ACTSC 372: Corporate Finance 2

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Contents

1 Introduction (Microeconomic Concept)

Page 5 - 18

Model: One-consumer $\Longrightarrow$ choose x_1 and x_2. such that the **satisfaction** (measured by utilities) from consumption of both goods is **max** subject to money income (m). There are two sides of this problem

$$\begin{cases} \text{Ability} \\ \text{Preference} \end{cases} \Longrightarrow \text{Optimal Choice}$$

- Ability: Budget constraint

  Assumptions – 2 goods: x_1 and x_2
  - Rationality
  - m = money income, P_1 = price of good 1, P_2 = price of good 2

  short-run analysis $\Longrightarrow$ P_1, P_2 and m are constants.

$$\begin{aligned} \text{income} & \qquad \text{expenses} \\ m \geq & \qquad P_1x_1 + P_2x_2 \\ \text{no saving} \Longrightarrow m & = P_1x_1 + P_2x_2 \end{aligned}$$

  We can draw this function on the x_1-x_2 coordinate and the slope is the following

$$\text{slope} = \frac{\Delta x_2}{\Delta x_1} = \frac{dx_2}{dx_1} = -\frac{P_1}{P_2}$$

  Example: $m = \$10$, $P_1 = 1$, $P_2 = 2$.

- Preference: Utility Note: Consider good X, TU = Total Utility, MU = Marginal Utility.

x	TU_x	$MU_x = \frac{\Delta TU}{x}$
1	20	20
2	36	16
3	46	10
4	46	0
5	42	-4

  By the first derivative test, we can find the local maximum is 46. Also by the law of diminishing marginal utility, $x \nearrow \Longrightarrow MU_x \searrow$

  Modeling Preference

- $U(x) \implies$ utility function
- Axioms of Preferences: Define A and B are bundles $A(x_1, x_2)$.

  Notation: If bundle A is at least as preferred to bundle B, then we write $A \succeq B$. If $A \succeq B$ and $B \succeq A$, then $A \sim B$. Then A is indifferent to B.

  Axiom 1. Completeness, for any two bundles A, B, we could have one of the following:

  - $A \succeq B$
  - or $B \succeq A$
  - or $A \sim B$

  This implies consumer can compare between bundles.

  Axiom 2. Transitivity: For any 3 bundles A, B, and C. If $A \succeq B$ and $B \succeq C$ $\implies A \succeq C$. Hence we can form a chain to rank our choice.

  Axiom 3. Continuity of Preference: If $x \succeq y$, then $U(x) \geq U(y)$.

  Axiom 1, 2, 3 ensure the representation of preference in a $U(\cdot)$ function.
- Monotonicity (more is always better)
- Diminishing MRS (Marginal Rate of Substitution): MRS is just simply the ratio of change of two goods. The MRS with respect to good x_1 means increase 1 unit of that good causes a increase of MRS of good 2. This implies the convexity of indifferent curve.

Note on continuity:

Suppose that X = Apple and Y = Bananas. $\{x_n\} \to x$ and $\{y_n\} \to y$. Hence we can model x as a continuous function. If $X \succeq Y$, x is at least as preferred to Y, then $u(X) \geq U(y)$. If $X \succ y$, then $U(x) > U(y)$. If A is a bundle: $A(x_1, x_2)$, B is another $B(x_1, x_2)$ and $A \succeq B \implies U(A) \geq U(B)$.

If Axiom 1 to 5 are satisfied, then well-behaved preference. The function that satisfies all these assumptions is called Cobb-Douglas utility function

$$U(x_1, x_2) = x_1^\alpha x_2^\beta$$

where $\alpha > 0, \beta > 0$. For a target utility level, $\bar{U}$, then the $U(\cdot)$ becomes $\bar{U} = x_1^\alpha x_2^\beta$ graphically presented by an indifference curve. The slope of the function is the MRS, $\frac{\Delta x_2}{\Delta x_1}$. It can be shown that

$$MRS_{1,2} = -\frac{MU_1}{MU_2} = -\frac{\partial U}{\partial x_1} / \frac{\partial U}{\partial x_2}$$

Proof.

$$\bar{U} = U(x_1, x_2)$$
$$d\bar{U} = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2$$
$$0 = MU_1 dx_1 + MU_2 dx_2$$

Hence the slope is above. □

Consumer Choice Problem

$$max_{\{x_1, x_2\}} U(x_1, x_2)$$

such that $m = p_1 x_1 + p_2 x_2$.

$\mathcal{L} = U(x_1, x_2) + \lambda[m - p_1 x_1 - p_2 x_2]$

FOC:

- $\frac{\partial \mathcal{L}}{\partial x_1} = 0$
- $\frac{\partial \mathcal{L}}{\partial x_2} = 0$
- $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$

From the division of the first two conditions, we get $\frac{p_1}{p_2} = \frac{MU_1}{MU_2}$ (tangency condition). The third condition is called feasibility condition.

2 Choice Under Uncertainty

Page 49

How to model risk?

Denote Risk by $\tilde{Z}$ = random variable. Risky investment was regarded as a gamble or lottery when it was first invented.

For example, $\tilde{Z} = \begin{cases} 6000 & \text{with a probability } \alpha = 1/2 \\ 0 & \text{with a probability } (1-\alpha) = 1/2 \end{cases}$. Then we get the expected return is $E[\tilde{Z}] = 1/2 \times 6000 + 1/2 \times 0 = 3000$. If you are offered \$5000. Then you will possibly gain \$2000 in the future. However, this \$2000 return means differently with regards to different individuals. Certainly, we should not sue mathematical expectations to value risky alternatives. Rather we should use the expected utility from the outcome. For example, the gains to a poor and a rich are the same but the utility of this gain will be different. Here is another example, suppose your initial wealth is $w = 4000$. You have merchandise abroad worth 8000 with a probability 1/10 that the ship will sink and 9/10 that the ship will make it.

$$\tilde{Z} = \begin{cases} -8000 & 1/10 \\ 8000 & 9/10 \end{cases}$$

The risky wealth after the gamble is

$$\tilde{w} = \begin{cases} 4000 & 1/10 \\ 12000 & 9/10 \end{cases}$$

$$\tilde{w}(w_1, w_2; \alpha) = (4000, 12000; 1/10)$$

$$E[\tilde{w}] = (4000 \times 1/10) + (12000 \times 9/10) = 11200$$

Alternatively, we can split the merchandise into two ships.

St. Petersburg Paradox

Outcomes $\{H, T\}$.

Heads	1st	2ed	3rd	4th	$\cdots$
probability	1/2	$(1/2)^2$	$(1/2)^3$	$(1/2)^4$	$\cdots$

$E[\tilde{w}] = 2(1/2)+4(1/2)^2+8(1/2)^3+\cdots = 1+1+1+\cdots = \infty$. Therefore, the willingness to pay is ∞. However, the actual willingness to pay should be $< \infty$.

Bernoulli's Explanation: Any lottery should be valued according to the EU that it generates. This implies treat wealth as a commodity so that we could have $U(w)$ such that total utility increases at a decreasing rates. This is called the law of marginal diminishing. If $U(w)$is increasing at a decreasing rate then the $MU(w)$ is decreasing. $U(w)$ satisfies the law of diminishing marginal utility. This implies that instead of $E[\tilde{w}]$, we should use $E[U(\tilde{w})] = \frac{1}{2}U(\$2) + (\frac{1}{2})^2 U(\$4) + \frac{1}{2}^3 U(\$8) + \cdots = o < \infty$ because $U(w)$ is increasing at decreasing rate. Buildling this idea, for a gamble $\tilde{w}(w_1, w_2; \alpha)$. We need to construct

$$U(\tilde{w}) = \alpha U(w_1) + (1-\alpha)U(w_2)$$

This is called VN-M utility.

Example: Tutorial 2, problem 3: $U(w) = \ln w$ Suppose that you are facing a lottery

$$\tilde{L}_1(50000, 10000; 1/2)$$

Determine the lottery

$$\tilde{L}_2(x, 0; 1)$$

such that $\tilde{L}_2 \sim \tilde{L}_1$. So what is the value x?

$$E[U(\tilde{L}_1)] = E[U(\tilde{L}_2)]$$

Use the VN-M utility. Hence $x = e^{1/2(\ln 50000 + \ln 10000)}$

Theorem 1. The expected utility theorem: $U(\tilde{w}) = \alpha \cdot U(w_1) + (1-\alpha)U(w_2)$. Preference Axioms needed to construct VN-M utility function.

1. Completeness
2. Transitivity
3. Continuity

Guarantee the existence of a real-valued $U(\cdot)$ over monetary outcomes or $x \succ y \implies U(x) > U(y)$, $x \succeq y \implies U(x) \geq U(y)$.

Axiom 4. Independence of irrelevant alternatives. Consider two lotteries

$$\tilde{w}_x(\delta, x; \alpha), \tilde{w}_y(\delta, y; \alpha)$$

given δ, if lottery $\tilde{w}_x \succeq \tilde{w}_y \implies x \succeq y$.

Axiom 5. Ranking, consider 2 outcomes $x\&y$ such that $a \succeq x \succeq b$ and $a \succeq y \succeq b$. Then, if $x \sim \tilde{w}_x(a, b; \alpha)$ and $y \sim \tilde{w}_y(a, b; \beta)$, this implies if $x \succeq y \implies \tilde{w}_x \succeq \tilde{w}_y \iff \alpha \geq \beta$. Hence $U(x) = \alpha, U(y) = \beta$.

