rate i applies to all future cash flow regardless of when they occur.

$$v(t) = v^t = e^{-\delta t}$$

In practice, we actually have a "term structure of interest rate" where there are different rates for different lengths of investment.

- Spot rates: y_t is t-year spot rate means \$1 at time 0 grows to $(1 + y_t)^t$ if invested for t years
- then \$1 at time t is worth $\frac{1}{(1+y_t)^t}$ at time 0

$$v(t) = \frac{1}{(1 + y_t)^t}$$

  Typically, the yield curve will be concave increasing.

- implied forward rates

$$f(t, t + k) = \text{k-year forward rate for an investment in t years}$$

  \$1 at time t grows to $(1 + f(t, t + k)^k)$ by time $t + k$.

  Thus $(1 + f(t, t + k))^k = \frac{(1+y_{t+k})^{t+k}}{(1+y_t)^t}$

In the context of insurance, we replace the discount factor v^t with the more realistic $\frac{1}{(1+y_t)^t}$. Tus a discrete annuity due to (x) under the yield curve y would have EPV

$$a(x)_y = \sum_{k=0}^{\infty} v(k)_t p_x = \sum_{k=0}^{\infty} \frac{1}{(1+y_t)^k} {}_k p_x$$

Similarly a continuous insurance has EPV

$$\bar{A}_{(x)_y} = \int_0^{\infty} v(t)_t p_x \mu_{x+t} dt$$

Diversifiable vs non-diversifiable risks usually an insurer relies on large numbers of identical/similar policies that are independent. By the "law of large numbers", things will tend to turn out as expected as the number of policies gets larger. Say we have N policies and some r.v. X_i $(i = 1, \cdots, N)$ that we are interested in for each policy. Then $E[\sum_{i=1}^N X_i] = \sum_{i=1}^N E[X_i] = N_\mu$ where μ is the mean of the X_i's.

$$Var(\sum_{i=1}^{N} X_i) = \sum_{i=1}^{N} Var(X_i) + \sum\sum_{i<j} Cov(X_i, X_j)$$

. Let σ^2 be the variance of X_i and ρ be the correlation between X_i and X_j. Hence it becomes $N\sigma^2 + N(N-1)\rho\sigma^2$. If the X_i's are independent, we just have $Var(\sum_{i=1}^N X_i) = N\sigma^2$ and $SD = \sigma\sqrt{N}$. By the CLT

$$\frac{\sum_{i=1}^N X_i - N_\mu}{\sigma\sqrt{N}} \sim N(0,1)$$

as $N \to \infty$.

The average risk would be $\frac{\sum_{i=1}^N X_i}{N}$. As $N \to \infty$, $Var \to 0$ if $Var(\sum_{i=1]^N X_i)}$ is a linear function of N.

Definition. *A risk X_i is diversifiable if* $\lim_{N\to\infty} \frac{SD(\sum_{i=1}^N X_i)}{N} = 0$ *and non-diversifiable if not.*

Mortality risk is diversifiable. Hence N increase will decrease risk but interest rate risk is non-diversifiable.

5.2 Profit Testing

With computers, we can easily project the cash-flows for portfolios of contracts and possibly incorporate more flexible. Assumptions (yield curve, changes in mortality, etc.).

Uses

- identify where profits come from

- set premiums (explicitly loading for profit).
- stress test (see how profits are affected by worsening assumptions)
- determine how much reserve to hold
- measure profitability
- for more complex contracts, determine dividends paid to policyholders

Profit testing is done on the profit testing basis

1. basis: set of mortality, interest, and expense assumptions
2. profit testing basis could be the same or different from the premium basis and the policy value basis.

Example:

10 year annual term insurance to (60) $P = 1500, S = 100,000$. Policy basis, 5.5% interest, 400 plus 20% of first premium at inception and 3.5% of premiums 2-10 and $q_{60+t} = 0.01 + 0.0001t$

- Generally assume initial expenses happen before first premium so surplus at time 0 is generally negative
- each line is only looking at income/outgo in that year.
- each line assumes the contract is in force of the beginning of period. so all cash flows (P_{t-1}, DB_t, etc) are assuming the policyholder is alive at $t-1$.

We had surplus at t $P_{t-1} - E_{t-1} + I_t - DB_t$ year $(t-1, t)$ is in isolation and assuming contract in force at $t-1$, but cash flows are not isolated since we do hold reserves. We want to incorporate them into the profit test. Suppose we hold reserves equal to NPPV on the basis. 4% interest, $q_{60+t} = 0.011 + 0.001t$ (worse mortality, worse interest, no expense).