Axiom 6. Measurability: an outcome could be expressed a lottery

$$a \succeq x \succeq b$$

Exists a unique probability, α, such that

$$x \sim (a, b; \alpha)$$

Axiom 7. Bounded set: for lottery: $a =$ most preferred outcome and $b =$ least preferred outcome.

If Axioms 1 - 3 are satisfied, then we can model $x, y, \cdots$ as utilities. If Axioms 4 -7 are satisfied (in addition to 1- 3), then for any lottery

$$\tilde{w}(x, y; \alpha)$$

Hence $U(\tilde{w}) = U(x)\alpha + U(y)(1-\alpha)$

For optimal choice,

1. Tangency:
$$\frac{MU_1}{MU_2} = \frac{P_1}{P_2}$$
2. Feasibility:
$$m = P_1 x_1 + P_2 x_2$$

The question is what is the meaning of $\frac{MU_1}{MU_2} = \frac{P_1}{P_2}$. The marginal utility is basically the marginal benefit of the last unit of good 1 relative to good 2. Marginal cost of the last unit of 1 relative to 2. By assuming contradiction, if $MB > MC$is $\frac{MU_1}{MU_2} > \frac{P_1}{P_2} \implies$ good 1 is more appealing, Then we need to increase the consumption of x_1, but the marginal utility of good 1 is going to decrease until $\frac{MU_1}{MU_2} = \frac{P_1}{P_2}$. Hence this condition is for the optimal choice.

3 Risk Aversion

3.1 Financial System & financial contracts

Borrowers (investors, Business sectors, producers) on the supply side and lenders (savers, household, consumers) on the demand side are basic components of the financial system. In between, there are stocks, bonds sold to lenders and borrowers prepare funds for lenders. Those are financial contracts. The financial market has equilibrium rice of securities.

The motivation for this section: why consumers are engaged in such financial contracts? consumption smoothing. For example, there is one consumer with period 0 and period 1. Suppose there are two income:y_0 and y_1 and consumption:c_0 and c_1 and saving:s_0 and s_1. In period 0, $\bar{y}_0 = s_0 + c_0$. If you have more saving today,i.e. you have to sacrifice consumption to generate some return in the future (period 1): additional income $(1 + r)$. In order to increase c_1, what happens to s_1. Well, $s_1 = 0$ because people don't live for another period. This implies in period 1 $y_1 + s_0(1+r) = c_1$. Anyone would prefer a bundle $A = (c_0, c_1) = (9, 9)$ over bundle $B = (c_0, c_1) = (8, 10)$.

$$A \succ B \implies U(A) > U(B)$$

$$U(9) + U(9) > U(10) + U(8) \implies U(9) > 0.5[U(8) + U(10)]$$

The utility of any saver is strictly concave and to be consistent with consumption smoothing. Mathematically, $U'(\cdot) > 0, U''(\cdot) < 0$.

3.2 Formal definition of risk-aversion

Consider the risky alternative $\tilde{w}(8, 10; \frac{1}{2})$.

- Certainty wealth: $E[\tilde{w}] = \frac{1}{2} \times 8 + \frac{1}{2} \times 10 = 9$
- Utility from certainty: $U(9) = U[E(\tilde{w})]$
- Utility from uncertainty: $U(8)$ and $U(10)$

$$[\frac{1}{2}U(8) + \frac{1}{2}U(10)] = E[U(\tilde{w})]$$

- Any risk averse consumer would favor certainly over uncertainty. This implies $U[E[\tilde{w}]] > E[U(\tilde{w})]$ where the utility from certainty is greater than the utility from uncertainty.
- If $U[E[\tilde{w}]] < E[U(\tilde{w})]$, then they are risk lover.
- otherwise, they are risk neural.

Example on risk aversion

Suppose $U(w) = \ln w$, This ensures concavity. Suppose you are focused with the following:

$$\text{lottery} = \begin{cases} +1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{cases}$$

$E[\bar{Z}] = 0 \implies$ actuarially fair gamble

Suppose your initial wealth is \$10. $\implies \tilde{w} = w + \tilde{Z} \begin{cases} 11 & 1/2 \\ 9 & 1/2 \end{cases}$

- Certainty outcome $E[\tilde{w}] = \frac{1}{2} \times 11 + \frac{1}{2} \times 9 = 10$.
- Uncertainty outcomes \$11, \$9
- Utility from uncertainty is $E[U(\tilde{w})] = \frac{1}{2} \ln 9 + \frac{1}{2} \ln 11$.
- Utility from certainty is $U(E[\tilde{w})] = \ln 10$.

3.2.1 Risk Premium: π

The risk premium is the maximum amount that a consumer is willing to pay to avoid risk. Need to find certainty equivalence wealth, w_{CE} = wealth level if the gamble is avoided. To define this, we need to start with the utility of the gamble and work backward. This means $w_{CE} = U^{-1}(E[U(\tilde{w}))$.

$$\pi = E[\tilde{w}] - w_{CE}$$

An alternative way to compute π is to find the certainty equivalence, CE, as follows:

- $CE(\tilde{Z}, w)$ is called certainty or cash equivalence. It is the sure increase in wealth that has the same effect on welfare as bearing the risk of the gamble. Or, equivalently, it is the asking price of the risk premium.

  Recall:

  $$E[U(\bar{w})] = U(w_{CE}) = U[E[\tilde{w}] - \pi]$$

  where $\tilde{w} = w + \tilde{Z}$. $\therefore$ $E[U(\tilde{w})] = U(w + E[\tilde{Z}] - \pi)$ where $CE(\tilde{Z}, w) = E[\tilde{Z}] - \pi$. Hence $CE = E[\tilde{Z}] - \pi$ and then $\pi = E[\tilde{Z}] - CE$. CE will be zero if you have an actuarially fair gamble.

- Example on page 65:

Mr U	Mr W
$U(w) = \sqrt{w}$	$V(w) = \ln w$
$w = \$4000$	$w = \$4000$

$$\tilde{Z} = \begin{cases} -2000 & 1/2 \\ 2000 & 1/2 \end{cases} \implies \tilde{w} = \begin{cases} 2000 & 1/2 \\ 6000 & 1/2 \end{cases}.$$

1. Verify that U and V are well-behaved by checking the utility function is positive, increasing and concave ($U''(\cdot) < 0$)
2. Compute for π_U and π_V. We can just use the equation above to solve.

3.2.2 How to find the Degree of Risk Aversion

Arrow-Patt measure of risk aversion. π approximation

Set-up

1. $U(\cdot)$ is well behaved.
2. W = initial wealth.
3. $E[\tilde{Z}] = 0, Var(\tilde{Z}) = E[(\tilde{Z} - E[\tilde{Z}])^2] = E[\tilde{Z}^2] = \sigma^2$.

The risk premium should satisfy:

1. $E[U(\tilde{w})] = U(\tilde{w} - \pi)$
2. $E[U(w + \tilde{Z})] = U(w + E[\tilde{Z}] - \pi)$. Do the second order Taylor approximation about $\tilde{Z}$ and first order taylor approximation about π.

Hence $E[U(w) + \tilde{Z}U'(w) + \frac{1}{2}\tilde{Z}^2U''(w)] \approx U(w) - \pi U'(w)$.

Apply the "E" operator, then $U(w) + \frac{1}{2}U''(w)E[\tilde{Z}^2] \approx U(w) - \pi U'(w)$ and then $\pi \approx \frac{1}{2}\sigma^2\frac{-U''(w)}{U'(w)} \implies ARA(w) = \frac{-U''(w)}{U'(w)}$ absolute risk aversion. Hence $\pi \approx \frac{1}{2}\sigma^2 ARA(w)$

Observations:

1. π from A-P measure is useful only if $\tilde{Z}$ is small. Then for larger risks, use the Markowitz definition of risk aversion

$$U[E(\tilde{[}w)] > E[U(\tilde{w})]$$

2. (a) If $ARA(w) > 0 \implies \pi > 0 \implies$ Risk aversion (concave)
   (b) If $ARA(w) < 0 \implies \pi < 0 \implies$ Risk lover (convex)

(c) If $ARA(w) = 0 \implies \pi = 0 \implies$ Risk neutral (linear)

$$ARA(w) = \frac{-U''(w)}{U'(w)}$$

Risk averse $\implies U''(\cdot) < 0 \implies ARA(w) > 0$. $\pi \approx \frac{1}{2}\sigma^2 ARA$. If $ARA > 0 \implies \pi > 0 \implies$ risk averse

3. $[\Delta U, \text{same } w] \implies$ How ARA is affected?. Suppose there are two agents, A and B. Same w: $w_A = w_B = w$. Then w increase and ARA decrease (DARA feature, decreasing absolute risk aversion).

3.2.3 Relative Risk Aversion RRA(w)

Note: DARA, wealth increase, but ARA(w) decreases. $ARA_{poor} > ARA_{rich}$.

For example, $[\Delta w, \text{same } U]$. A is poor and with initial wealth 10 and $U^A = \ln w$. B is rich and with initial wealth 1000, and $U^B = \ln w$. Then $ARA^A(w) = \frac{-v''}{v'} = \frac{1}{w} = 0.1$. $ARA^B = \frac{1}{w} = \frac{1}{1000}$. Therefore, $ARA^A > ARA^B$. Hence w increases but ARA decreases. To verify DARA, then $\frac{d[ARA]}{dw} < 0$. Note: monotonic transformations of utility functions. Say $V = 2w$, Then $U = w$. Claim: V is a monotone transformation of U. To verify this claim.

1. Step 1: express $V = f(U) \implies V = 2U$.
2. Step 2: check the sign of $\frac{dV}{du} \implies \frac{dV}{dU} = 2 > 0$. Yes V is a monotonic transformation of U. This implies if a utility function V is a monotonic transformation of another function U. Therefore, both represent the same preference. i.e. $(MRS)^V = (MRS)^U$.

   Recall, $|MRS| = \frac{MU_1}{MU_2}$. Example: $U = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}$. Define $V = \frac{1}{2}\ln x_1 + \frac{1}{2}\ln x_2$. Claim: V is a monotonic transformation of U. Hence $MRS^V = MRS^U$.

   Proof. Take logs on both sides of U

   $$\ln U = \ln x_1^{\frac{1}{2}} + \ln x_2^{\frac{1}{2}}$$

   $$\ln U = \frac{1}{2}(\ln x_1 + \ln x_2)$$

   Therefore $V = \ln U, \frac{dV}{dU} = \frac{1}{U} > 0$ since $U > 0$. Therefore, V is a monotonic transformation of U. $\square$

Note: show that A=P approximation of π is equivalent to Morkowitg analysis if and only if the risk $\tilde{Z}$ is small. For example, $U = \ln w$ initial wealth is \$10, $\tilde{Z}(+1, -1; \frac{1}{2})$. This implies $\tilde{w}(11, 9; \frac{1}{2})$. π is from Morkowitg: $E[U(\tilde{w})] = U(E[\tilde{w}] - \pi)$. Hence $E[U(\tilde{Z} + w)] =$

$U(E[\tilde{Z}]+w-\pi)$. Hence this will be $\frac{1}{2}U(9)+\frac{1}{2}U(11) = U(10-\pi)$ using $U = \ln w$. Therefore, $\pi = 0.05$.

π from A-P approximation:

$$\pi \approx \frac{1}{2}\sigma^2_{\tilde{Z}} ARA(w)$$

$$\begin{aligned}
\sigma^2_{\tilde{Z}} &= E[\tilde{Z} - E[\tilde{Z}]]^2 \\
&= \frac{1}{2}(1^2 + (-1)^2 \\
&= 1 \\
ARA(w)\frac{-U''}{U'} &= \frac{1}{w} = \frac{1}{10} = 0.1 \\
\pi &= \frac{1}{2}1\frac{1}{10} = 0.05
\end{aligned}$$

Relative Risk Aversion

It is unit-less measure.

$$RRA = \frac{-\%\Delta MU}{\%\Delta w} = \frac{-\frac{\Delta MU}{MU}}{\frac{\Delta w}{w}}` = \frac{-\frac{dU'(w)}{U'(w)}}{\frac{dw}{w}} = w \cdot ARA$$

gives you the percentage of wealth that an investor is willing to pay to get rid of a proportional risk.

Let $\tilde{Z}_R = \frac{\tilde{Z}}{w}$ = proportional risk, $\pi_R = \frac{\pi}{w}$ = proportional premium. Note $\tilde{Z} = \tilde{Z}_R \cdot w$. Since $var[\tilde{Z}] = \sigma^2_{\tilde{Z}}$ in absolute setup, then $var[\tilde{Z}] = var[\tilde{Z}_R w], \sigma^2 = w^2\sigma^2_R$ or $\sigma^2_{\tilde{Z}} = w^2\sigma^2_{\tilde{Z}_R}$. Also recall $\pi \approx 1/2\sigma^2_{\tilde{Z}}ARA(w)$. Hence $\pi_R = 1/2w^2\sigma^2_{\tilde{Z}_R}RRA(w)$.

Note on RRA: Pratt's argument: RRA might be constant or perhaps increase. Recently, evidence shows the RRA has a U-shape. This can be explained by time series setup $w_t = f(w_{t-1}, w_{t-2}, \cdots, w_{t-k})$. This is called logistic smooth transition regression model. The consent is RRA is constant. Hence we call it CRRA. Conclusion:

- well-behaved: $U'() > 0, U''() < 0$.
- DARA
- CRRA

Q5: $\pi_R \approx \frac{1}{2}\sigma^2_R RRA(w)$

Example: 2 alternatives

outcome	x_1	Probability x_1	F_1	x_2	Probability x_2	F_2
	10	0.4	0.4	10	0.4	0.4
	1000	0.6	1	1000	0.4	0.8
	2000	0	1	2000	0.2	1

Mean-variance criterion $\implies u_A > u_B \implies$ select A or $\sigma_A < \sigma_B \implies$ Select A. The highest return and/or lowest risk. There are two projects,

Project 1 : $u_1 = E[x_1] = \sum P_i x_i = 64$. $\sigma_1 = \sqrt{E[(x_1 - E[x_1])^2} = 44$

Project 2 $u_2 = 444$; $\sigma_2 = 779$.

Hence $u_2 > u_1 \implies$ select 2. Then $\sigma_1 < \sigma_2 \implies$ Select 1. Instead of $4 - \sigma^2$, we could use the stochastic dominance criterion based on the concept of probability matching using C.D.F.

3.3 Stochastic Dominance

3.3.1 First-order stochastic Dominance (FSD)

Definition. *If F_2 F.S.D. F_1 if and only if F_2 is everywhere below to the right of F_1.*

Theorem 2. For random payoffs x_1 and x_2, $F_2(x_2)$ F.S.D. $F_1(x_1)$ if and only if $E[U_2(x_2)] \geq E[U_1(x_1)]$ for all utility functions that are non-decreasing.

This implies FSD is a good criterion to select projects if we don't know the risk attitude of the investor.

3.3.2 Second-order stochastic Dominance

Definition. *If CDF's cross, use the S.S.D. criterion. F_2 S.S.D. F_1 if and only if $\int_{-\infty}^{x} [F_1(t) - F_2(t)] \geq 0$ for any x.*

Theorem 3. F_2 S.S.D. F_1 if and only if $E[U_2(\cdot)] \geq E[U_1(\cdot)]$ for all utility functions that are non-decreasing and concave. S.S.D. is a better criterion if and only if the investor is a risk averse.

How to compute S.S.D.? For example, 2 projects.

x_1	probability 1	x_2	probability 2
1	1/3	4	1/4
6	1/3	5	1/2
8	1/3	9	1/4

We can compute the $\int_0^x [F_1 - F_2]dt \geq 0$ for all values. Therefore, 2 S.S.D. 1

4 Portfolio Theory (Markowitz Analysis)

Mean-variance Analysis of Portfolio

Connection: $(\mu - \sigma^2)$ analysis and EU (choice theory under uncertainty

Notation:

- $\tilde{r}_i$ = uncertain rate of return on Asset i, $i = 1, \cdots, n$.
- r_f = certain rate of return, risk free rate of return
- Mean of a risk rate of return
- Variance of a risk rate of return= $var[\tilde{r}_i]$
- Covariance between two risky rate of returnσ_{ij}
- Matrix notation: vector of assets returns= $\tilde{r} = \begin{pmatrix} \tilde{r}_1 \\ \vdots \\ \tilde{r}_n \end{pmatrix}$
- $n \times 1$ vector of assets weights

  $$w \begin{pmatrix} \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{pmatrix}$$

  where $\sum w_i = 1$.

Portfolio : portfolio of n assets.

Mean of portfolio return : $E[\tilde{r}_p] = \mu_p = E[w_1\tilde{r}_1 + w_2\tilde{r}_2 + \cdots + w_n\tilde{r}_n] = E[w'\tilde{r}] = w'E[\tilde{r}]$

Variance of Portfolio return $Var[\tilde{r}_p] = \sigma_p^2 = w'\Omega w$ where Ω is the covariance matrix.

Assume there are two periods: period 0 and period 1. Wealth for period 0 is y_0 and for period 1 is y_1. Let P_{i0} = current price of asset i and P_{i1} = period 1's price of asset i. For each r_i in $\tilde{r}$ where $\tilde{r}_i = \frac{P_{i1} - P_{i0}}{P_{i0}}$ At time 0, y_0 is allocated on n assets such that

$$y_0 = a_1P_{10} + a_2P_{20} + \cdots + a_nP_{n0} = \sum a_iP_{i0}$$

Definition.

$$w_i = \frac{a_iP_{i0}}{y_0}$$

$$r_p = w'\tilde{r}$$

$$\tilde{y}_1 = \text{consumer's wealth at the end of period 1}$$

$$\tilde{y}_1 = \sum a_i P_{i1}$$

We can express it as

$$\tilde{y}_1 = \sum a_i(P_{i1} - P_{i0}) + \sum a_i P_{i0} = y_0$$

$$\tilde{y}_1 = \sum a_i P_{i0} \frac{P_{i1} - P_{i0}}{P_{i0}} + y_0 = y_0(1 + \frac{\sum a_i P_{i0}}{y_0}(\frac{P_{i1} - P_{i0}}{P_{i0}})) = y_0(1 + \sum w_i \tilde{r}_i)$$

$$\tilde{y}_1 = y_0(1 + w'\tilde{r})$$

End of period wealth depends on the portfolio weights and rates of returns. Take expectations

$$E[\tilde{y}_1] = y_0(1 + w'E[\tilde{r}])$$

$$Var = y_0^2 w'\Omega w = y_0^2 \sigma_p^2$$

4.1 Agent's problem

For a given $\tilde{r}_p$, the consumer chooses $w_{n\times 1} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ such that $var(\tilde{r}_p) = \sigma_p^2$ is minimized.