To incorporate reserves,

1. in first year there is no reserve.
2. for row 2 and on, we add a column to include the reserve held at the start of the year ("income" of ${}_{t-1}V$ at the start of the t^{th} year, which accrues interest so $I_t = i({}_{t-1}V + P_{t-1} - E_{t-1})$)
3. For line 1 and onwards, we have to add a column for the "cost" of setting up next year's reserve but we only need the reserve for the policy holder who are alive at time t. Therefore, the cost in the year t row is

$$ {}_tV p_{60+t-1} $$

Now the surplus at time t is

$$ {}_{t-1}V + P_{t-1} - E_{t-1} + I_t - DB_t - {}_tVp_{60+t-1} $$

4. The collected surplus values for $t = 0, 1, \cdots, 10$ is known as the profit vector **Pr**.

$$ Pr_0 = \text{initial setup cost} $$

$$ Pr_1 = \text{surplus at t}|\text{alive at } t-1 $$

5. If we want to have profits that are not conditional on anything, we can just multiply each element by ${}_{t-1}p_{60}$ (probability of being alive at $t-1$). The resulting vector is called the profit signature, π.

$$ \pi_0 = Pr_0 = \text{initial setup cost} $$

$$ \pi_1 = Pr_1 = \text{surplus at time 1} $$

$$ \pi_t = Pr_{tt-1}P_{60} = \text{surplus at time t} $$

With the profit signature, we can do any analysis we like to assess the profitability, such as

- NPV at the company's interest rate for future cash flows.

$$ NPV = \sum_{k=0}^{n} \pi_k v^k $$

- IRR of cash flows, (the rate such that NPV is zero)
- Partial NPV's (each year t, find

$$ \sum_{k=0}^{t} \pi_k v^k $$

for some t they will be positive. (pay back period)

6 Final Review

ACTSC 232
- Survival Models including life tables
- EPV of insurances and annuities
- Premium calculation including expense

Policy Values
- definition of L_t
  - future benefits - expenses given alive at t.
- Policy value ${}_tV = E[L_t | T_x \geq t]$
- NPPV v.s. GPPV (artificial premium with no expense v.s. actual premium including expenses). Both are calculated on P.V. basis.
- Simplified formula for NPPV ${}_tV = 1 - \frac{a_{x+1}}{a_x}$ for whole life etc.
- prospective v.s. retrospective (EPV of future benefits - premiums v.s. EPV at 0 of premium-benefits in (0, t) divided by ${}_tE_x$)
- Recursive relationship

  $$({}_tV + P)(1+i) = q_{x+t}S + p_{x+t t+1}V$$

  for annual. Similar for $\frac{1}{m}$ly.
- $NAAR_{t+1} = S - {}_{t+1}V$ measures mortality risk.
- Policy value between payment dates at $t+s$
  - exact (forwards from t or backwards from $t + \frac{1}{m}$
  - interpolation
  - interpolation with interest

    $${}_tV + P$$

  - Continuous payments
    - L_t has nice distribution
    - can calculate ${}_tV$ exactly for $t \in \mathbb{R}$
    - graphs have no discontinuities
    - Thiele DE

      $$\frac{d}{dt}{}_tV = \delta_t V + P - (S - {}_tV)\mu_{x+t}$$

      boundary condition matter.
- Can use as an approximation for small h

  $$\frac{d}{dt}{}_tV = \approx \frac{{}_{t+h}V - {}_tV}{h} = \delta_t V + P - NAAR_t \mu_{x+t}$$

- Asset shares AS_t = amount per policyholder in force at t the company has. ${}_tV$ = amount they need
- analysis of surplus - split $AS_t - {}_tV$ into pieces caused by interest, mortality and expenses.
- policy alternations, cash value C_t. Determine unknowns using equivalence principle

$$C_t + \text{EPV future premiums} = \text{EPV future benefits}$$

MSM

- assumptions and notations: $\mu_x^{ij}, {}_tp_x^{ij}, {}_tp_x^{\overline{ii}}$
- ${}_tp_x^{\overline{ii}} = e^{-\int_0^t \mu_{x+r}^{i\cdot} dr}$
- calculate other ${}_tp_x^{ij}$'s by fixing the time of first transition at r, and $\int_0^t dr$
- KFE's

$$\frac{d}{dt}{}_tp_x^{ij} = \sum_{k \neq j} ({}_tp_x^{ik}\mu_{x+t}^{kj} - {}_tp_x^{ij}\mu_{x+t}^{jk})$$

- can useKFE as approximation for small h
- benefit EPVs $\bar{a}_x^{ij}$ and $\bar{A}_x^{ij}$

sum/integral over pmt dates of amt paid $\times$ discount function $\times$ prob of pmt

- premiums and policy values are the same but policy value can depend on current state.
- ${}_tV^{(i)}$ = EPV at t of future benefits − premiums given in i at time t. Thiele DE for ${}_tV^{(i)}$ in MSM
- MDM
  - one active state, n absorbing states
  - tables
  - fractional age assumptions
  - EPVs, premiums, policy values similarly
  - dependent is independent probability
  - assumptions for splitting/reassembling MD Tables into SD tables.

Multiple Lives

- "xy" and "$\overline{xy}$"

$$\overline{xy} = x + y - xy$$

  for p's and q's, A's, $\ddot{a}$'s.
- reversionary annuities (pay to x after death of y)
- contingent insurances (order of death matters)
- independent lives
- Common Shock