Hence $\min \sigma_p^2 \iff \max E[U(\tilde{y}_1)]$.

Proof. Let $U(\tilde{y}_1)$ be well-behaved. Perform a second order Taylor approximation about $E[\tilde{y}_1] \implies U(\tilde{y}_1) \approx U[E[\tilde{y}_1]] + U'()(\tilde{y}_1 - E[\tilde{y}_1]) + \frac{1}{2}U''()(\tilde{y}_1 - E[\tilde{y}_1])^2 + R$ We isn the remainder hopefully negligible.

Take expectations from both side. $E[U(\tilde{y}_1)]U[E[\tilde{y}_1]] + \frac{1}{2}U''()var[\tilde{y}_1] + R$. Hence we get the link between consumer choice theory and mean-variance portfolio theory. □

Observations:

1. $\max E[U(\tilde{y}_1)] = \min \sigma_p^2 \iff U''()$ is negative. This implies that utility function is concave and the consumer is risk-averse.

2. R has to go to zero. $R = 0$ exactly if $U()$ is quadratic.

3. Quadratic: U is not desirable. Then an alternative way is putting a distribution or $\tilde{r}_i$. If $\tilde{r}_i \sim \tilde{N} \implies E[U(\tilde{y}_i)]$ can be maximized and the probability would be equivalent to minimize σ_p^2. This implies we can represent $E[U(\tilde{y}_1)]$ in $(\mu_p - \sigma_p)$ space.

4.2 Morkowitz Analysis

Decision Problem: choose an optimal weight, w, such that σ_p^2 is min given $E[\tilde{r}_p] = \mu_p$.

$$\min_{\{w_{n\times 1}\}} \sigma_p^2 = w'_{1\times n}\Omega_{n\times n}w_{n\times 1}$$

$$\text{such that } w'\mu = \mu_{p1\times 1}$$

$$(w_1, \cdots, w_n)\begin{pmatrix}\mu_1\\ \vdots \\ \mu_n\end{pmatrix} = \sum w_i\mu_i = \mu_p$$

The section to this problem is a vector $(n \times 1)$ of optimal weights $\hat{w}$ that defines the min variance portfolio-given μ_p. The solution of the Markowitz problem is a set of minimum variance frontier such that each optimal σ_p^2 corresponds to a different target μ_p. Set of minimum variance frontier is just a parabola. Let O be the minimum value of portfolio. The upper leg of the parabola eliminates the lower leg. Therefore we call the set of portfolio's starting from MVP. and going up along the parabola (the dashed part of the figure). We call the set of efficient frontier.

4.2.1 The Algebra of the Portfolio Frontier

$$\min \sigma_p^2 = \frac{1}{2}w'\Omega w$$

subject to

1. $w'\mu = \mu p$
2. $w'1 = 1$.

where Ω is a covariance matrix.

Define $\mathcal{L} = \frac{1}{2}w'\Omega w + \lambda_1[\mu p - w'\mu] + \lambda_2[1 - w'1]$.

FOC

1. $$\frac{\partial \mathcal{L}}{\partial w_{n\times 1}} = 0_{n\times 1} \implies \Omega w - \lambda_1\mu - \lambda_2 1 = 0$$
2. $$\frac{\partial L}{\partial \lambda_1} = 0_{1\times 1} \implies \mu p - w'\mu = 0 \implies w'\mu = \mu p \iff \mu' w = \mu p$$
3. $$\frac{\partial \mathcal{L}}{\partial \lambda_2} = 0_{1\times 1} \implies w'1 = 1 \iff 1'w = 1$$

   from 1), $\Omega w = \lambda_1\mu + \lambda_2 1$. Times Ω^{-1} (assume it exists) $w = \lambda_1\Omega^{-1}\mu + \lambda_2\Omega^{-1}1$ We need to define λ_1 and λ_2. Take equation 2), and multiply by u'. Take equation 4) and do the following:

- $u' \implies u'w = \lambda_1 u'\Omega^{-1}\mu + \lambda_2 u'\Omega^{-1}1$ (5)
- $1' \implies 1'w = \lambda_1 1'\Omega^{-1}\mu + \lambda_2 1'\Omega^{-1}1$ (6)
- By setting 2) = 5) and 3) = 6), then $\mu_p = \lambda_1\alpha + \lambda_2 b$ and $1 = \lambda_1 b + \lambda_2 c$.

In a compact notation

$$\begin{pmatrix} \mu_p \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} (7)$$

where $\Psi = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

Solve $\lambda's$, Then $\times\Psi^{-1}$, More explicitly,

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \Psi^{-1} \begin{pmatrix} \mu p \\ 1 \end{pmatrix}$$

Then we can solve for w. Hence

$$\hat{w} = (\frac{c\mu_p - b}{d})\Omega^{-1}\mu + \frac{a - b\mu_p}{d}\Omega^{-1}1_{n\times 1}$$

Therefore,

$$\hat{w} = \Phi + \Theta\mu_p$$

where

(a) $a = \mu'\Omega^{-1}\mu$

(b) $b = \mu'\Omega^{-1}1$

(c) $c = 1'\Omega^{-1}1$

(d) $d = ac - b^2$

(e) $\Phi = \frac{a\Omega^{-1}1 - b\Omega^{-1}\mu}{d}$

(f) $\Theta = \frac{c\Omega^{-1}\mu - b\Omega^{-1}1}{d}$

What is the min-variance corresponding to $\hat{w} \implies \sigma_p^2$. Therefore, $\hat{\sigma^2}_p = \hat{w}'\Omega\hat{w}$. Therefore, $\hat{\sigma}_p^2 = \frac{c}{d}(\mu_p - \frac{b}{c})^2 + \frac{1}{c}$.

This is a parabola in $(\sigma^2 - \mu_p)$ space. Mean-variance frontier equation in case of n risky asset. Digression on parabolas and hyperbolas: $(\sigma_p^2 - \mu_p) \implies$ parabola and $\sigma - \mu_p$ is hyperbola.

For MVP portfolio, $(\mu_p)_{MVP} = \frac{d}{c}$ and $(\sigma_p^p)_{MVP} = \frac{1}{p}$ and $\hat{w} = \Phi + \Theta(\mu_p)_{MVP}$. If your expected return of portfolio is zero, then $\hat{w} = \Phi$.

Material Covered: Chapter 1 (only what was covered in class), Chapter 2 (All), Chapter 3 (till today's lecture).

4.3 Portfolio Theory

$$min\sigma_p^2 = maxE[U(\tilde{y}_1)]$$

How can we compute the rate of return on any individual financial asset (stock)? And then, minimize the σ_p^2 of a portfolio formed from n of these stocks?

Now, let $n = 2$, then stock 1: apple stock, stock 2: google stock. Period 0 price: P_{10}, P_{20}, spot prices Period 1 prices: state contingent prices (state of nature that could exist next period. Say we have 3 states $\Theta_1 = \text{expansion}, \Theta_2 = \text{steady}, \Theta_3 = \text{Recession}$. (Contingent means dependent) $(P_{11})_{\Theta_1}$ = apple stock price in period 1 contingent on Θ_1, $(P_{11})_{\Theta_2}$ and $(P_{11})_{\Theta_3}$

Prob	State	Stock 1	Stock 2
1/3	θ_1	$(P_{11})_{\Theta_1}$	$(P_{21})_{\Theta_1}$
1/3	θ_2	$(P_{11})_{\Theta_2}$	$(P_{21})_{\Theta_2}$
1/3	θ_3	$(P_{11})_{\Theta_3}$	$(P_{21})_{\Theta_3}$

1. $\Theta_1 : (r_1)_{\Theta_1} = \frac{(P_{11})_{\Theta_1} - P_{10}}{P_{10}} \times 100\%$
2. $\Theta_1 : (r_1)_{\Theta_2} = \frac{(P_{11})_{\Theta_2} - P_{10}}{P_{10}} \times 100\%$
3. $\Theta_1 : (r_1)_{\Theta_3} = \frac{(P_{11})_{\Theta_3} - P_{10}}{P_{10}} \times 100\%$

Then we can get $\mu_1 = E[r_1]$ and $var(\tilde{r}_1) = E[(\tilde{r}_1 - \mu_1)^2]$. We can use the data to get all the metrics we need to solve the portfolio question.

4.4 Shape of the portfolio frontier

Case 1 $n = 2$ risky-assets with $P_{12} = +1$ (correlation coefficient). Recall:

$$P_{12} = \frac{cov(r_1, r_2)}{\sigma_1 \cdot \sigma_2} = \frac{\sigma_{12}}{\sigma_1 \cdot \sigma_2}$$

This implies $\sigma_{12} = P_{12} \cdot \sigma_1 \cdot \sigma_2(1)$

Note: $P_{12} = +1 \implies \sigma_{12} = \sigma_1\sigma_2(2)$

2 assets w_1, w_2 such that $w_1 + w_2 = 1 \implies w_2 = 1 - w_1$. THerefore $E[\tilde{r}_p]\mu_p = w_1\mu_1 + (1 - w_1)\mu_2$ can be expressed as

$$\mu_p = \mu_1 + (1 - w_1)(\mu_2 - \mu_1)$$

.

$$Var(\tilde{r}_p] = \sigma_p^2 = w_1^2\sigma_{11} + (1 - w_1)^2\sigma_{22} + 2w_1(1 - w_1)\sigma_{12}$$

From 1 we get

$$\sigma_2 = w_1^2\sigma_{11} + (1-w_1)^2\sigma_{22} + 2w_1(1-w_1)\sigma_1\sigma_2(4)$$

If $P_{12} = +1$, then σ_p^2 is a perfect square

$$\sigma_p^2 = (w_1\sigma_1 + (1-w_1)\sigma_2)^2$$

Therefore

$$\sigma_p = w_1\sigma_1 + (1-w_1)\sigma_2(5)$$

from 5 we get

$$w_1 = \frac{\sigma_p - \sigma_2}{\sigma_1 - \sigma_2}(6)$$

$$(1-w_1) = \frac{\sigma_1 - \sigma_p}{\sigma_1 - \sigma_2}(7)$$

$$\mu_p = \mu_1 + \frac{\sigma_2 - \mu_1}{\sigma_2 - \sigma_1}(\sigma_p - \sigma_1)$$

Equation of mean-variance frontier in mean-variance space is a straight line with the slope

$$\frac{d\mu_p}{d\sigma_p} = \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}$$

$w_1 \implies$ from $5, \sigma_p = \sigma_1$ also from 3 we get $\mu_p = \mu_1$ (A). if $w_1 = 0$, we get $\sigma_p = \sigma_2, \mu_p = \mu_2$ (B).

If $w_1 \neq 0, w_1 \neq 1$, in case $P_{12} = +1$, Mean variance frontier and the efficient frontier are the same represented by line A-B. . If $-1 < P_{12} < +1$, then from 4, σ_p^2 will be smaller, This implies the line AB will move to the left. If $|P_{12} < 1$, then the mean-variance frontier will be different than the efficient frontier; instead, it curve to the left with A and B two ending points.

Case 2 $-1 < P_{12} < +1$. Note in mean and variance space, the min-variance frontier is a parabola.

Case 3 $P_{12} = -1$. Perfectly negatively correlated. Again $\sigma_p^2 = [w_1\sigma_1 - (1-w_1)\sigma_2]^2 \implies \sigma_p = \pm[w_1(\sigma_1+\sigma_2) - \sigma^2]$ solve for $w_1 : \implies w_1 = \frac{\pm\sigma_p + \sigma_2}{\sigma_1 + \sigma_2}$. Plug w_1 in the portfolio mean equation

$$\tilde{r}_p = w_1\tilde{r}_1 + (1-w_1)\tilde{r}_2$$

$$\mu_p = w_1\mu_1 + (1-w_1)\mu_2$$

Substitute w_1 in μ_p and simplify

$$\mu_p = (\frac{\sigma_2}{\sigma_1+\sigma_2}\mu_1 + \frac{\sigma_1}{\sigma_1+\sigma_2}\mu_2) \pm \frac{\mu_1 - \mu_2}{\sigma_1+\sigma_2}\sigma_p$$

$[\mu_p = \alpha \pm \beta\sigma_p]$ equation of the min-variance frontier in case $P_{12} = -1$. Note MVP in case $p = -1$ is a risk-free portfolio ($\sigma_p = 0$).

Case 4 A risky-asset

$$-1 < P_{ij} < +1$$

No perfect correlation between any two assets i and j, $i \neq j$. Our general case:

$$\sigma_p^2 = \frac{c}{d}(\mu_p - \frac{b}{c})^2 + \frac{1}{2}$$

This is the equation of the min-variance frontier in case of n risky assets.

Case 5 $n = 2$, 1 is risk free and 2 is risky. $\mu_1 = r_f$ and $E[\tilde{r}_2] = \mu_2$ such that $\mu_2 > r_f$. $\sigma_0 = 0, \sigma_{11} = 0, \sigma_2 > \sigma_1, \sigma_{22} > \sigma_{11}$. Claim: in this case the min-variance frontier is a straight line starting the r_f rate on the vertical axis.

$$\tilde{r}_p = w_1\tilde{r}_1 + (1 - w_1)\tilde{r}_2$$

$$var[\tilde{r}_p] = \sigma_p^2 = w_1^2\sigma_{11} + (1 - w_1)^2\sigma_{22} + 2w_1(1 - w_1)\sigma_{12}$$

Hence $\sigma_p = (1 - w_1)\sigma_2$. Solve for $w_1 \implies [w_1 = 1 - \frac{\sigma_p}{\sigma_2}]$. Plug w_1 in μ_p of the portfolio:

$$\mu_p = w_1 r_f + (1 - w_1)\mu_2$$

$$\mu_p = (1 - \frac{\sigma_p}{\sigma_2})r_f + (1 - 1 + \frac{\sigma_p}{\sigma_2})\mu_2$$

$$\mu_p = r_f + \frac{\mu_2 - rf}{\sigma_2}\sigma_p$$

Case 6 A risky assets and 1 risk free. Why risk free? to ensure a guaranteed rate for borrowing and learning securities.

Claim: In this case, min-variance frontier is a hyperbola but the efficient frontier is a straight-line.

Proof. Efficient frontier is AM? Consider risk free and portfolio E. Hence AE is the min-variance frontier. if a risk-free asset with portfolio F, then AF is the min-variance frontier. AF is preferred. AF is preferred to AE and so on. $AM \succ AF \succ AE \cdots$ and AM is the best constructed min-variance portfolio. i,e, AM is the efficient frontier. Implications (optimal market portfolio).

Question: what is the optimal portfolio that will maximize the investor's mean-variance utility.

Everyone would like to hold risk free and M regardless of the degree of risk aversion. This implies that you could have 3 equilibria. There exist three investors, aggressive, moderate and conservative. This is known as the two-fund separation theorem: everyone will be on the efficient frontier (line from risk free and passing by M) regardless of their risk aversion.

Assumptions:

1. borrowing is the same as lending
2. homogeneous beliefs regarding mean and covariance matrix.

□

4.5 Midterm

- Chapter 1: P5 - P17: Consumer Choice Under Certainty
  - Optimal Choice
  - $\frac{MU_1}{MU_2} = \frac{P_1}{P_2}$ and $m = P_1x_1 + P_2x_2$
  - Explanation of the tangency condition.
  - P18 is not required
  - P19 to 48 is not required
- Chapter 2
  - Implication of the Axioms
  - No need for the proof of VN-M EU function
  - Page 80 and 81 Pnat's 3 proposition are not required
- Chapter 3
  - Markowitz analysis (P83 - 90)
  - Page 91: EU as a function of mean-variance.
  - mid 90-94 Only the graph is required.
  - 95-102
  - Case 1: P 103, 104 only.

4.6 3 Stocks Universe

Stock	μ	σ	GM	IBM	Apple
GM	1.1	6	38	16	23
IBM	1.3	7	16	40	23
Apple	1.7	9	23	23	94

5 CAPM

5.1 Rationale of CAPM

Markowitz Analysis

1. Individual stocks are represented by a point (each) in the $(\mu - \sigma)$ space.
2. The market portfolio is approximated as a market index. (SP 500, Russell 2000).
3. Combine n risky stocks in one portfolio. This implies from Markowitz, the portfolio will be presented by one point on the min-variance frontier, which is a hyperbola in this case.
4. The upper leg of the frontier is better (higher return for the same risk). This is the efficient frontier.
5. Rational investors should like to be on the farthest NW point in $(\mu - \sigma)$ space. This implies investors would like to be on the efficient frontier. Note: Rational investors are those who prefer more return for the same risk or less risk for the same return.
6. Forming a market using Markowitz analysis $\Longrightarrow$ requires a rate at which investors can borrow and lend $\Longrightarrow$ this market could be constructed by adding a risk free asset (TBs).

   Consider the mean-variance space, let AM be efficient frontier. Then all rational investors would like to be on AM but where?

   Portfolio A 100% of your wealth is in TB

$$w_A = \text{weight of your wealth invested in rf}$$

$$\tilde{r}_A = 1 \cdot rf + \text{zero}$$
$$E[\tilde{r}_A] = rf$$
$$\sigma[\tilde{r}_A] = 0$$

.

Portfolio M $E[\tilde{r}_M] = \mu$ and $[\tilde{r}_m] = \sigma_M$

Portfolio L w_L in the market, $w_L < 1$ and $(1 - w_L)$ in the risk free.

$$\tilde{r}_L = w_L \tilde{r}_M + (1 - w_L) r_f$$
$$E[\tilde{r}_L] = w_L^2 \sigma_M^2 + (1 - w_L)^2 0 + 2 w_L (1 - w_L) 0$$

Therefore, $\sigma_L^2 = w_L^2 \sigma_M^2$ and $\sigma_= w_L \sigma_M$.

7. In general ($\sigma_p = w_p \sigma_\mu$). For any portfolio p on the efficient frontier.

   Portfolio B On the extended line all. $w_B > 1$ w_B must be greater than 2 is he is borrowing. $\tilde{r}_B = w_B \tilde{r}_\mu] - (w_B - 1) rf = w_B \mu_\mu - (w_B - 1) rf$
   $\sigma_B^2 = w_B^2 \sigma_\mu^2 \implies \sigma_B = w_B \sigma_\mu$.

8. The Sharpe Ratio: for any portfolio P, the Sharpe ratio is the slope of the efficient frontier at p.

$$\text{Sharpe Ratio} = \frac{E[\tilde{r}_p] - rf}{\sigma_p}$$

   where the denominator is risk of portfolio p and the numerator is excess return of portfolio p. Sharpe Ratio measures the performance of portfolio p. For portfolio managers, the Sharpe ratio is a measure of their value added service.

   Note: Sharpe Ratio at x has to be greater than Sharpe ratio at point p.

9. M is the market portfolio forced by All investors.This implies the line AM is called Capital Market Line (CML).

$$\text{Slope of CML} = \frac{E[\tilde{r}_M] - rf}{\sigma_M}$$

   and the CML equation: for any portfolio P on the efficient frontier:

$$E[\tilde{r}_p] = rf + \frac{E[\tilde{r}_M] - rf}{\sigma_M} \sigma_p$$

   CML reprint the risk-return relation of efficient portfolios only. What about individual stock?

10. In this analysis, risk is measured by σ. What about individual risky assets that are inefficient? CAPM is needed. The risk of individual assets is measured by its beta coefficient.

11. CAPM: $E[\tilde{r}_j] = rf +$ asset's risk premium.

    - $$\text{Amount of risk due to stock j} = \beta_j \sigma_M$$

      where σ_M is market risk

    - $$\text{Price of risk} = \frac{(E[\tilde{r}_M] - rf)}{\sigma_M}$$

- $$E[\tilde{r}_j] = rf + \beta_j\sigma_M \times \frac{E[\tilde{r}_M] - rf)}{\sigma_M}$$

  E.x. Say the market consists of 2 stocks: 1 and 2

  $$\tilde{r}_M = w_1\tilde{r}_1 + w_2\tilde{r}_2$$

  where $E[\tilde{r}_M] = w_1E[\tilde{r}_1] + w_2E[\tilde{r}_2]$ and $var[\tilde{r}_M] = \sigma_M^2 = w_1^2\sigma_{11} + w_2^2\sigma_{22} + 2w_1w_2\sigma_{12}$. Say you want to hold asset $j = 1$. $E[\tilde{r}_1] =?$ required rate of return on 1. and $E[\tilde{r}_1] = rf +$ (premium).

- The risk contribution of asset 1: $\Delta\sigma_M$ as a result of increase asset 1's share in the market. $\frac{d[\sigma_M]}{dw_1} = \frac{1}{\sigma_M}(w_1\sigma_{11} + w_2\sigma_{22})$
  Note:

  $$\begin{aligned} cov(\tilde{r}_1, \tilde{r}_M) &= cov(\tilde{r}_1, w_1\tilde{r}_1 + w_2\tilde{r}_2) \\ &= w_1 cov(\tilde{r}_1, \tilde{r}_1) + w_2 cov(\tilde{r}_1, \tilde{r}_2) = w_1\sigma_{11} + w_2\sigma_{12} \end{aligned}$$

  Hence

  $$\frac{d\sigma_M}{dw_1} = \frac{cov(\tilde{r}_1, \tilde{r}_M)}{\sigma_M}$$

  Therefore, amount of risk due to asset 1 = $\beta_{1,M}{\cdot}\sigma_M$. Hence general price of risk = $\frac{E[\tilde{r}_M - rf}{\sigma_M}$. Therefore, $E[\tilde{r}_1] = rf + \beta_{1,M}(E[\tilde{r}_M] - rf)$
  In general, for any stock j, such that the market consists of n stocks, then

  $$E[\tilde{r}_j] = rf + \beta_{j,M}(E[\tilde{r}_M] - rf)$$

  where $\beta_{j,M} = \frac{cov(\tilde{r}_j, \tilde{r}_M)}{\sigma_M^2}$.
  Decision: if the actual rate of return is great than required. Then invest in j.Otherwise, don't invest.

  $$\beta_{r\delta} = 0$$

5.2 Observations on CAPM

1. Asset j's risk contribution: $\beta_j \cdot \sigma_M$.
2. Asset j's systematic risk β_j. Risk that you cannot diversify (market risk).
3. Sign of beta:

$$\begin{aligned} \beta_j = 0 &\implies E[\tilde{r}_j] = rf \\ \beta_j = 1 &\implies E[\tilde{r}_j] = E[\tilde{r}_u] \\ \beta_j > 1 &\implies E[\tilde{r}_j] > E[\tilde{r}_u] \\ \beta_j < 1 &\implies E[\tilde{r}_j] < E[\tilde{r}_u] \\ \beta_j < 0 &\implies \text{Possible is } Cov(\tilde{r}_j, \tilde{r}_u) \text{ is negative} \end{aligned}$$

High risk associated with high return. Less risk means less return. We would want this stock in our portfolio just in case Market crashes (this stocks resets inversely. (Os these stocks exist, when the market crashes, these stocks aren't correlated with the market and demand increase, the price increase and rate drops). Hence a stock with negative beta is appealing. Everyone would demand that stock and this drives the stock price increase.

$$r = \frac{P_1 - P_0}{P_0} \times 100\% \implies r\text{decrease}$$

Ideal case: To send a stock with a negative beta and a positive $\tilde{r}$. In practice: the betas of the stocks traded in the market range between $[0.5 - 1.5]$. Negative beta is rare.

If β_j, for stock j, is extremely negative. $\tilde{r}_j < 0$. and investors are holding it?

The idea here is risk aversion. The investors are willing to pay an amount (negative return) in order to hold that stock to avoid (hedge) market risk.

If $\beta_j \approx 0 \implies$ does this mean that volatility $= 0$? No volatility $= \sigma_j =$ measure of total risk. Systematic risk is market risk and nondiversible risk. Non-systemmatic risk is firm specific risk or idiosyncratic risk. Note: $\sigma_j^2 = var[\tilde{r}_j]$ and $\sqrt{\sigma_j^2} = \sigma_j =$ measure of volatility. (systematic risk: due to common factors, This cannot be dominated by diversification; non-systematic risk: firm-specific risk can be eliminated by diversification).

Example:

Consider two stocks: AAPL: A, Gillette: G. Monthly rates of return on both stocks. Suppose $\beta_A = 1.4, \beta_G = 0.6$ and $rf = 5\%$ and market premium $= (E[\tilde{r}_M] - rf) = 6\%$.

What is the required rate of return on each stock?

From CAPM, $E[\tilde{r}_A] = rf + \beta_A(E[\tilde{r}_M - rf])$. $E[\tilde{r}_A] = 5\% + 1.4(6\%) = 13.4\%$.

Decision: required v.s. actual (realized) rate. If

$$13.4\% < \text{actual } \tilde{r}_A \implies \text{buy it}$$

$$13.4\% > \text{actual } \tilde{r}_B \implies \text{don't buy it}$$

Similarly for G

- CAPM in practice: 3 applications
  (a) CAPM can be used to find the required return on risky portfolios instead of one single asset .n stocks

$$r_p = w_1\tilde{r}_1 + w_2\tilde{r}_2 + \cdots + w_n\tilde{r}_n$$

$$\beta_p = \frac{cov[\tilde{r}_p, \tilde{r}_M]}{var[\tilde{r}_M]} = \frac{cov[w_1\tilde{r}_1 + w_2\tilde{r}_2 + \cdots + w_n\tilde{r}_n]}{var[\tilde{r}_M]}$$
$$= \frac{w_1 cov(\tilde{r}_1, r_M + w_2 cov(\tilde{r}_2, \tilde{r}_M) + \cdots + cov(\tilde{r}_n, \tilde{r}_M)}{var(\tilde{r}_M}$$
$$\beta_p = w_1\beta_1 + w_2\beta_2 + \cdots + w_n\beta_n$$

This implies that we can model $E[\tilde{r}_p]$ and β_p using CAPM. For any portfolio p:

$$E[\tilde{r}_p] = rf + \beta_p(E[\tilde{r}_M] - rf)$$

(b) Measuring performance of portfolio using CAPM [Alpha of the portfolio]. Example: average rate of return (annually) on fund $p = 14.85\%$
   - Historical date, $rf = 5\%$ and market premium $= 6\%$. $E[\tilde{r}] = rf + \beta$ market premium and $E[\tilde{r}] = 5\% + \beta 6\%$.
   - Market: M, $\beta_M = 1 \implies E[\tilde{r}_M] = 11\%$.
   - for $b = -0.025 \implies E[\tilde{r}] = 5\% - 0.025(6\%) = 4.85\%$. This is the required rate of return.

(c) Using the beta of corporation to compute NPV, i.e., using β as a discount rate.

5.3 Empirical Tests of CAPM

$$E[\tilde{r}_i] - rf = \beta_i(E[\tilde{r}_M - rf)$$

(Ex-ante) Apply the following transformation: the realized rate of return on stock i is approximately expected rate of return. On average $\tilde{r}_{i_t} \approx E[\tilde{r}_i]$ for $i = 1, \cdots, T$. This is called the fair game equation. Given this fact, the fair game equation can be written as follows:

$$\tilde{r}_{i_t} = E[\tilde{r}_i] + \beta_i[\tilde{r}_{M_t} - E[\tilde{r}_{M_t}]]$$

the realized rate of return on i is the expected return plus β_i times the difference of realized return on market and expected market return. Take expectations:

$$E[\tilde{r}_{i_t}] = E[\tilde{r}_i]$$

the average realized return is equal to expected rate of return. Hence

$$\tilde{r}_{i_t} = rf + (E[\tilde{r}_M] - rf)\beta_i + \beta_i(\tilde{r}_{M_t} - E[\tilde{r}_{M_t}])$$

satisfy

$$(\tilde{r}_{i_t} - rf)_t = \beta_i(\tilde{r}_{M_t} - rf_t)$$

(Ex-post CAPM)

Add α_i = intercept and $\tilde{\epsilon}_i$ = error-term to render the ex-post CAPM Stochastic.

Hence

$$(\tilde{r}_{i_t} - rf_t) = \alpha_1 + \beta_1(\tilde{r}_{M_t} - rf_t) + \tilde{\epsilon}_{i_t}$$

stochastic ex-post CAPM.

There are two usages for this equation:

1. Academic: Test if $\alpha_1 = 0$ or not (hypothesis test)

2. Practice (fund manager): Care about 3 characteristics of the portfolio

   (a) Beta: beta of the portfolio: the stochastic ex-post CAPM can be expressed for a portfolio p of n stocks as

   $$\tilde{r}_{pt} - rf_t = \alpha_p + \beta_p(\tilde{r}_{M_t} - rf_\epsilon) + \tilde{\epsilon}_{p_\epsilon}$$

   where $\beta_p = w_1\beta - 1 + \cdots + w_n\beta_n$ and $\tilde{\epsilon}_p = w_1\tilde{\epsilon}_1 + \cdots + w_n\tilde{\epsilon}_n = \sum w_i\tilde{\epsilon}_i$.
   A portfolio manager can estimate β_p and this can be a measure of systematic risk of the portfolio.

   (b) Alpha: α_p. Step 1: find β_p from above equation and step 2: plug β_p in the CAPM (original equation) and compute $E[\tilde{r}_p]$ = required rate of return on the portfolio. For example $\beta_p = 1.5$ and $rf = 5\%$ and market premium is 6%. CAPM: $E[\tilde{r}_p] = 5\% + 1.5(6\%) = 14\%$. This is required rate of return. Step 3: Compute the average rate of return of the portfolio over the life time of the fund. and take it as actual rate of return. For example, say over the last 10 years, $(r_p)_{actual} = 18\%$. Then α_p = actual – required = 4%. This is the measure of performance.

   (c) Sigma: The volatility of the portfolio: consider the stochastic ex-post CAPM and assuming that rf_t is fixed. This implies $\tilde{r}_{p_t} = \Gamma + \beta_p\tilde{r}_{M_t} + \tilde{\epsilon}_{p_t}$. The middle part is called systematic risk and the last part is called idiosyncratic or firm specific risk. This is due to Market (common) factors. If w_i is very small and n is very large, then we can get rid of $\tilde{\epsilon}_{p_t}$.

   Proof.

   $$var[\tilde{r}_p] = \beta_p^2\sigma_M^2 + var[\sum w_i\tilde{\epsilon}_i]$$

   if $w_i \approx \frac{1}{n}$, then the lost term:

   $$\begin{aligned} var[\sum \frac{1}{n}\tilde{\epsilon}_i] &= \frac{1}{n^2}var[\sum \tilde{\epsilon}_i] \\ &= \frac{1}{n}\frac{1}{n}\sum var(\tilde{\epsilon}_i) = \frac{1}{n}\bar{\epsilon}_i^2 = \frac{1}{n}\bar{\sigma}^2 \\ n \to \infty \lim \frac{1}{n}\bar{\sigma}^2 &\to 0 \\ \implies \tilde{r}_p &= \Gamma + \beta_p\tilde{r}_{M_t} \end{aligned}$$

Note: although we diversified away the non-systematic risk, there might still be other factors than $(\beta_p r_{M_t})$ that can affect $\tilde{r}_p$. Those factors are captured by Γ. Then ${}_p^2 = \beta^2 \sigma_M^2 \implies \sigma_p = \beta \sigma_M$. $\square$

5.4 Formal Derivation of the CAPM

See p. 142 in the Course Notes.

Also remark that an alternative formulation is to maximize the slope of the CML or the Sharpe ratio. That is

$$\begin{aligned} \max_{\{w_i\}} \frac{(\mu_p)_T - r_f}{\sigma_p} &= \frac{w^t \mu - r_f}{(w^t \Omega w)^{1/2}} \\ \text{subject to} & \quad w^t 1 = 1 \end{aligned}$$

after some tedious algebra,

$$(\hat{w})_T = \frac{\Omega^{-1}(\mu - r_f \mathbf{1})}{\mathbf{1}^t \Omega^{-1}(\mu - r_f \mathbf{1})}$$

5.5 Arbitrage Pricing Theory

Recall the assumptions for CAPM

1. Everyone calculates the same market portfolio
2. No friction in the market (no transaction costs)
3. Supply = Demand (market is always in equilibirum)
4. Everyone has access to the risk-free rate

and the only factor that explains the variations in $\tilde{r}_{it}$ is the market return $\tilde{r}_{Mt}$.

In the *Arbitrage Pricing Theory,* we assume that $k > 1$ factors can explain the variation in $\tilde{r}_{it}$. Examples can include changes in

- GDP
- Inflation
- Interest Rate
- Cost of Labour

Let's say that the GDP decreases. Then economic activities decrease, aggregate demand decrease, and sales of corporations will drop. This implies that future expected cash flows will drop. So the net present value of corporations will drop and their overall values drop. The price of a corporation will then drop and so will $\tilde{r}_{it}$.

Consider utility companies and transportation companies. As GDP goes down, the effect is more profound on transportation companies because the demand for utility will remain relatively constant.

However, with regards to a change in interest rates (bank rates), if it decreases our demand for transportation will increase (lower borrowing costs) however, utility companies will not be affect much due to stable demand. In conclusion, we have k factors affecting $\tilde{r}$ in different magnitudes according to the sector in which the corporation is operating.

5.5.1 The Idea of arbitrage

Exmaple: consider a single factor model. Two well-diversified portfolios: A and B (all non-systematic risks are diversified). Assume that they have the same betas but different expected return.

$$E[\tilde{r}_A] = 14\%, \beta_A = 1.5$$

$$E[\tilde{r}_B] = 10\%, \beta_B = 1.5$$

How to arbitrage?

$$\tilde{r}_p = E[\tilde{r}_p] + \beta_p \tilde{F} + \tilde{\epsilon}_p$$

Well diversified $\implies n \to \infty \implies \tilde{\epsilon}_p \to 0$.

$$\begin{cases} \tilde{r}_A &= E[\tilde{r}_A] + \beta_A \tilde{F} \\ \tilde{r}_B &= E[\tilde{r}_B] + \beta_B \tilde{F} \end{cases}$$

Long position A: $14\% + 1.5\tilde{F}$ and short position in B: $-10\% - 1.8\tilde{F}$. Net proceeds: 4%. THere is arbitrage.

Conclusion: well diversified portfolios with equal betas must have equal expected return if no arbitrage is assumed.

Example 2: 2 portfolios well-diversified such that $E[\tilde{r}_M] = 11\%, \beta_M = 1, E[\tilde{r}_D] = 7\%, \beta_D = 0.5$. $rf = 5\%$. Is this consistent with no arbitrage assumption? SML: $E[\tilde{r}_p] = rf + \beta_p(E[\tilde{r}_M] - rf)$ and $E[\tilde{r}_p] = 5 + 6\beta_p$.

Portfolio D: $\beta_D = 0.5$, then from SML, required rate of return $D = 5 + 6(0.5) = 8$.

How can we construct N? $E[\tilde{r}_N] = 8\% = w_{rf} \cdot rf + w_M E[\tilde{r}_M]$. Therefore, $w_{rf} + w_M = 1$. Therefore, $w = 1/2$. Therefore, $E[\tilde{r}_N] = 8\%$ and $\beta_N = 0.5$.

Conclusion: Two well-diversified portfolio with different betas and different $E[\tilde{r}]$ should be on the SML to be consistent with no arbitrage condition.

The arbitrage pricing theory is based on this condition!

It is based on multi factor models:

$$\tilde{r}_i = E[\tilde{r}_i] + b_{1i}\tilde{F}_1 + b_{2i}\tilde{F}_2 + \cdots + b_{ki}\tilde{F}_k + \tilde{\epsilon}_i$$

Note: $(\beta_i)_{CAPM} = \frac{cov(\tilde{r}_i, \tilde{r}_n)}{\sigma_M^2} \neq b_i$. Here $b_{ik} = \frac{d[\tilde{r}_i]}{d[\tilde{F}_k]}$. This can be exteneded to portfolio p:

$$\tilde{r}_p = E[\tilde{r}_i] + b_{1i}\tilde{F}_1 + b_{2i}\tilde{F}_2 + \cdots + b_{ki}\tilde{F}_k + \tilde{\epsilon}_i$$

where $\tilde{\epsilon}_p = \sum w_i \tilde{\epsilon}_i$

APT is based on the following:

1. The capital (financial) market is in equilibrium.
2. Homogeneous beliefs.
3. It requires a inch market (large n).

Such that portfolios constructed in this inch market should satisfy 3 properties:

1. Property 1: self financed is just zero cost portfolio
2. Property 2: Well-diversified is $\tilde{\epsilon}_p \to 0, n \to \infty$
3. Property 3: zero sensitivity to factor loadings.

$$\tilde{r}_p = E[\tilde{r}_p] + b_{p1}\tilde{F}_1 + b_{p2}\tilde{F}_2 + \epsilon_p$$

By property 3 we know that $\sum w_i b_{ik} = 0$ for each factor.

Therefore, 2 implies $\tilde{\epsilon}_p = 0$ and 3 implies $\sum w_i b_i = 0$ for each factor $w'b = 0$.

Multifactor model

$$\tilde{r}_p = E[\tilde{r}_p] + b_{1p}\tilde{F}_1 + \cdots + b_{kp}\tilde{F}_k + \sum w_i \tilde{\epsilon}_i$$

From 1 and 3, we can define $\tilde{r}_p$ as follows. Then

$$\tilde{r}_p = \sum w_i E[\tilde{r}_i] + \sum \sum w_i b_{ij} \tilde{F}_i + \sim w_i \tilde{\epsilon}_i$$

From 2 and 3, then $\tilde{r}_p = \sum w_i E[\tilde{r}_i]$ we got rid of both types of risks. However, the zero cost assumption and the no arbitrage implies that 3 is zero. Thus $\sum w_i u_i = 0 \implies w'u = 0$.

Conclusion: 3 orthogonal relations:

1. $w'1 = 0$
2. $w'b = 0$
3. $w'u = 0$

$\mu = \lambda_0 1 + \lambda_1 b$.

6 Arrow-Debrew Economy

6.1 Digression on Competitive Equilibrium

Environment:

1. Assume 2 period: 0 (today) and 1 (tomorrow)
2. Preference of consumers: $U(C,l)$ where C is consumption, and l is leisure.

$$24 = 12 + 12$$

$$h = N^s + l = \text{time constraint}$$

   Total time available is equal to labor supply plus leisure.

3. Income: work; dividends = 0 (for simplicity).

   $y_0 = w_0 N_0^s$ and $y_1 = w_1 N_1^s$. Note: $N_0^s = h - l_0$. Therefore, $y_0 = w_0(h - l_0), y_1 = w_1(h - l_1)$.

4. Budget constraint;

   Period 0: $y_0 - t_0 = c_0 + s_0$

   Period 1: $y_1 - t_1 + s_0(1 + r) = c_1 + s_1$ where r could be the interest rate and return on stocks (financial market)

   Deriving the inter temporal budget over time.

   From equation 1 $s_0 = (y_0 - t_0) - c_0$. Plug 3 in 2, we get $(y_1 - t_1) + [(y_0 - t_0) - c_0](1 + r)$ =c_1. Therefore

$$(y_0 - t_0) + (y_1 - t_1)/(1 + r) = c_0 + c_1/(1 + r)$$

   Therefore consumer problem: given t_0, t_1, h, r, w, the consumer will choose $c_0, c_1, N_0^s, N^s, l_0, l_1$, such that $U(c_0, l_0)$ and $U(c_1, l_1)$ are maximized subject to

   (a) IBC: $(y_0 - t_0) + (y_1 - t_1)/(1 + r) = c_0 + c_1/(1 + r)$
   (b) Time: $h = N_0^s + l_0, h = N_1^s + e_1$

5. N consumers
6. producers (no firms for simplicity)
7. Government: object is to increase the welfare of people. This implies government will try to balance its budget. Government budget constrain in period 0: $B_0 + T_0 = G_0$. Period 1: $T_1 = G_1 + B_0(1 + r)$.

Derive the inter temporal B.C. for the government: isolate $B_0 = G_0 - T_0$ and plug it in and simplify:

$$G_0 + \frac{G_1}{1+r} = T_0 + \frac{T_1}{1+r}$$

8. Markets

   (a) Labour market: $N^d = N^s =$ fixed.

   (b) Financial Market (credit market): individuals and government can issue bonds at the market rate. total borrowings + lending = 0.

   $$S_0^T + S_0^G = 0$$

   (c) Goods market

   $$AD = AS$$

6.2 Competitive Equilibrium (CE)

A CE is a price vector and a set of rules such that:

1. Given t_0, t_1 and r, each consumers choose C_0^* and C_1^* such that $U(C_0, C_1)$ is max subject to IBC.

2. Government satisfies its IBC

$$G_0 + \frac{G_1}{1+r} = T_0 + \frac{T_1}{1+r}$$

3. All markets clear:

6.3 CE in an A-D economy; homogeneous Agents example

$$v^i = \frac{1}{2}(c_0^i) + \delta^i(\pi_1 v^i(c_1^i) + \pi_2 v^i(c_2^i))$$

$\pi_1 = \frac{1}{3}, \delta^1 = \delta^2 = 0.9$ and $\pi_2 = 2/3$. Endowments: $y_0^1 = 10, y_1^1 = 1, y_2^1 = 2$ and $y_0^2 = 5, y_1^2 = 4, y_2^2 = 6$. Fixed variables: $y_0, y_1, \delta, \pi_1, \pi_2$ and $p_1 \& p_2$. choice variables: c_0, c_1, c_2.

Solve optimal consumption choices: $(c_0^i)^* = f(P_1, P_2)$ and $(c_1^i)^* = f(P_1, P_2)$ and $(c_2^i)^* = f(P_1, P_2)$. Do this for both consumers: $i = 1, 2$. Use the market clearing conditions to solve for P_1^* and P_2^*.

Alternative solutions:

Consumer 1:

$$\frac{\partial L}{\partial c_0^1} = 0 \implies \lambda = \frac{1}{2}$$

$$\frac{\partial L}{\partial c_1^1} = 0 \implies \frac{0.3}{c_1^1} = \lambda P_1$$

$$\frac{\partial L}{\partial c_2^1} = 0 \implies \frac{0.6}{c_2^1} = \lambda P_2$$

$$\frac{\partial L}{\partial \lambda} = 0 \implies 10 + P_1 + 2P_2 = c_0^1 + P_1 c_1^1 + P_2 c_2^1$$

Therefore $(c_1^1)^* = \frac{0.6}{P_1}$ and $(c_2^1)^* = \frac{1.2}{P_2}$. Therefore, $10 + P_1 + 2P_2 = c_0^1 + P_1(\frac{0.6}{P_1}) + P_2(\frac{1.2}{P_2})$. Then $(c_0^1)^* = 8.2 + P_1 + 2P_2$.

Similarly for individual 2:

$\lambda = \frac{1}{2}$ and $(c_1^2)^* = \frac{0.6}{P_1}$ and $(c_2^2)^* = \frac{1.2}{P_2}$.

Budget Constraints for individual 2: $5 + 4P_1 + 6P_2 = c_0^2 + P_1 c_1^2 + P_2 c_2^2$. Rearrange the term, we get $(c_0^2)^* = 3.2 + 4P_1 + 6P_2$. Use any market clearing condition to solve for $A - D$ prices:

$$C_1 = y_1$$

$$c_1^1)^* + (c_1^2)^* = y_1^1 + y_1^2$$

Then $\frac{0.6}{P_1} + \frac{0.6}{P_1} = 5 \implies P_1 = 0.24$ and $(c^1)_2)^* + (c_2^2)^* = 2+ \implies P_2 = 0.3$.

6.3.1 Example on CE under certainty

Problem 6, page 39 in the textbook: 2 consumers (A and B), 2 periods (0 and 1), certainty (no states in period 1), heterogeneous.

Endowments:

- $y_0^A = 11, y_1^A = 10$
- $y_0^B = 11, y_1^B = 10$
- $t_0^A = 2, t_0^B = 2, T_0 = 4, G_0 = 4$
- $t_1^A = 1, t_1^B = 1, T_1 = 2, G_1 = 2$

CE: a set of price and decision rules such that

1. given t_0, t_1, y_0, y_1, r, each consumer chooses c_0 and c_1 such that $V(c_0, c_1)$ is max subject to the IBC.
2. Government should satisfy: $G_0 + \frac{G_1}{1+r} = T_0 + \frac{T_1}{1+r}$
3. Market clearing:
   (a) Goods: $C_0 + G_0 = Y_0, C_1 + G_1 = Y_1$.
   (b) credit: $S_0^P + S_0^G = 0$

Solve for r^*?

Using a general framework: $(C_0^A)^*, (C_1^A)^*, (C_0^B)^*, (C_1^B)^*$ as functions of r. use any market clearing condition to solve for r^*.

Consumer A $\max v^A = \ln c_0^A + 0.5 \ln c_1^A$ and $\{c_0^A, c_1^A\}$. subject to $[y_0^A - t_0^A + \frac{y_1^A - t_1^A}{1+r}] = c_0^A + \frac{c_1^A}{1+r}$. Therefore $[9 + \frac{9}{1+r}] = c_0^A + \frac{c_1^A}{1+r}$.

$$\frac{MU_0}{MU_1} = (1+r)$$

a tangency condition. Then $W^A = c_0^A + \frac{c_1^A}{1+r}$

Same thing for B.

Use the credit market clearing condition:

$$S_0^P + S_0^G = 0$$

$$S_0^A + S_0^B + S_0^G = 0$$

$$[y_0^A - t_0^A - (c_0^A)^*] + [y_0^B - t_0^B - (c_0^B)^*] + T_0 - G_0 = 0$$